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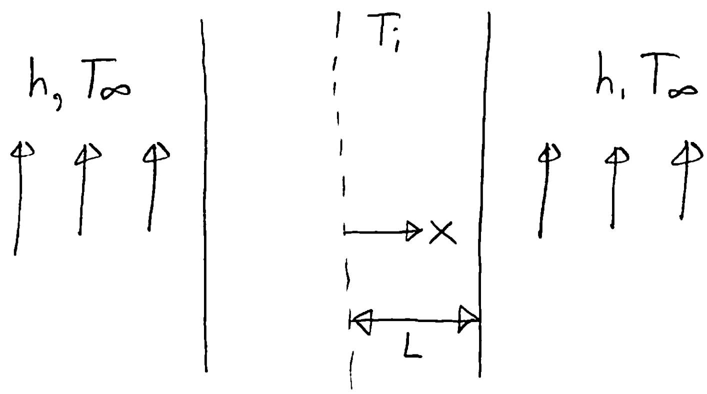
Transient Heat Conduction in Finite Bodies

So far, we've covered the cases of: 1) Lumped capacitance, $T \neq f(x)$
2) Infinite media, $0 \leq x \leq \infty$

How do we handle bodies that are finite in size & have $Bi_L > 0.1$?

Let's solve the plane wall problem. Slab initially placed in a medium with T_∞ & h. The slab has initial temp. T_i.



What is $T(x,t)$?

Looking at our heat equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Assuming: 1) 1D
2) $Q''' = 0$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

We need 2 B.C.'s and an IC to solve:

1) $T(x, t=0) = T_i$ (Initial condition)

2) $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ (1'st B.C.)

3) $-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h(T(x=L) - T_\infty)$ (2'nd B.C.)

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To make our solution more tractable, we will non-dimensionalize

Let: $\bar{x} = \frac{x}{L}$ (Dimensionless distance from center, $0 \le \bar{x} \le 1$)

$$\frac{\partial \bar{x}}{\partial x} = \frac{1}{L} \Rightarrow \partial \bar{x} = \frac{\partial x}{L}$$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \quad \text{(Dimensionless temperature, } 0 \le \theta \le 1\text{)}$$

$(T_i, T_\infty = \text{constant})$

$$\frac{\partial \theta}{\partial T} = \frac{\partial}{\partial T}\left(\frac{T}{T_i - T_\infty}\right) - \cancel{\frac{\partial}{\partial T}\left(\frac{T_\infty}{T_i - T_\infty}\right)}^{\,0} = \frac{1}{T_i - T_\infty} \Rightarrow \partial \theta = \frac{\partial T}{T_i - T_\infty}$$

How about dimensionless time?

$$\frac{\partial T}{\partial x} = \frac{(T_i - T_\infty)}{L} \frac{\partial \theta}{\partial \bar{x}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_i - T_\infty}{L^2} \frac{\partial^2 \theta}{\partial \bar{x}^2}$$

$$\frac{\partial T}{\partial t} = (T_i - T_\infty) \frac{\partial \theta}{\partial t}$$

Back substituting into our PDE

$$\frac{T_i - T_\infty}{L^2} \frac{\partial^2 \theta}{\partial \bar{x}^2} = (T_i - T_\infty) \frac{\partial \theta}{\partial t} \cdot \frac{1}{\alpha}$$

$$\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{L^2}{\alpha} \frac{\partial \theta}{\partial t} \quad (1)$$

→ We still have a dimentional time here.

Let's use Fo # or dimensionless time (look on pg. 59 of notes)

$$Fo = \tau = \frac{\alpha t}{L^2} \equiv \frac{\text{diffusive transport rate}}{\text{storage rate}}$$

$$t = \frac{L^2 \tau}{\alpha} \Rightarrow \frac{\partial t}{\partial \tau} = \frac{L^2}{\alpha} \Rightarrow \partial t = \frac{L^2}{\alpha} \partial \tau$$

⇒ back substitute into (1)

Now our equation becomes:

$$\frac{\partial^2\theta}{\partial\bar{x}^2} = \frac{\cancel{L^2}}{\cancel{\alpha}}\cdot\frac{\partial\theta}{\partial\tau}\cdot\frac{\cancel{\alpha}}{\cancel{L^2}} \Rightarrow \boxed{\frac{\partial^2\theta}{\partial\bar{x}^2} = \frac{\partial\theta}{\partial\tau} \quad \text{or} \quad \frac{\partial\theta}{\partial Fo}}$$

↳ Dimensionless 1D, transient heat eq.

Now we can non-dimensionalize our IC & B.C.'s

1) Our IC becomes: $\boxed{\theta(\bar{x}, \tau=0) = 1}$

2) Our 1'st B.C. becomes: $\boxed{\left.\frac{\partial\theta}{\partial\bar{x}}\right|_{\bar{x}=0} = 0}$

3) Our 2'nd B.C. becomes:

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h\left(T(x=L) - T_\infty\right)$$

$$-k\frac{\cancel{(T_i - T_\infty)}}{L}\cdot\left.\frac{\partial\theta}{\partial\bar{x}}\right|_{\bar{x}=1} = h\,\theta(\bar{x}=1)\cdot\cancel{(T_i - T_\infty)}$$

$$\boxed{\left.\frac{\partial\theta}{\partial\bar{x}}\right|_{\bar{x}=1} = -\underbrace{\frac{hL}{k}}_{Bi}\,\theta(\bar{x}=1)}$$

$Bi \Rightarrow$ Biot # $\equiv \frac{\text{conduction resistance}}{\text{convection resistance}}$

Now we've reduced the problem to:

$$T(x,t) = f(x, L, t, \alpha, h, T_\infty, T_i)$$

$$\Downarrow$$

$$\theta = f(\bar{x}, Bi, Fo) \Rightarrow \text{Very powerfull}$$

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Now we can solve using the separation of variables:

$$\Theta(\bar{x}, \tau) = F(\bar{x}) \cdot G(\tau)$$ $\Rightarrow$ Back substitute into our PDE and divide through by $F \cdot G$

$$\underbrace{\frac{1}{F}\frac{\partial^2 F}{\partial \bar{x}^2}}_{f(\bar{x})\text{ only}} = \underbrace{\frac{1}{G}\cdot\frac{\partial G}{\partial \tau}}_{f(\tau)\text{ only}} = \text{constant}$$ (Must be a constant only since two independent functions)

Assuming our constant $= -\lambda^2$

$$\frac{\partial^2 F}{\partial \bar{x}^2} + \lambda^2 F = 0 \quad (1)$$
$$\frac{\partial G}{\partial \tau} + \lambda^2 G = 0 \quad (2)$$

Homogeneous ODE's. Use character. equation.

Solving (1) first: $\lambda'^2 + \lambda^2 = 0$

$$\lambda' = \pm\sqrt{-1}\,\lambda = \lambda i$$

So our solution becomes:

$$F = C_1 e^{+i\lambda\bar{x}} + C_2 e^{-i\lambda\bar{x}}$$ $\Rightarrow$ since

Identities

$\sin\bar{x} =$	$\dfrac{e^{i\bar{x}} - e^{-i\bar{x}}}{2i}$
$\cos\bar{x} =$	$\dfrac{e^{i\bar{x}} + e^{-i\bar{x}}}{2}$

we can write our solution as:

$$F = C_1 \cos(\lambda\bar{x}) + C_2 \sin(\lambda\bar{x}) \quad (3)$$

Now solving (2): $\lambda' + \lambda^2 = 0 \Rightarrow \lambda' = -\lambda^2$

$$G = C_3 e^{-\lambda^2\tau} \quad (4)$$

Combining our two solutions, we obtain:

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$$\Theta = F \cdot G = C_3 e^{-\lambda^2 \tau}\left(C_1 \cos(\lambda \bar{x}) + C_2 \sin(\lambda \bar{x})\right)$$

$$\Theta = e^{-\lambda^2 \tau}\left(A \cos(\lambda \bar{x}) + B \sin(\lambda \bar{x})\right) \quad (5)$$

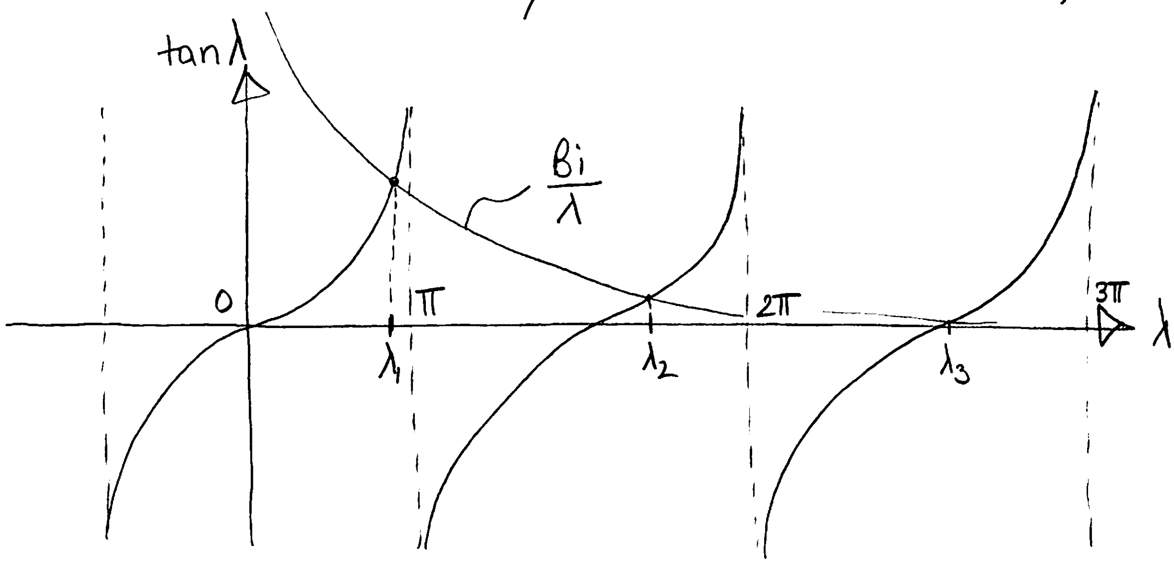
Now we apply our B.C.'s:

$$\left.\frac{\partial \Theta}{\partial \bar{x}}\right|_{\bar{x}=0} = 0 = -e^{-\lambda^2 \tau}\left(A\lambda \sin(0) + B\lambda \cos(0)\right) \Rightarrow \boxed{B = 0}$$

$$\left.\frac{\partial \Theta}{\partial \bar{x}}\right|_{\bar{x}=1} = -Bi\,\Theta(\bar{x}=1,\tau) \Rightarrow -A\cancel{e^{-\lambda^2 \tau}}\lambda \sin\lambda = -Bi\,A\cancel{e^{-\lambda^2 \tau}}\cos\lambda$$

$$\lambda \tan\lambda = Bi \quad (6)$$

We know $\tan\lambda$ is a periodic function with period π, so solution can lie anywhere between $0 \le \lambda \le \pi$, $\pi \le \lambda \le 2\pi$, etc...



Multiple roots exist, so we have multiple solutions

$$\lambda_n \tan\lambda_n = Bi \quad \text{(Eigenfunction with eigenvalues } \lambda_n\text{)}$$

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There exist an infinite number of solutions of the form $Ae^{-\lambda^2\tau}\cos(\lambda\bar{x})$. The final solution is a linear superposition of all of them:

$$\boxed{\theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau}\cos(\lambda_n\bar{x})} \quad (7)$$

Our IC allows us to determine our constants A_n:

$$\theta(\bar{x},\tau=0) = 1$$

$$1 = \sum_{n=1}^{\infty} A_n\cos(\lambda_n\bar{x})$$

Using orthoganality $\Rightarrow$ multiply both sides by $\cos(\lambda_m\bar{x})$ & integrate:

$$\int_0^1 \cos(\lambda_m\bar{x})d\bar{x} = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos(\lambda_n\bar{x})\cos(\lambda_m\bar{x})d\bar{x}}_{=0 \text{ if } n\neq m}$$

So our solution becomes:

$$\underbrace{\int_0^1 \cos(\lambda_n\bar{x})d\bar{x}}_{\sin(\lambda_n\bar{x})\Big|_0^1} = A_n\underbrace{\int_0^1 \cos^2(\lambda_n\bar{x})d\bar{x}}$$

$$= \sin(\lambda_n(1)) - \cancelto{0}{\sin(0)}$$

$$= \sin(\lambda_n)$$

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = (1+\cos 2x)/2$$

$$\Rightarrow \left[2\lambda_n\bar{x} + \sin(2\lambda_n\bar{x})\right]\Big|_0^1$$

$$= 2\lambda_n + \sin(2\lambda_n)$$

$$\boxed{A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)}} \quad (8)$$

$\Rightarrow$ Combine with eq. 7 & you're done!