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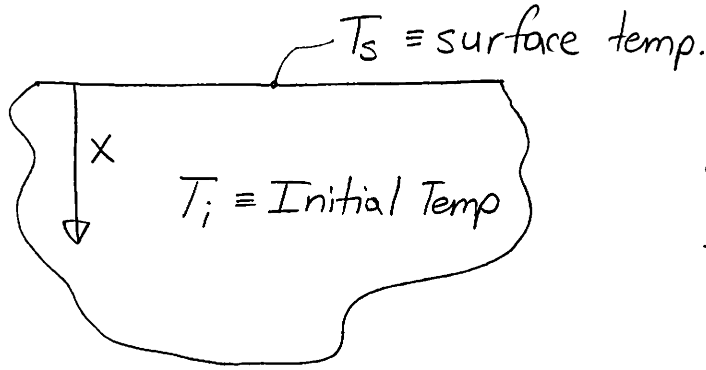
Transient Conduction

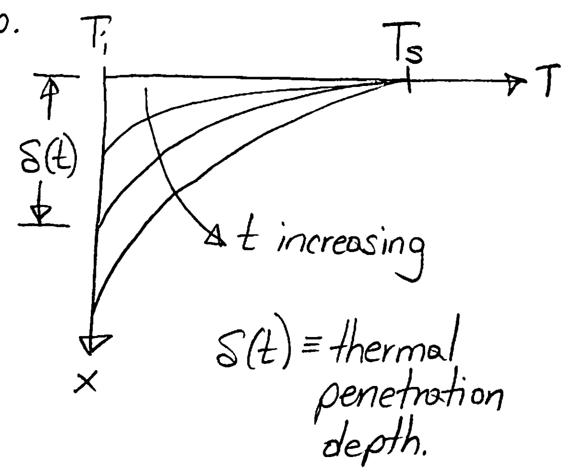
So far, we have been mainly dealing with steady-state conduction problems. What if transient effects are important and $Bi > 0.1$?

$$\frac{\partial^2 T}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 T}{\partial y^2}}}_{0\ 1D} + \underbrace{\cancel{\frac{\partial^2 T}{\partial z^2}}}_{0\ 1D} + \underbrace{\cancel{\frac{Q'''}{k}}}_{0\ (Q'''=0)} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ $\Rightarrow$ Second order PDE. Need 2 BC's & 1 IC.

Semi-Infinite Body





The surface of the semi-infinite body suddenly experiences a finite temperature (T_s), how does the body temperature $T(x)$ change with time (t)?

Let's non dimensionalise our temperature & solve our PDE:

$$\theta = \frac{T - T_s}{T_i - T_s}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

1) $x = 0,\ \theta = 0\ \ (T(x=0) = T_s)$
2) $x \to \infty,\ \theta = 1\ \ (T(x\to\infty) = T_i)$
3) $t = 0,\ \theta = 1\ \ (T(x, t=0) = T_i)$

↳ Note, separation of variables won't work due to infinite med.

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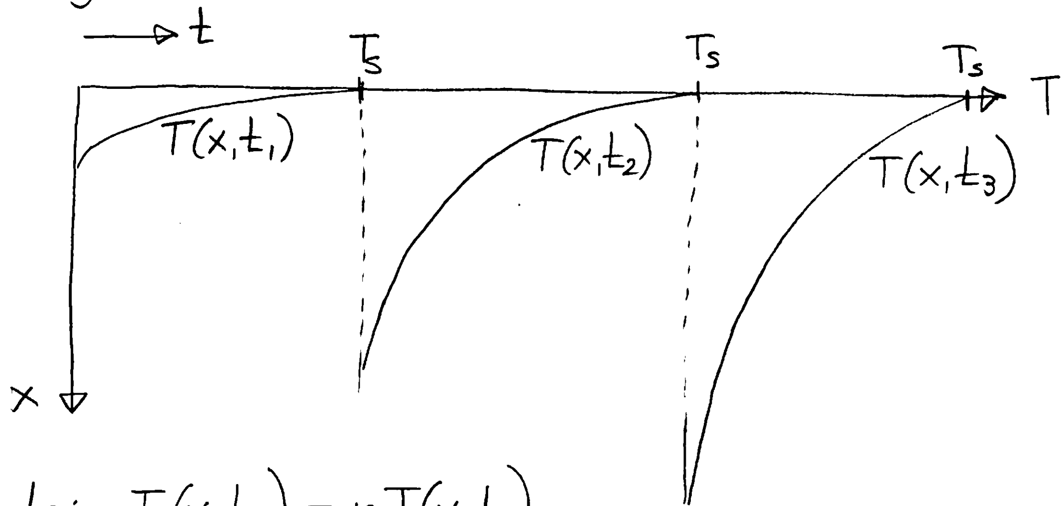
The best way to solve this equation is to look for a scaling parameter or a similarity variable.



<u>Note:</u> $T(x,t_2) = nT(x,t_1)$
$T(x,t_3) = nT(x,t_2) = mT(x,t_1)$

All of the temperature profiles look self similar in nature, just scaled up or down. This tells us there is a fundamental underlying similarity variable:

<u>Assume:</u> $\eta = \dfrac{x}{f(t)}$, so $\theta(\eta)$, $f(t) \equiv$ function of time

Transforming our equation:

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{1}{f}\frac{\partial \theta}{\partial \eta}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial \eta}\right) = \frac{1}{f}\frac{\partial}{\partial \eta}\left(\frac{\partial \theta}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \frac{1}{f^2}\frac{\partial^2 \theta}{\partial \eta^2}$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = -\frac{\partial \theta}{\partial \eta}\cdot x\cdot\frac{f'}{f^2} = -\eta\frac{\partial \theta}{\partial \eta}\frac{f'}{f}$$

Back substituting into our PDE

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$$\frac{1}{f^2}\frac{\partial^2\theta}{\partial\eta^2} + \frac{f'}{f\alpha}\eta\frac{\partial\theta}{\partial\eta} = 0$$

$$\frac{\partial^2\theta}{\partial\eta^2} + \underbrace{\left(\frac{f\cdot f'}{\alpha}\right)}\eta\frac{\partial\theta}{\partial\eta} = 0$$

Let this equal to a constant $= 2$ (Could chouse any number)

→ Note, this may seem arbitrary but it ends up working out & providing a solution.

$$\frac{ff'}{\alpha} = 2$$

$$\frac{f\partial f}{\partial t} = 2\alpha \Rightarrow \int f\partial f = \int 2\alpha\,\partial t$$

$$\frac{f^2}{2} = 2\alpha t$$

$$\boxed{f = 2\sqrt{\alpha t}} \Rightarrow \boxed{\eta = \frac{x}{2\sqrt{\alpha t}}}$$

So our equation becomes:

$$\frac{\partial^2\theta}{\partial\eta^2} + 2\eta\frac{\partial\theta}{\partial\eta} = 0 \Rightarrow \text{ODE only! Nice!}$$

B.C.'s: $\left.\begin{array}{l}\eta = 0,\ \theta = 0\\ \eta\to\infty,\ \theta = 1\end{array}\right\}$ Turned our 2BC's & IC into 2BC's only.

Rewriting our ODE:

$$\frac{\theta''}{\theta'} = 2\eta$$

$$\frac{\partial}{\partial\eta}(\ln\theta') = \frac{\theta''}{\theta}$$

$$\frac{\partial}{\partial\eta}(\ln\theta') = -2\eta \quad \text{(Integrate)}$$

$$\ln(\theta') = -\frac{\cancel{2}\eta^2}{\cancel{2}} + C = -\eta^2 + C_1 \quad \text{(Take exponent on both sides)}$$

$$\theta' = C_2 e^{-\eta^2}$$

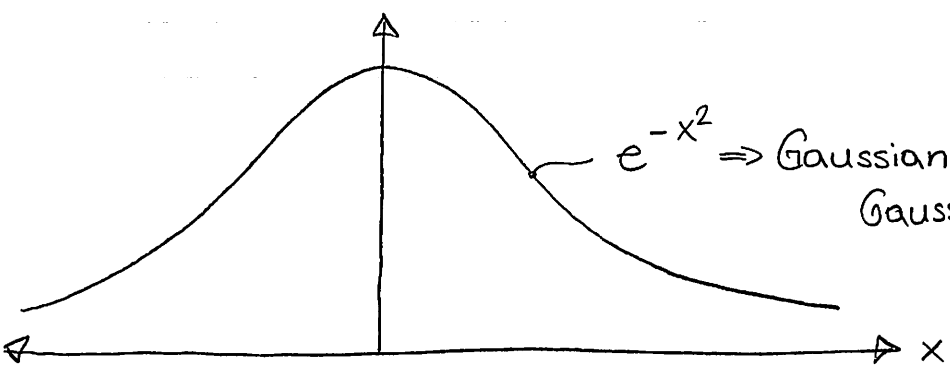
Integrating one more time:

$$\theta = C_2 \int_0^{\eta} e^{-\eta^2} d\eta + C_3$$

Applying our B.C.'s

$$\theta(\eta = 0) = 0 \Rightarrow \boxed{C_3 = 0}$$

$$\theta(\eta \to \infty) = 1 \Rightarrow 1 = C_2 \underbrace{\int_0^{\infty} e^{-\eta^2} d\eta}_{\text{Let's chat about this a bit!}}$$



$e^{-x^2} \Rightarrow$ Gaussian function or Gaussian Distribution

The function derscribes "error" or difference between a measurement and its unbiased estimator (or mean).

$\int_0^{x} e^{-x^2} dx$ cannot be calculated analytically, but it can be shown analytically that:

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}} \Rightarrow \boxed{\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}} \Rightarrow \text{Symmetric function about } x=0$$

So now we can define the following:

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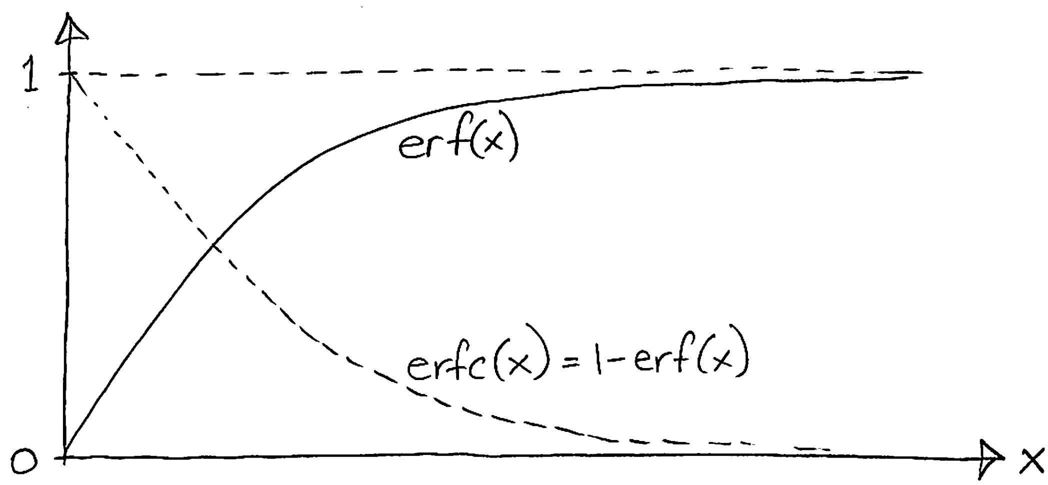
We define the "error" function as:

$$\boxed{\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt} \quad \text{Error Function}$$

$$\boxed{\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)} \quad \text{Complimentary Error Function}$$



So now we can go back and solve. We had the following:

$$1 = C_2 \underbrace{\int_0^\infty e^{-\eta^2}\,d\eta}_{\frac{\sqrt{\pi}}{2}} \Rightarrow \boxed{C_2 = \frac{2}{\sqrt{\pi}}}$$

$$\therefore \boxed{\Theta = \frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta = \mathrm{erf}(\eta) = \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}$$

↳ Tabulated results for erf(η) on pg. 1015 of Textbook Appendix B, Table B.2

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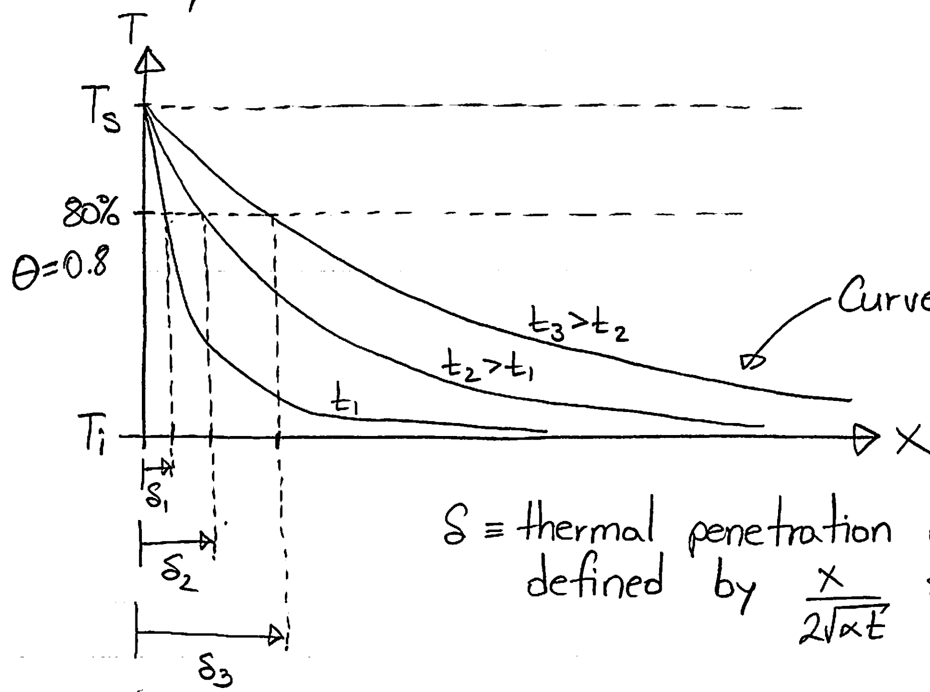
If we plot our result:



Curves are all self-similar

$\delta \equiv$ thermal penetration depth which is defined by $\dfrac{x}{2\sqrt{\alpha t}}$ since $\theta = 0.8$

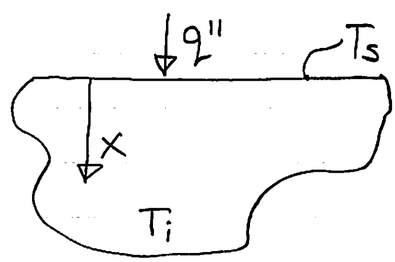
In our equation, we had:

$$\theta = \frac{T-T_s}{T_i-T_s} = 0.8 = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \operatorname{erf}\left(\frac{\delta}{2\sqrt{\alpha t}}\right)$$

$$\frac{\delta}{2\sqrt{\alpha t}} = \text{constant} \Rightarrow \delta \sim 2\sqrt{\alpha t}$$

So we can say that the heat penetration depth proceeds with $\sqrt{\alpha t}$. Usually we say δ is found at $\theta = 0.95$ or 0.99.

How do we determine heat flux?



$$q''\Big|_{x=0} = -k\frac{\partial T}{\partial x}\Big|_{x=0}$$

$$\theta = \frac{T-T_s}{T_i-T_s} \; ; \quad d\theta = \frac{1}{T_i-T_s}\,dT$$