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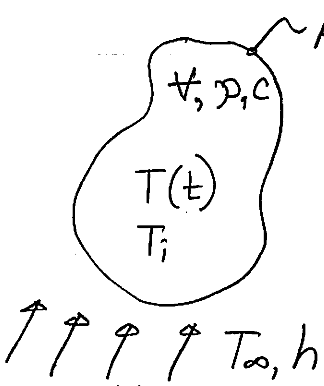
Lumped Capacitance Analysis (Transient Problems)

This analysis method is very valuable for a whole host of problems



A_s (outter surface area for convective heat transfer)

$\forall$ = Volume $[m^3]$
ρ = density $[kg/m^3]$
c = heat capacity $[J/kg \cdot K]$
T_i = initial temperature of whole body $[K]$

Doing an energy balance on our body of interest:

$$\underbrace{\cancel{E_{in}}}_{0} - E_{out} + \underbrace{\cancel{E_{gen}}}_{0} = E_{stored}$$

$$-hA_s(T - T_\infty) = \frac{\partial}{\partial t}(\rho \forall c T) \Rightarrow \text{Assume constant properties}$$

$$\forall \rho c \frac{\partial T}{\partial t} + hA_s(T - T_\infty) = 0$$

$$\frac{\partial T}{\partial t} + \frac{hA_s}{\forall \rho c}(T - T_\infty) = 0 \Rightarrow \text{Let } \theta = T - T_\infty$$

$$\frac{\partial \theta}{\partial t} + \frac{hA_s}{\forall \rho c}\theta = 0 \Rightarrow \text{Let } \lambda = \frac{hA_s}{\forall \rho c}$$

$$\frac{\partial \theta}{\theta} = -\int \lambda \, dt$$

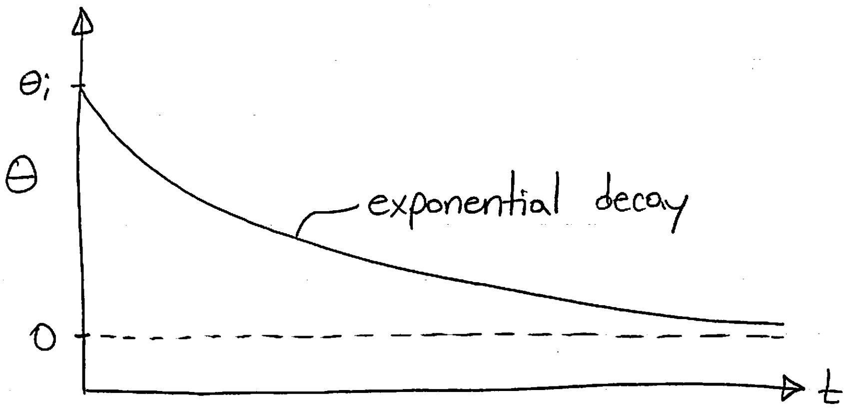
$$\ln \theta = -\lambda t + C_2$$

$$e^{\ln \theta} = e^{-\lambda t + C_2} \Rightarrow \theta(t) = C_3 e^{-\lambda t}$$

Our B.C. or I.C. is: $\theta(t=0) = \theta_i = T_i - T_\infty$

$$\theta(t=0) = C_3 \underbrace{e^{-(0)}}_{1} = \theta_i \Rightarrow \boxed{C_3 = \theta_i}$$

$$\boxed{\theta(t) = \theta_i \, e^{-\frac{h A_s t}{\forall \rho c}}}$$ ⇒ Lumped capacitance model. Exponential decay in body temp.



Note, the lumped capacitance model is only valid for bodies that change temperature uniformly as they cool or heat up.

$$\boxed{Bi < 0.1}$$

t ≡ slab thickness not time!

Remember, for a <u>slab</u>: $\boxed{Bi_t = \frac{ht}{k} \leq 0.1}$

What about a cylinder or sphere? What thickness do we use in our Bi?

$$t = \frac{\text{Volume}}{\text{Surface Area}}$$

Sometimes labeled L_c for length scale.

slab thickness

⇒ Slab ⇒ $\frac{t \cdot \cancel{L} \cdot \cancel{W}}{2(\cancel{L \cdot W})} = t$; $\boxed{Bi_{slab} = \frac{ht}{k}}$

Cylinder ⇒ $\frac{\cancel{\pi} R^{\cancel{2}} \cancel{L}}{2\cancel{\pi}\cancel{R}\cancel{L}} = \frac{R}{2}$; $\boxed{Bi_{cyl} = \frac{hR}{2k}}$

Sphere ⇒ $\frac{\frac{\cancel{4}}{3}\cancel{\pi} R^{\cancel{3}}}{\cancel{4\pi}\cancel{R^2}} = \frac{R}{3}$; $\boxed{Bi_{sph} = \frac{hR}{3k}}$

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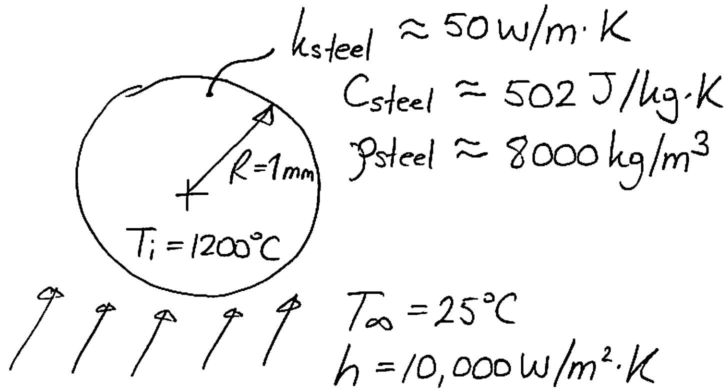
Example | A steel quenching process for ball bearings requires the ball bearing to reach 100°C in order to work. The initial ball temperature is 1200°C, and it is dipped in cold water at 25°C. The ball radius is $R = 1mm$. The heat transfer coefficient is $h = 10{,}000 \ W/m^2 \cdot K$. Find how long the quenching time should be.



First step : Check the $Bi_{sphere} < 0.1$

$$Bi_{sph} = \frac{hR_{sph}}{k} = \frac{(10{,}000 \ W/m^2 \cdot K)(0.001m)}{3(50 \ W/m \cdot K)} = 0.066 < 0.1$$

$\therefore \ Bi_{sph} \leq 0.1 \Rightarrow$ Lumped Capacitance OK to use.

We just solved that :

$$\theta(t) = T - T_\infty = (T_i - T_\infty)e^{-\frac{hA_st}{\forall \rho C}}$$

$$\frac{100°C - 25°C}{1200°C - 25°C} = e^{-\frac{(10{,}000)(3)t}{(0.001)(8000)(502)}}$$

$$0.064 = e^{-7.47t}$$

$$\ln(0.064) = -7.47t$$

$$\boxed{t = 0.37s}$$

If we look back at our solution for lumped capacitance

$$\underbrace{\frac{\theta}{\theta_i}}_{\text{Dimensionless}} = \exp\left[-\underbrace{\frac{hA_s t}{\forall \rho c}}_{\text{Dimensionless}}\right] \Rightarrow \text{Implies that:}$$

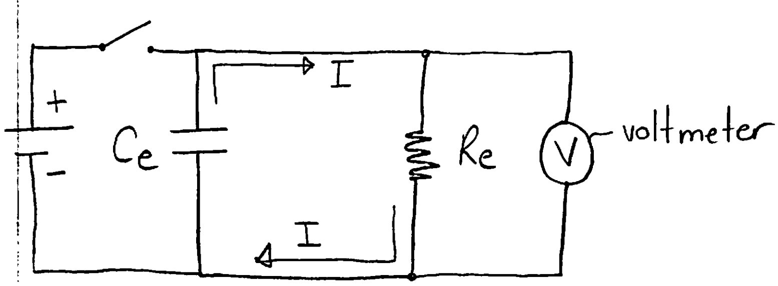
$$\tau = \underbrace{\left(\frac{1}{hA_s}\right)}_{\substack{\text{Convective Resistance} \\ \equiv R_t}} \underbrace{(\rho \forall c)}_{\substack{\text{Thermal Capacitance} \\ \equiv C_t}} = R_t \cdot C_t \quad \text{(Thermal time constant)}$$

$\Downarrow$

The larger τ is, the slower the response of the body.

Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment.

Anologous to the voltage decay when discharging a capacitor through a resistor in an RC circuit.



$$V(t) = V_0 e^{-\frac{1}{R_e C_e}} \quad \text{solution}$$

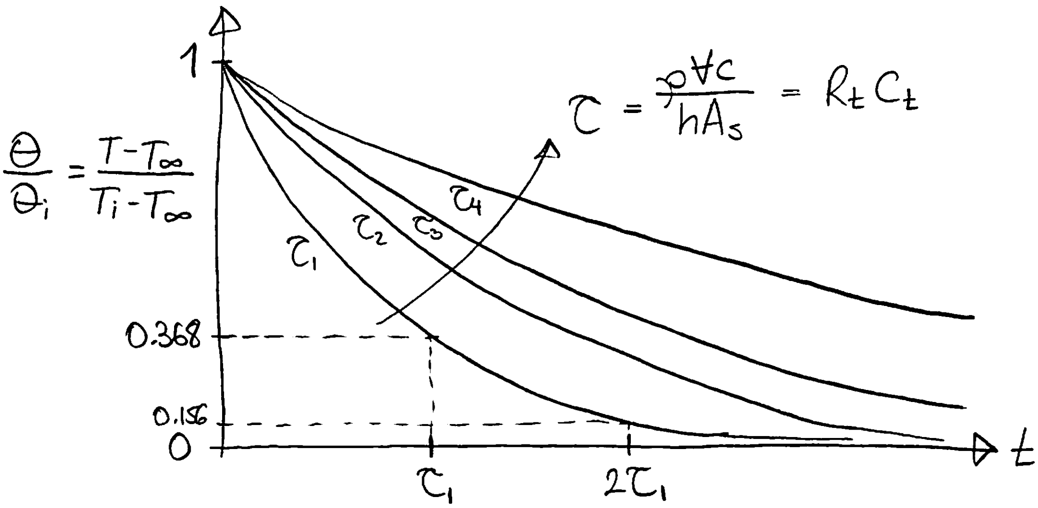
$$V = I R_e \quad \text{(Voltage)}$$

$$\underbrace{C_e \frac{dV}{dt}}_{\substack{\text{Current at} \\ \text{capacitor}}} + \underbrace{\frac{V}{R_e}}_{\substack{\text{Current through} \\ \text{resistor}}} = 0$$

(e)
- The larger the capacitance C_e, the more energy stored and the slower the voltage decay

(t)
- The larger the thermal capacitance $(\rho \forall c)$, the more energy stored and the slower the temperature decay
- The larger the resistor (electrical or thermal), the smaller the current (electrical or thermal) and the slower the temp decay

We can now draw our solutions in dimensionless form:



We can take the analysis one step further and say:

$$\frac{hA_s t}{\rho \forall c} = \frac{ht}{\rho c L_c} \Rightarrow L_c \equiv \frac{\forall}{A_s} \text{ (length scale)}$$

$$= \frac{hL_c}{k} \cdot \frac{k}{\rho c} \cdot \frac{t}{L_c^2} = \underbrace{\frac{hL_c}{k}}_{Bi} \cdot \underbrace{\frac{\alpha t}{L_c^2}}_{Fo} \Rightarrow \boxed{\frac{\theta}{\theta_i} = e^{-Bi\,Fo}}$$

↳ Lumped capacitance analysis in dimensionless form.

$\boxed{Fo \equiv \text{Fourier \#} = \frac{\alpha t}{L_c^2}}$ ⇒ Dimensionless time that characterises transient heat transfer prob.

$\boxed{\alpha = \text{Thermal diffusivity} = \frac{k}{\rho c}}$ ⇒ Describes how quickly a material responds to a change in temperature. (Thermal inertia)

Note:

$$\boxed{Fo \equiv \frac{\text{Diffusive transport rate}}{\text{Storage rate}}}$$

$$Fo = \frac{\alpha t}{L_c^2} = \frac{kt}{\rho c L_c^2} \cdot \left(\frac{\Delta T}{\Delta T}\right) = \left(\frac{k \Delta T}{L}\right) \Big/ \left(\frac{\rho c \Delta T L}{t}\right) = q''_{cond} / q''_{stored}$$