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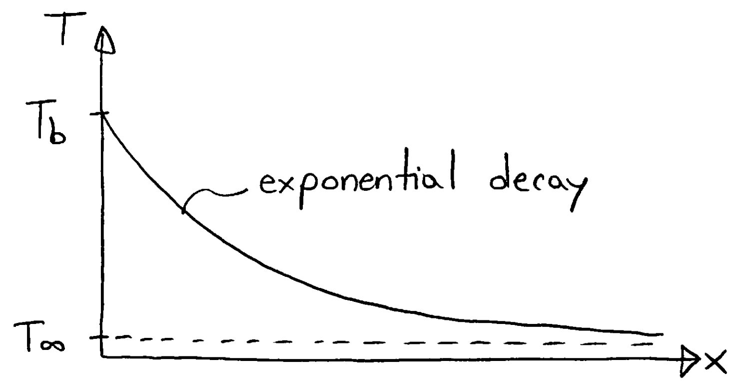
Applying our first B.C.:

$$\theta(x=0) = \theta_b = C_2 \underbrace{e^{-m(0)}}_{e^0 = 1} \Rightarrow C_2 = \theta_b$$

$$\boxed{\theta(x) = \theta_b e^{-mx}}$$

Or in dimensional form:

$$\boxed{\frac{T - T_\infty}{T_b - T_\infty} = e^{-\sqrt{\frac{hP}{kA}}\cdot x}} \Rightarrow \text{Fin temperature profile:}$$



So how do we calculate heat transfer?

Let's calculate it at the base because all of our energy comes from the base:

$$Q_{net} = Q(x=0) = -kA\frac{\partial \theta}{\partial x}\bigg|_{x=0} = +kA\theta_b \underbrace{e^{-m(0)}}_{1}$$

$$\boxed{Q_{net} = q_{net} = +kA\theta_b\sqrt{\frac{hP}{kA}} = \sqrt{kAhP}\,\theta_b}$$

Note, if we compare our heat transfer to that of the case with no fin, we get:

$$\frac{Q_{net}}{Q_0} = \frac{\sqrt{kAhP}\,\cancel{\theta_b}}{hA\cancel{\theta_b}} = \sqrt{\frac{kP}{hA}} = \sqrt{\frac{k}{ht}} = Bi_t^{-1/2} >> 1 \text{ since } Bi_t \ll$$

Hence adding the fin enhances the heat transfer very much!

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Note here I assumed: $\frac{P}{A} = \frac{1}{t}$

We can do another sanity check:

$$Q_{net} = \int_0^\infty hP\theta\, dx = \int_0^\infty hP\theta_b e^{-mx} dx$$

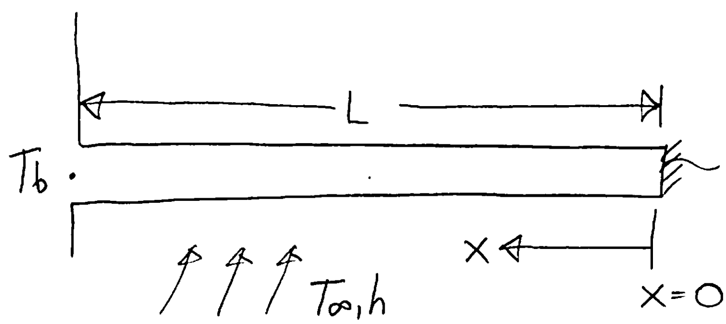
$$= hP\theta_b \left.\frac{e^{-mx}}{m}\right|_0^\infty = \frac{hP\theta_b}{m} = \sqrt{kAhP}\,\theta_b$$

$\Downarrow$

Works!

Same answer as before!

② <u>Insulated Tip</u> $\left(-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0\right)$



We know our fundamental differential fin equation remains the same, so we will have the same general solution:

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

We can re-write this as:

$$\theta(x) = C_1 \cosh(mx) + C_2 \sinh(mx)$$

<u>Remember:</u>

$$\sinh(mx) = \frac{e^{mx} - e^{-mx}}{2}$$

$$\cosh(mx) = \frac{e^{mx} + e^{-mx}}{2}$$

Our new B.C.'s are:

1) $\theta(x=L) = \theta_b$; 2) $-k\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0$

Back substituting our B.C.'s

$$\frac{\partial\theta}{\partial x} = C_1 m \sinh(mx) + C_2 m \cosh(mx)$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = C_1 m \underbrace{\sinh(0)}_{0} + C_2 m \underbrace{\cosh(0)}_{1} = 0$$

$$\boxed{C_2 = 0}$$

Applying our first B.C.

$$\theta(x=L) = \theta_b = C_1 \cosh(mL)$$

$$\boxed{C_1 = \frac{\theta_b}{\cosh(mL)}}$$

Back substituting C_1 & C_2 into our solution:

$$\theta(x) = \frac{\theta_b \cosh(mx)}{\cosh(mL)}$$

$$\boxed{\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh(mx)}{\cosh(mL)}} \Rightarrow$$ Temperature profile along the fin for insulated tip.

How about heat transfer?

$$Q_{net} = -kA \left.\frac{\partial\theta}{\partial x}\right|_{x=L} = +kA\,\theta_b\, m \frac{\sinh(mL)}{\cosh(mL)} = kA\,\theta_b \sqrt{\frac{hP}{kA}} \underbrace{\frac{\sinh(mL)}{\cosh(mL)}}_{\tanh(mL)}$$

$$\boxed{Q_{net} = \sqrt{kAhP}\,(T_b - T_\infty) \tanh(mL)}$$

↳ Heat transfer from a fin with an insulated tip.

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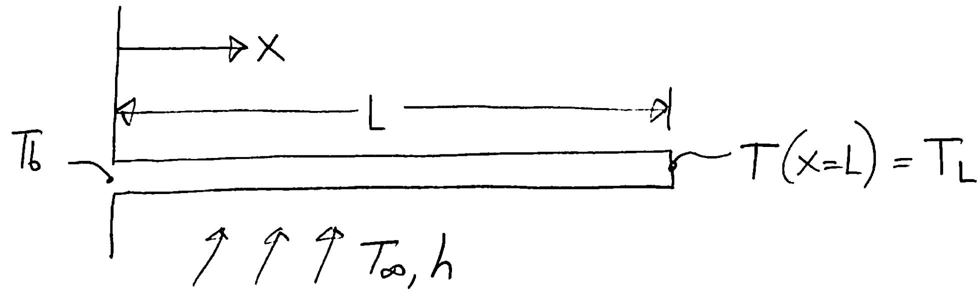
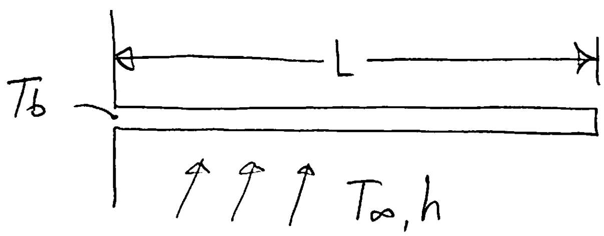
(3.) Perscribed Tip Temperature $\left(T(x=L)=T_L\right)$



$$\boxed{\frac{T-T_\infty}{T_b-T_\infty}=\frac{T_L-T_\infty}{T_b-T_\infty}\cdot\frac{\sinh(mx)+\sinh\left[m(L-m)\right]}{\sinh(mL)}}$$

(4.) Convection at the Tip $\left(h\left(T(x=L)-T_\infty\right)=-k\left.\frac{\partial T}{\partial x}\right|_{x=L}\right)$



$$\boxed{\frac{T-T_\infty}{T_b-T_\infty}=\frac{\cosh\left(m(L-x)\right)+\frac{h}{mk}\sinh\left(m(L-x)\right)}{\cosh(mL)+\frac{h}{mk}\sinh(mL)}}$$

Fin Efficiency

We can define a fin efficiency as:

$$\eta_{fin}=\frac{\text{Actual Heat Transfer with Fin}}{\text{Heat Transfer if } \theta=\theta_b \text{ everywhere on Fin}}$$

$$=\frac{q_{actual}}{q_{ideal}}$$

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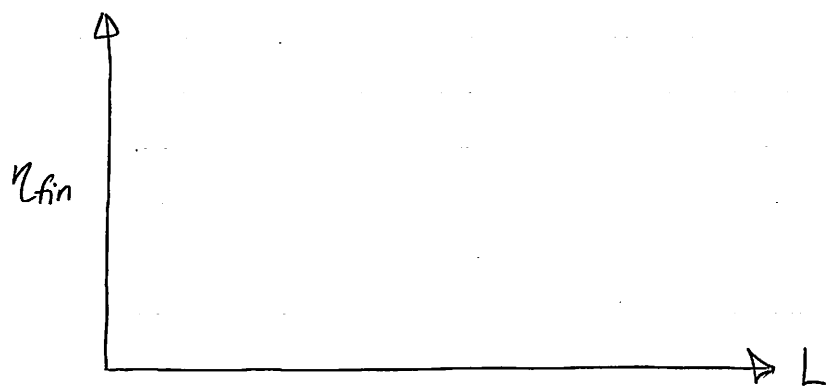
$$q_{ideal} = h(PL)(T_b - T_\infty)$$ ⇒ Note this is for the insulated tip case.

$$q_{actual} = \int_0^L hP(T - T_\infty)\,dx = \sqrt{kAhP}\,\theta_b \tanh(mL)$$

$$\eta_{fin} = \frac{\sqrt{kAhP}\,\cancel{\theta_b}\tanh(mL)}{hPL\cancel{\theta_b}} = \underbrace{\frac{\sqrt{kA}}{\sqrt{hP}}}_{1/m} \cdot \frac{\tanh(mL)}{L}$$

$$\boxed{\eta_{fin} = \frac{\tanh(mL)}{mL}}$$ ⇒ Insulated Tip Fin Efficiency
For others, see Table 3.5 of Textbook (pg. 168)



Fin Resistance ($R = \Delta T / Q$)

$$R_{fin} = \frac{\theta_b}{q_{actual}} = \frac{T_b - T_\infty}{q_{actual}} = \frac{\cancel{\theta_b}}{\sqrt{kAhP}\tanh(mL)\cancel{\theta_b}} = \frac{1}{hA_{fin}\eta_{fin}}$$

$$\boxed{R_{fin} = \frac{1}{hA_{fin}\eta_{fin}}}$$ ⇒ A_{fin} = fin outside area (PL)
η_{fin} = fin efficiency

Fin Effectiveness (ε_{fin})

$$\varepsilon_f = \frac{\text{actual heat transfer}}{\text{heat transfer if no fin}} = \frac{\cancel{q_{ideal}}\cdot\eta_{fin}}{\cancel{q_{ideal}}\left(\frac{A}{PL}\right)} \Rightarrow \boxed{\varepsilon_f = \eta_{fin}\cdot\frac{PL}{A}}$$