Nenad Miljkovic

There exist an infinite number of solutions of the form $Ae^{-\lambda^2\tau}\cos(\lambda\bar{x})$. The final solution is a linear superposition of all of them:

$$\boxed{\theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n \bar{x})} \quad (7)$$

Our IC allows us to determine our constants A_n:

$$\theta(\bar{x}, \tau=0) = 1$$

$$1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n \bar{x})$$

Using orthoganality $\Rightarrow$ multiply both sides by $\cos(\lambda_m \bar{x})$ & integrate:

$$\int_0^1 \cos(\lambda_m \bar{x})\,d\bar{x} = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos(\lambda_n \bar{x})\cos(\lambda_m \bar{x})\,d\bar{x}}_{=0 \text{ if } n \neq m}$$

So our solution becomes:

$$\underbrace{\int_0^1 \cos(\lambda_n \bar{x})\,d\bar{x}}_{\frac{1}{\lambda_n}\sin(\lambda_n \bar{x})\Big|_0^1} = A_n \underbrace{\int_0^1 \cos^2(\lambda_n \bar{x})\,d\bar{x}}$$

$$= \frac{1}{\lambda_n}\left(\sin(\lambda_n(1)) - \cancelto{0}{\sin(0)}\right) = \sin(\lambda_n) \cdot \frac{1}{\lambda_n}$$

$$\cos^2 x + \sin^2 x = 1$$
$$\cos^2 x - \sin^2 x = \cos 2x$$
$$\cos^2 x = (1 + \cos 2x)/2$$
$$\Rightarrow A_n\left[\frac{1}{2} + \frac{\sin(2\lambda_n)}{4\lambda_n}\right]$$

$$\boxed{A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)}} \quad (8) \Rightarrow \text{Combine with eq. 7 \& you're done!}$$

Since the solution involves an infinite series, not very usefull from an analytical standpoint.

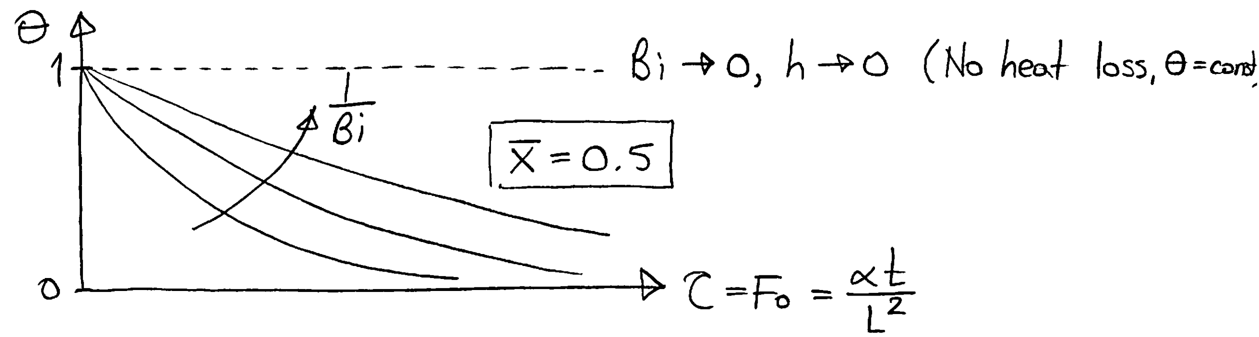
Turns out a good approximation is available by using only the first few terms, since the exponential decay term diminishes the higher order terms.

Tabular Solutions

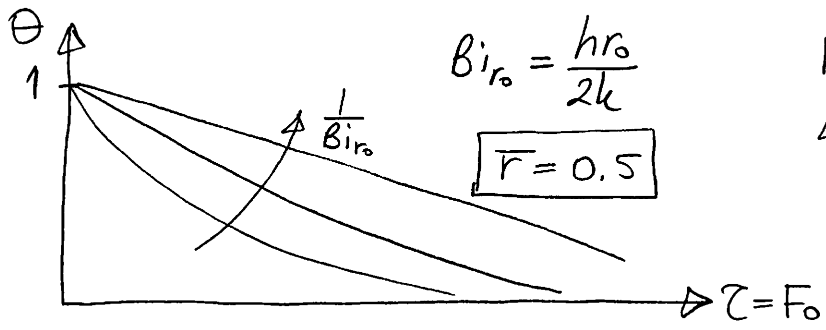
People have convenienty tabulated the solutions for specific geometries:

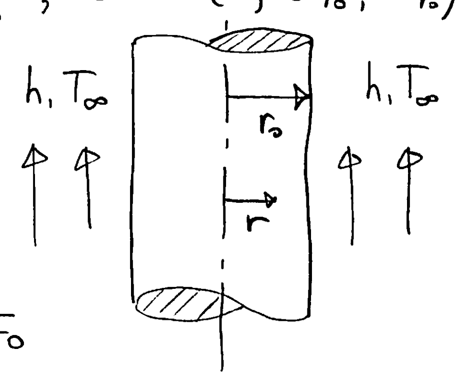
We know: $\theta = f(\bar{x}, Bi, \tau \text{ or } Fo)$ for a slab

We can plot our results for a given $\bar{x}$ as a function of Bi & Fo



Same for a cylinder: for a given $\bar{r} = \frac{r}{r_o}$, $\theta = f(\bar{r}, Bi_{r_o}, Fo_{r_o})$





In general, these tabular solutions are provided & simple to use.

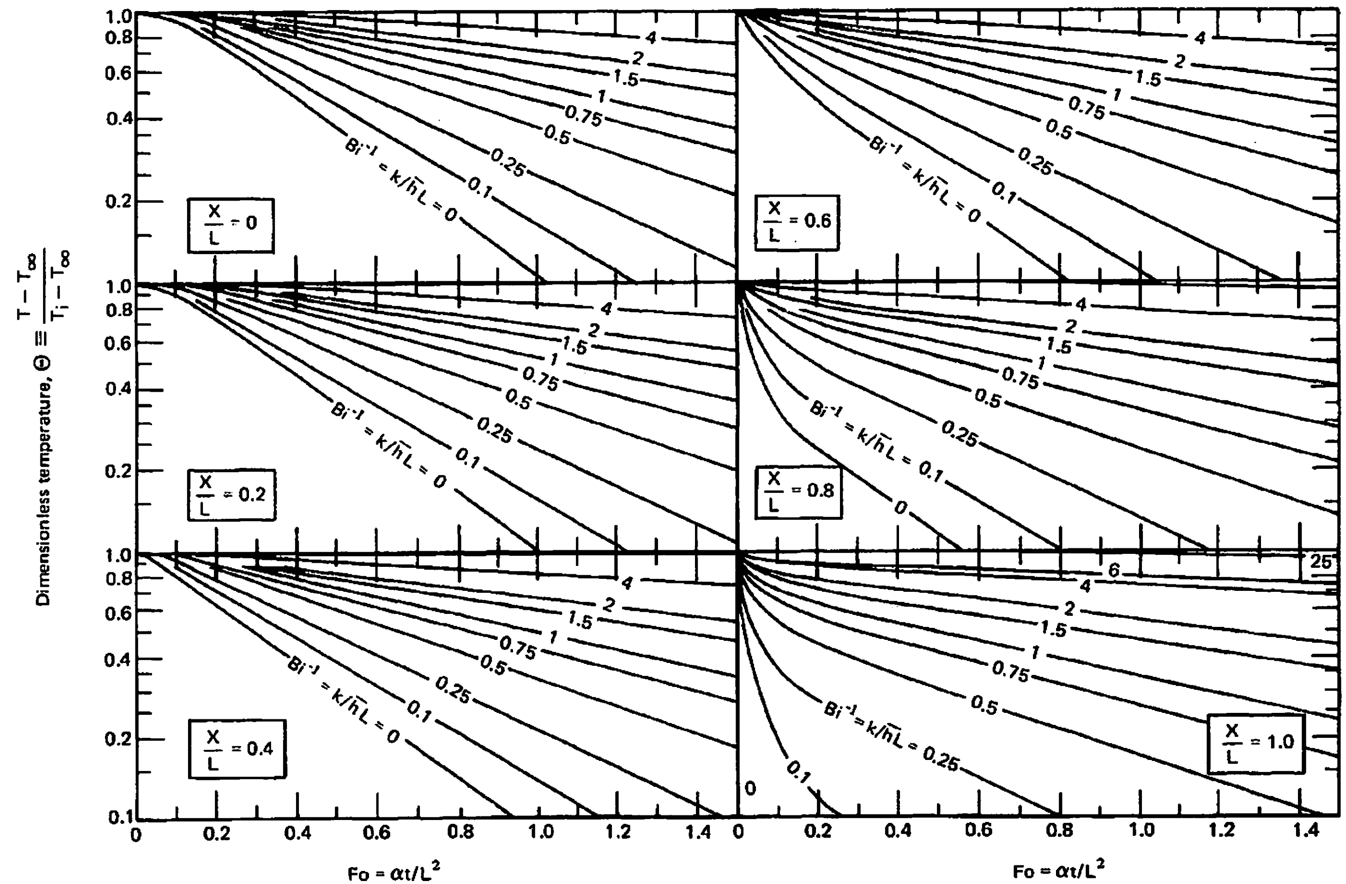


Figure 5.7 The transient temperature distribution in a *slab* at six positions: $x/L = 0$ is the center, $x/L = 1$ is one outside boundary.

77

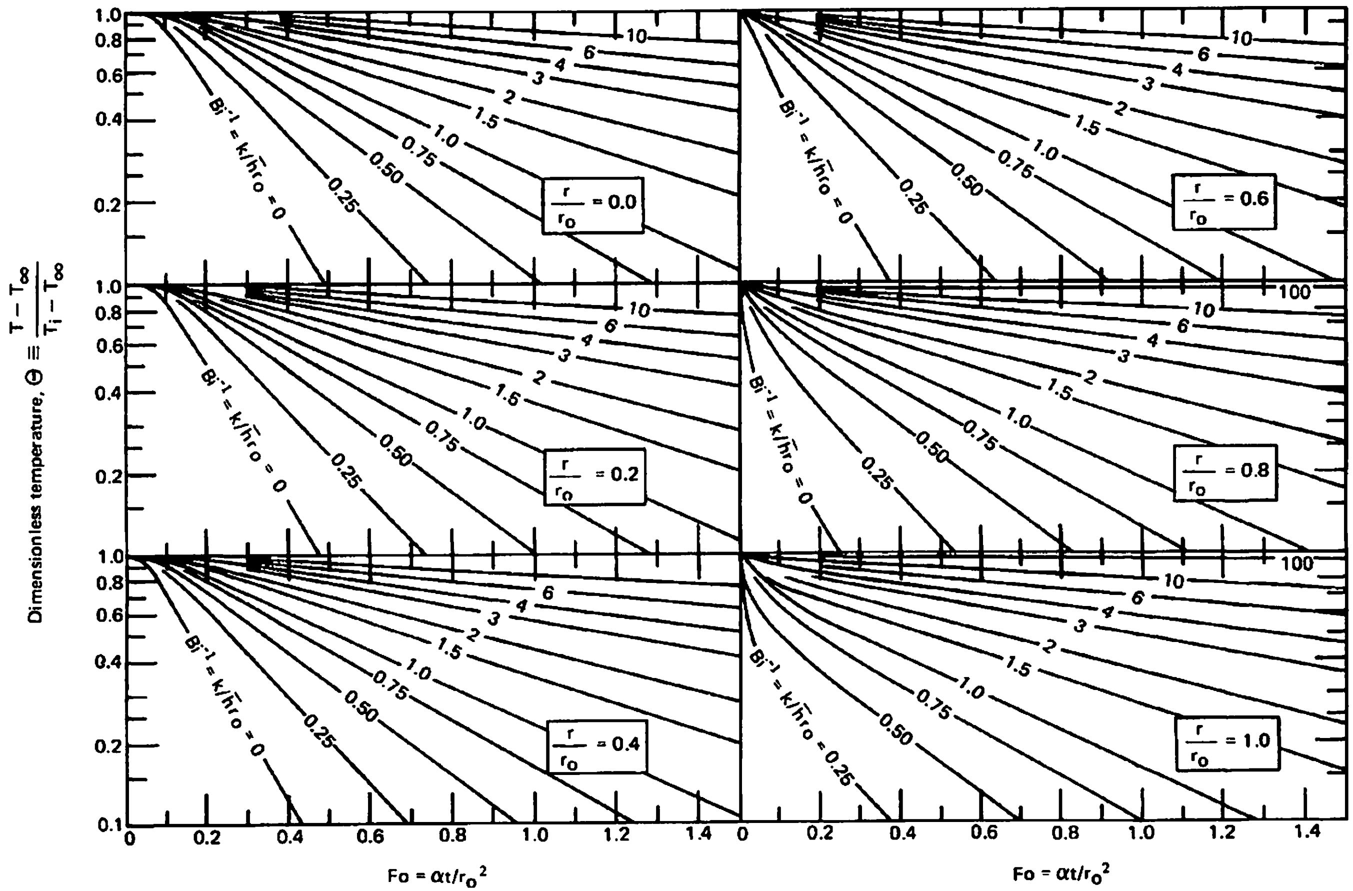


Figure 5.8 The transient temperature distribution in a long *cylinder* of radius r_o at six positions: $r/r_o = 0$ is the centerline; $r/r_o = 1$ is the outside boundary.

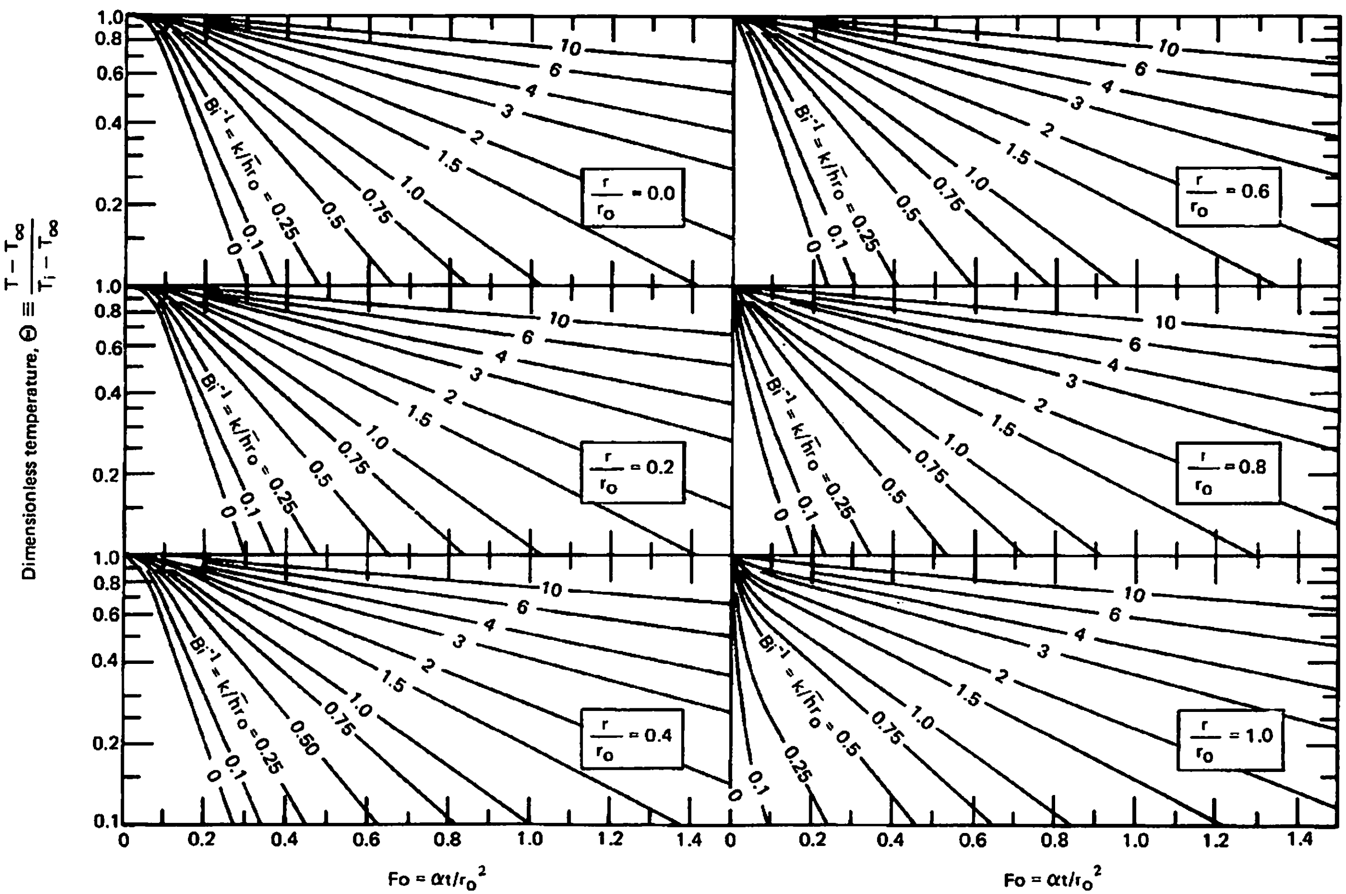


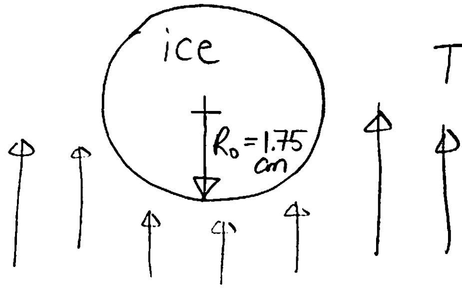
Figure 5.9 The transient temperature distribution in a *sphere* of radius r_o at six positions: $r/r_o = 0$ is the center; $r/r_o = 1$ is the outside boundary.

79

Nenad Miljkovic

Example A large hailstone (ice sphere), 3.5 cm in diameter & initially at $T_i = -20°C$ falls in the surrounding air at $T_\infty = 5°C$ at teminal velocity.

$k_{ice} = 2.22 W/m \cdot K$
$\alpha_{ice} = 1.26 \times 10^{-6} m^2/s$



(a) How long will it take before the outer surface begins to melt?
(b) What is the temperature of the center when it begins to melt at the surface?

To solve we can simply use our chart on pg. 79 of notes.

$$Bi_{R_o} = \frac{\bar{h} R_o}{k} = \frac{(125 W/m^2 \cdot K)(0.0175 m)}{(2.22 W/m \cdot K)} = 0.985 \approx 1$$

At the instant the surface begins to melt, the surface temperature is $T_s = 0°C$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} = \frac{0°C - 5°C}{-20°C - 5°C} = 0.2$$

From our chart at $\frac{r}{r_o} = \frac{R}{R_o} = 1$ (surface), & $\theta = 0.2$, $Bi^{-1} = 1$

$$Fo = 0.6 = \frac{\alpha_{ice} t}{R_o^2} \Rightarrow \boxed{t = \frac{R_o^2 Fo}{\alpha_{ice}} = 146 \text{ seconds}}$$

For part (b) we can use the same Bi_{R_o}, Fo, and solve for θ_c

$$\theta_c = \frac{T_c - T_\infty}{T_i - T_\infty} = 0.3 \Rightarrow \boxed{T_c = -2.5°C}$$