**Numerical Analysis of Heat Conduction**

So far, we have been using strictly analytical methods to solve heat transfer problems.

Sometimes, we are presented with non-trivial geometries which require numerical solution of the heat equation.

For example:

![Diagram](image)

So how do we solve numerically:

First, we need to break up our domain into a finite set of discrete points:

![Control Volume Diagram](image)

Assume $T_{m,n}$ is the average temperature of the control volume.
Applying an energy balance on our control volume:

\[ E_{in} - E_{out} = 0 \]  
(Assuming SS, unit depth (20), \( Q'' = 0 \))

Applying Fourier's Law:

\[ q = -kA \frac{dT}{dx} \]

\[ q_{m-1, n \rightarrow m, n} = k\Delta y \frac{T_{m-1, n} - T_{m, n}}{\Delta x} \]

\[ q_{m, n \rightarrow m+1, n} = k\Delta y \frac{T_{m, n} - T_{m+1, n}}{\Delta x} \]

We can do the same for the y-direction & substitute into our energy balance:

\[ k \frac{\Delta y}{\Delta x} (T_{m-1, n} - T_{m, n}) - k \frac{\Delta y}{\Delta x} (T_{m, n} - T_{m+1, n}) + k \frac{\Delta x}{\Delta y} (T_{m, n-1} - T_{m, n}) - k \frac{\Delta x}{\Delta y} (T_{m, n} - T_{m, n+1}) = 0 \]
Assuming $\Delta x = \Delta y$, we obtain:

$$\frac{T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1}}{4} = T_{m,n}$$

$\Rightarrow$ The nodal temp. is the average of its neighbours.

We would apply this equation for all interior nodes.

**Boundary Conditions** (Applied to all surface or boundary nodes)

1. **Fixed Temperature** - set node temperature to equal the boundary condition.

   $T_{m,n+1} = T_s$

   $T_{m,n} = T_s$

   $T_{m,n-1} = T_s$

2. **Uniform Heat Flux** ($q''$)

   Applying an energy balance:

   $$E_{in} - E_{out} = 0$$

   $$k \frac{\Delta y}{\Delta x} (T_{m-1,n} - T_{m,n}) + \frac{k \Delta x}{2 \Delta y} (T_{m,n-1} - T_{m-1,n}) - k \frac{\Delta x}{2 \Delta y} (T_{m,n} - T_{m,n+1}) + q'' \Delta y = 0$$

   If $\Delta x = \Delta y$:

   $$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} + \frac{2q'' \Delta x}{k} - 4T_{m,n} = 0$$

   Special case ($q'' = 0$): $T_{m,n} = \frac{1}{4} (2T_{m-1,n} + T_{m,n-1} + T_{m,n+1})$
Table 4.2 of Incropera

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Finite-Difference Equation for $\Delta x = \Delta y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$</td>
<td></td>
</tr>
<tr>
<td><strong>Case 1.</strong> Interior node</td>
<td></td>
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<tr>
<td>$2(T_{m-1,n} + T_{m,n+1}) + (T_{m+1,n} + T_{m,n-1})$</td>
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<tr>
<td>$+ 2 \frac{h \Delta x}{k} T_{\infty} - 2 \left( 3 + \frac{h \Delta x}{k} \right) T_{m,n} = 0$</td>
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<tr>
<td><strong>Case 2.</strong> Node at an internal corner with convection</td>
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<tr>
<td>$(2T_{m-1,n} + T_{m,n-1} + T_{m,n-1}) + \frac{2h \Delta x}{k} T_{\infty} - 2 \left( \frac{h \Delta x}{k} + 1 \right) T_{m,n} = 0$</td>
<td></td>
</tr>
<tr>
<td><strong>Case 3.</strong> Node at a plane surface with convection</td>
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<tr>
<td>$(T_{m,n-1} + T_{m,n-1}) + 2 \frac{h \Delta x}{k} T_{\infty} - 2 \left( \frac{h \Delta x}{k} + 1 \right) T_{m,n} = 0$</td>
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<tr>
<td><strong>Case 4.</strong> Node at an external corner with convection</td>
<td></td>
</tr>
<tr>
<td>$(2T_{m-1,n} + T_{m,n-1} + T_{m,n-1}) + \frac{2q' \Delta x}{k} - 4T_{m,n} = 0$</td>
<td></td>
</tr>
<tr>
<td><strong>Case 5.</strong> Node at a plane surface with uniform heat flux</td>
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</tbody>
</table>

$^{ab}$To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set $h$ or $q'$ equal to zero.
Solution Methodology

1. Discretize the solution domain

   M = # of nodes in x-direction
   N = # of nodes in y-direction

   \[
   \Delta x = \frac{1}{M-1} = 0.2 \text{ m} \\
   \Delta y = \frac{1}{N-1} = 0.2 \text{ m}
   \]

   \[m = 6, \ n = 6\]

   \# of interior nodes = 16 \quad \# of exterior nodes = 20 \quad \# of total nodes = 36

2. Form finite volume equations for all internal nodes

   \[2 \leq m \leq M-1, \quad 2 \leq n \leq N-1\]

3. Form finite volume equations for all boundary nodes

4. Solve \(M \times N\) equations and unknowns using:
   i) Gaussian elimination (direct)
   ii) Gauss-Seidel method (indirect and iterative)
   iii) Software (i.e. Excel, Matlab, Maple, etc...)
Example: A 1 x 1 m square concrete column, $k = 1 \text{ W/m.K}$. Find the temperature distribution in the column.

$$T_s = 500\text{K}$$

$$T = 500\text{K}$$

$$\Rightarrow h = 10\text{W/m}^2\cdot\text{K}$$

$$\Rightarrow T_\infty = 300\text{K}$$

1. Discretize the solution domain:

   $M = 5$, $N = 5$, $\Delta x = \Delta y = 0.25\text{m}$

   Ambiguous (what do we assign here?)
   we'll discuss this later.

2. Interior nodes: $2 \leq m \leq 4$, $2 \leq n \leq 4$

   $$T_{m,n} = \frac{1}{4}(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1})$$

   $\Rightarrow 9$ equations

3. Exterior nodes:
   For: $m=1$, $2 \leq n \leq 5$
   $n=5$, $1 \leq m \leq 5$
   $m=5$, $2 \leq n \leq 5$
   $n=1$, $2 \leq m \leq 4$ 
   => Convective boundary condition
For the convective boundary condition, look at Table 4.2, eq.4.42:

\[
(2T_{m,n+1} + T_{m-1,n} + T_{m+1,n}) + \frac{2h\Delta y}{k}T_{\infty} = 2\left(\frac{h\Delta y}{k} + 2\right)T_{m,n} = 0
\]

Back substituting \( h \) & \( T_{\infty} \), & solving for \( T_{m,n} \):

\[
T_{m,n} = \frac{1}{q}(2T_{m,n+1} + T_{m-1,n} + T_{m+1,n} + 1500)
\]

\( \Rightarrow 3 \) equations

But what about the corners?

\[
T_s = 500 K
\]

\( \Rightarrow \)

\[
\begin{align*}
\text{Assume:} & \\
\frac{T_{1,1}}{T_{s,1}} &= T_s \\
\frac{T_{s,1}}{T_s} &= T_s
\end{align*}
\]

\( \Rightarrow 2 \) equations

4. Assemble the equation set: 25 equations, 25 unknowns
5. Solve the set of equations (we will use Excel)