Lumped Capacitance Analysis (Transient Problems)

This analysis method is very valuable for a whole host of problems.

\[ A_s \text{ (outer surface area for convective heat transfer)} \]

\[ V = \text{Volume } [m^3] \]
\[ \rho = \text{density } [\text{kg/m}^3] \]
\[ c = \text{heat capacity } [\text{J/kg.K}] \]
\[ T_i = \text{initial temperature of whole body } [\text{K}] \]

Doing an energy balance on our body of interest:

\[ E_{\text{in}} - E_{\text{out}} + E_{\text{gen}} = E_{\text{stored}} \]

\[ -hA_s(T - T_\infty) = \frac{\partial}{\partial t}(\rho V cT) \Rightarrow \text{Assume constant properties} \]

\[ \forall \rho c \frac{\partial T}{\partial t} + hA_s(T - T_\infty) = 0 \]

\[ \frac{\partial \theta}{\partial t} + \frac{hA_s}{\forall \rho c} (T - T_\infty) = 0 \Rightarrow \text{Let } \theta = T - T_\infty \]

\[ \frac{\partial \theta}{\partial t} + \frac{hA_s}{\forall \rho c} \theta = 0 \Rightarrow \text{Let } \lambda = \frac{hA_s}{\forall \rho c} \]

\[ \frac{\partial \theta}{\partial t} = -\lambda \theta dt \]

\[ \ln \theta = -\lambda t + C_2 \]

\[ e^{\ln \theta} = e^{-\lambda t + C_2} \Rightarrow \theta(t) = C_3 e^{-\lambda t} \]
Our B.C. or I.C. is: \( \Theta(t=0) = \Theta_i = T_i - T_a \)

\[ \Theta(t=0) = C_3 e^{\frac{-t}{\tau}} = \Theta_i \Rightarrow C_3 = \Theta_i \]

\[ \Theta(t) = \Theta_i e^{-\frac{hA_s t}{\rho c}} \Rightarrow \text{Lumped capacitance model.} \]

Exponential decay in body temp.

Note, the lumped capacitance model is only valid for bodies that change temperature uniformly as they cool or heat up.

\[ Bi \leq 0.1 \]

Remember, for a slab: \( Bi_{slab} = \frac{ht}{k} \leq 0.1 \)

What about a cylinder or sphere? What thickness do we use in our \( Bi \)?

\[ l = \frac{\text{Volume}}{\text{Surface Area}} \Rightarrow \text{Slab} \Rightarrow \frac{t \cdot l - w}{2(l - w)} = t; \quad Bi_{slab} = \frac{ht}{k} \]

\[ \text{Cylinder} \Rightarrow \frac{\pi R^2 K}{2\pi RK} = \frac{R}{2}; \quad Bi_{cyl} = \frac{hR}{2k} \]

\[ \text{Sphere} \Rightarrow \frac{4}{3} \pi R^3 K = \frac{R}{3}; \quad Bi_{sph} = \frac{hR}{3k} \]
Example 1: A steel quenching process for ball bearings requires the ball bearing to reach 100°C in order to work. The initial ball temperature is 1200°C, and it is dipped in cold water at 25°C. The ball radius is \( R = 1 \text{mm} \). The heat transfer coefficient is \( h = 10,000 \text{W/m}^2\cdot\text{K} \). Find how long the quenching time should be.

\[
k_{\text{steel}} \approx 50 \text{W/m}\cdot\text{K} \\
C_{\text{steel}} = 502 \text{J/kg}\cdot\text{K} \\
\rho_{\text{steel}} = 8000 \text{kg/m}^3 \\
T_i = 1200°C \\
T_\infty = 25°C \\
h = 10,000 \text{W/m}^2\cdot\text{K}
\]

First step: Check the \( B_{\text{sphere}} \leq 0.1 \)

\[
B_{\text{sph}} = \frac{hR_{\text{sph}}}{k} = \frac{(10,000 \text{W/m}^2\cdot\text{K})(0.001 \text{m})}{3(50 \text{W/m}\cdot\text{K})} = 0.066 < 0.1
\]

\( B_{\text{sph}} \leq 0.1 \Rightarrow \text{Lumped Capacitance OK to use.} \)

We just solved that:

\[
\Theta(t) = T - T_\infty = (T_i - T_\infty) e^{-\frac{hA(t)}{4\pi\rho c}}
\]

\[
\frac{100°C - 25°C}{1200°C - 25°C} = e^{-\frac{(10,000)(3)t}{(0.001)(8000)(502)}}
\]

\[
0.064 = e^{-7.47t}
\]

\[
\ln(0.064) = -7.47t
\]

\[
t = 0.375
\]
If we look back at our solution for lumped capacitance

\[ \frac{\Theta}{\Theta_i} = \exp \left[ -\frac{hA_s t}{V_0 c} \right] \]

Dimensionless \quad \text{Dimensionless} \implies \text{Implies that}:

\[ C = \left( \frac{1}{hA_s} \right) (\rho + c) = R_t \cdot C_t \quad \text{(Thermal time constant)} \]

\[ \text{Convective} \quad \text{Thermal} \]
\[ \text{Resistance} \quad \text{Capacitance} \]
\[ = R_t = C_t \]

The larger \( C \) is, the slower the response of the body.

Any increase in \( R_t \) or \( C_t \) will cause a solid to respond more slowly to changes in its thermal environment.

Analogous to the voltage decay when discharging a capacitor through a resistor in an \( RC \) circuit.

- The larger the capacitance \( C_e \), the more energy stored and the slower the voltage decay.
- The larger the thermal capacitance \((\rho + c)\), the more energy stored and the slower the temperature decay.
- The larger the resistor (electrical or thermal), the smaller the current (electrical or thermal) and the slower the temperature decay.
We can now draw our solutions in dimensionless form:

\[
\frac{\Theta}{\Theta_i} = \frac{T - T_\infty}{T_i - T_\infty}
\]

\[
C = \frac{\varphi A_c}{\rho h A_s} = R_t C_t
\]

We can take the analysis one step further and say:

\[
\frac{h A_s t}{\rho c v^2} = \frac{ht}{\rho c L_c} \Rightarrow L_c = \frac{v}{A_s} \text{ (length scale)}
\]

\[
= \frac{h L_c}{k} \cdot \frac{k}{\rho c} \cdot \frac{t}{L_c^2} = \frac{h L_c}{k} \cdot \frac{\alpha t}{L_c^2} \Rightarrow \frac{\Theta}{\Theta_i} = e^{-\frac{8}{\text{Bi} F_0}}
\]

Lumped capacitance analysis in dimensionless form.

\[
F_0 = \text{Fourier } \# = \frac{\alpha t}{L_c^2} \Rightarrow \text{Dimensionless time that characterises transient heat transfer prob.}
\]

\[
\alpha = \text{Thermal diffusivity} = \frac{k}{\rho c}
\]

\[
\text{Describes how quickly a material responds to a change in temperature. (Thermal inertia)}
\]

Note:

\[
F_0 = \frac{\text{Diffusive transport rate}}{\text{Storage rate}} = \frac{\alpha t}{L_c^2} = \frac{k t}{\rho c L_c^2} \frac{(\Delta T)}{\Delta T} = \frac{(k \Delta T)}{L} \left( \frac{\rho c \Delta T L}{t} \right)^{-1} = \frac{q_{\text{cond}}}{q''_{\text{stored}}}
\]