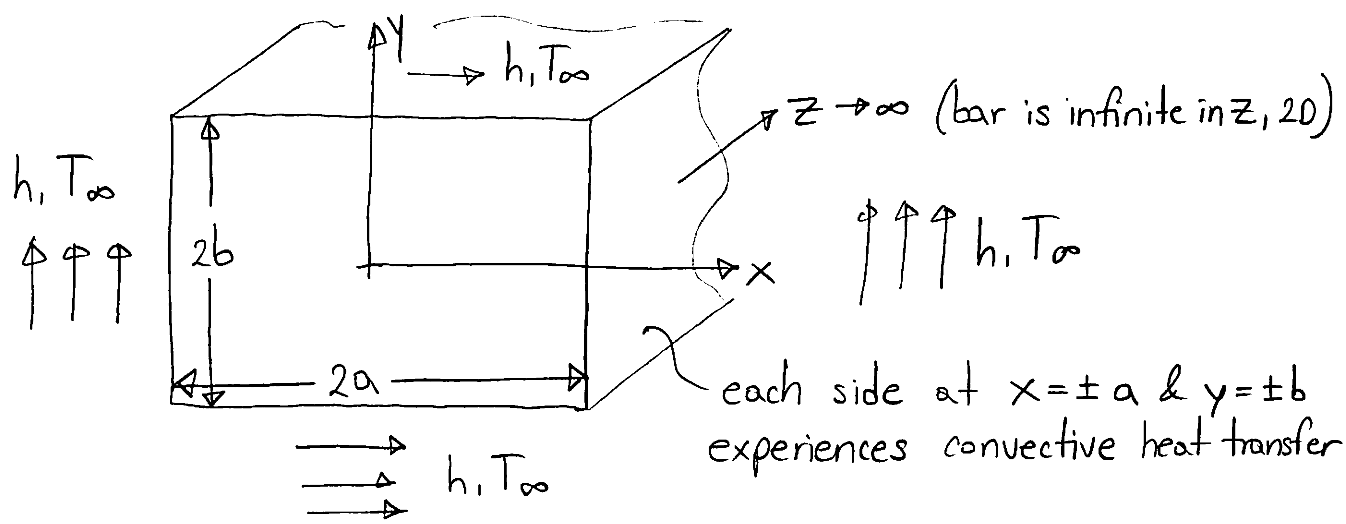
Nenad Miljkovic

Conduction in 2D & 3D bodies of Finite Extent

So far we've learned to hande 1D, transient, finite body problems. How do we deal with 2D or 3D problems?

It's easier than it seems.

Assume we have the following 2D bar being cooled:



Our governing equation is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \Rightarrow \text{Needs 4 B.C.'s and 1 IC to solve}$$

B.C.'s

1) $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ (Symmetry)

2) $-k\left.\frac{\partial T}{\partial x}\right|_{x=a} = h\left(T(x=a) - T_\infty\right)$ (Energy balance at the wall)

3) $\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0$ (symmetry)

4) $-k\left.\frac{\partial T}{\partial y}\right|_{y=b} = h\left(T(y=b) - T_\infty\right)$ (Energy balance at the top wall)

Nenad Miljkovic

Our I.C. is: 1) $T(x,y,t=0) = T_i$

Note, for the analysis, the h on each side (Top & side) need not be the same.

If we apply the separation of variables again:

$$T(x,y,t) = X(x,t) \cdot Y(y,t) \Rightarrow \text{Back substitute into PDE}$$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = \frac{Y}{\alpha} \frac{\partial X}{\partial t} + \frac{X}{\alpha} \cdot \frac{\partial Y}{\partial t}$$

Divide both sides by $X \cdot Y$ & rearange

$$\underbrace{\frac{1}{X} \cdot \left(\frac{\partial^2 X}{\partial x^2} - \frac{1}{\alpha} \frac{\partial X}{\partial t} \right)}_{f(x,t)} = - \underbrace{\frac{1}{Y} \left(\frac{\partial^2 Y}{\partial y^2} - \frac{1}{\alpha} \frac{\partial Y}{\partial t} \right)}_{g(y,t)} = h(t)$$

Where $h(t) = 0$ since the x & y solutions are fundamentaly similar in character:

$$\left.\begin{aligned} \frac{\partial^2 X}{\partial x^2} - \frac{1}{\alpha} \frac{\partial X}{\partial t} &= 0 \\ \frac{\partial^2 Y}{\partial y^2} - \frac{1}{\alpha} \frac{\partial Y}{\partial t} &= 0 \end{aligned}\right\}$$

Note, these are the exact same equations as in the 1D case we just did.

It turns out the separate directions can be treated independently. The only condition is that for the perscribed Fo in each direction, that t is the same since time is same for each direction.

Hence we can use our <u>graphical approach</u>!

Hence for our 2D case:

$$\theta = f(\bar{x}, \bar{y}, Bi_x, Bi_y, Fo_x, Fo_y) = \frac{T - T_\infty}{T_i - T_\infty}$$

$$\theta = \theta_{x,t} \cdot \theta_{y,t} \quad \text{(non-dimensional form of } T(x,y,t) = X \cdot Y\text{)}$$

$$\theta_{x,t} = f(\bar{x}, Bi_x, Fo_x) \qquad \theta_{y,t} = f(\bar{y}, Bi_y, Fo_y)$$

$$\bar{x} = \frac{x}{a} \qquad \bar{y} = \frac{y}{b}$$

$$Bi_x = \frac{ha}{k} \qquad Bi_y = \frac{hb}{k}$$

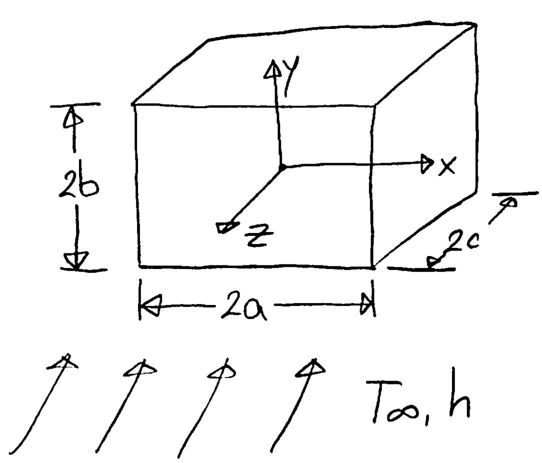
$$Fo_x = \frac{\alpha t}{a^2} \qquad Fo_y = \frac{\alpha t}{b^2}$$

note*, t must be the same.

Use charts on pages 77 to 79 of notes to solve for each independent direction and then combine & multiply.

$$\boxed{\theta = \frac{T - T_\infty}{T_i - T_\infty} = \theta_{x,t} \cdot \theta_{y,t}}$$

The same would be true for a 3D slab or other geometries:



$$\boxed{\theta = \frac{T - T_\infty}{T_i - T_\infty} = \theta_{x,t} \cdot \theta_{y,t} \cdot \theta_{z,t}}$$

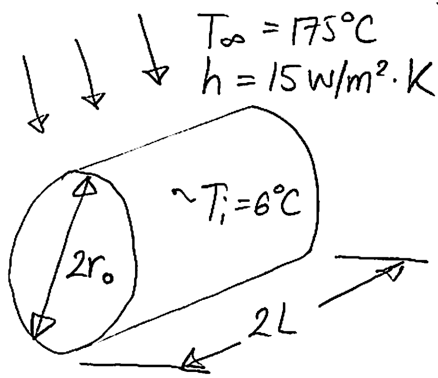
↳ Remember, use same t for all Fo calculations.

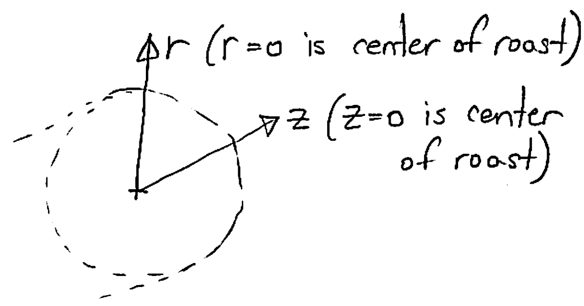
Nenad Miljkovic

Example | To cook a roast, all portions of the roast must attain a temperature of 80°C. How long will it take to cook a 2.25 kg roast?

$T_i = 6°C$
$T_{oven} = T_\infty = 175°C$
$h = 15\ W/m^2 \cdot K$

Assumptions:
1) Treat roast as cylinder with $D = 2L$
2) Roast properties similar to water
3) Constant properties.





We can model this as 2D conduction in r & z coordinates
We need to solve for:

$$T(r,z,t) \Rightarrow T(0,0,t) = 80°C$$

We just learned that:

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} = \underbrace{\theta_{r,t}}_{\text{Cylinder solution}} \cdot \underbrace{\theta_{z,t}}_{\text{Slab solution}}$$

$$\frac{T(0,0,t) - T_\infty}{T_i - T_\infty} = \frac{80°C - 175°C}{6°C - 175°C} = 0.56$$

$$0.56 = \left.\frac{T(0,t) - T_\infty}{T_i - T_\infty}\right|_{slab} \cdot \left.\frac{T(0,t) - T_\infty}{T_i - T_\infty}\right|_{cylinder}$$

Nenad Miljkovic

We just learned for each solution, we need Bi & Fo

$$Bi = \frac{h r_o}{k} = \frac{hL}{k} = \frac{(15\,W/m^2\cdot K)(L)}{0.6\,W/m\cdot K}$$ ← Water thermal conductivity

But we need L & r_o :

$$M = \rho \forall = \rho \cdot 2L \cdot \pi r_o^2$$

$$r_o = L = \left[\frac{M}{2\pi\rho}\right]^{1/3} = \left[\frac{2.25\,kg}{2\pi(1000\,kg/m^3)}\right]^{1/3}$$

$$\boxed{L = r_o = 0.0712m}$$

Plugging back into our Bi # :

$$Bi = 1.68 \quad \text{or} \quad \boxed{Bi^{-1} = 0.6}$$

Now for Fourier # (Fo): Note*: here $L=r_o$ but not always the case, hence Bi & Fo may differ.

$$Fo = \frac{\alpha t}{r_o^2} = \frac{\alpha t}{L^2} = (1.53\times10^{-7} m^2/s)\cdot t/(0.0712m)^2$$

$$\boxed{Fo = 3.02\times10^{-5}\cdot t}$$

Now we need to use trial and error to guess from our tabular solutions:

Trial	Fo	t (hours)	$\theta_o/\theta_i\|_{slab}$	$\theta_o/\theta_i\|_{cylinder}$	$\theta_{slab}\cdot\theta_{cyl}$
1	0.4	3.68	0.8 (pg. 77)	0.52 (pg. 78)	0.42
2	0.3	2.75	0.82	0.65	0.53

So on our second trial, we have good agreement with $\theta = 0.56$
∴ It takes 2 3/4 hours to roast the meal.