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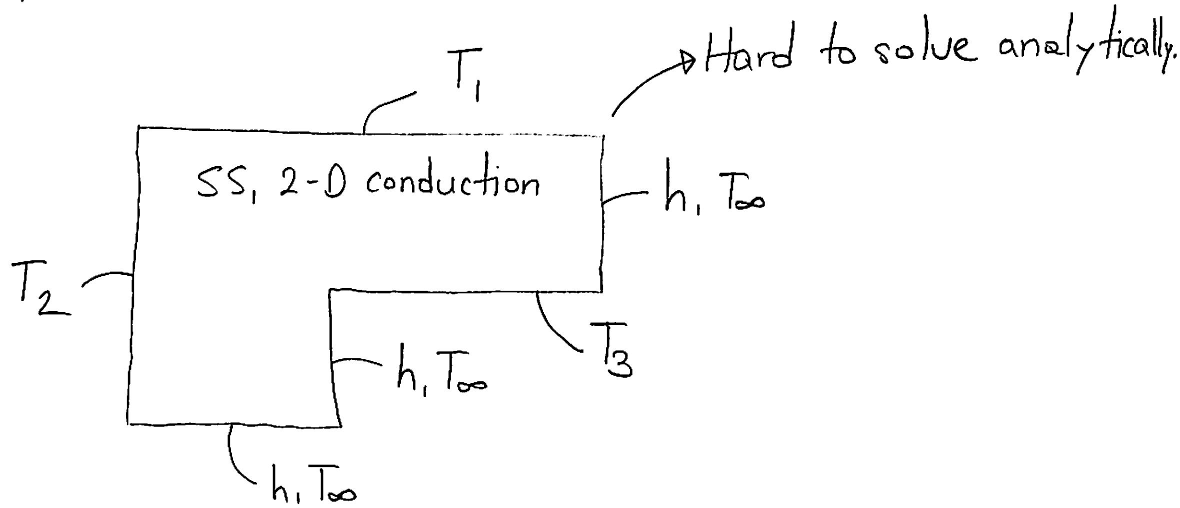
Numerical Analysis of Heat Conduction

So far, we have been using strictly analytical methods to solve heat transfer problems.

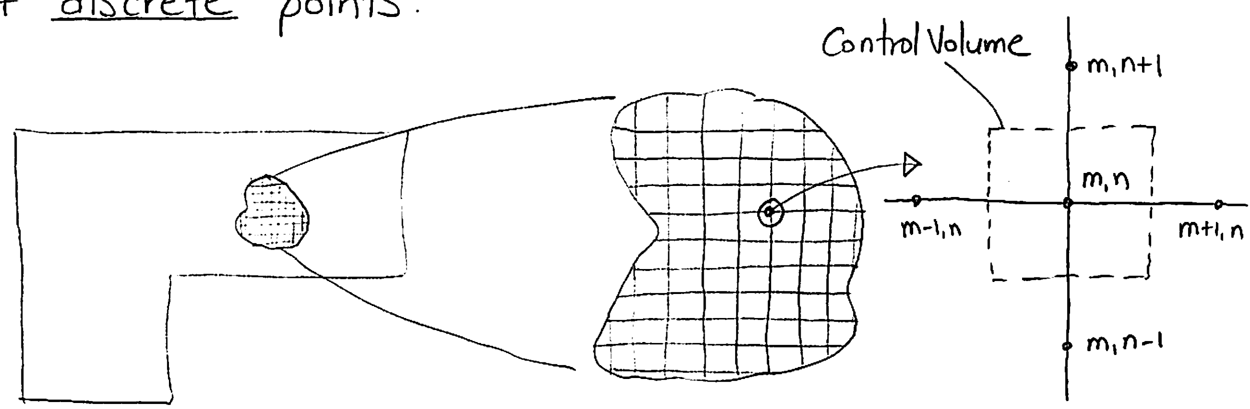
Sometimes, we are presented with non-trivial geometries which require numerical solution of the heat equation.

For example:



So how do we solve numerically:

First we need to break up our domain into a finite set of discrete points:



Assume $T_{n,m}$ is the average temperature of the control volume.

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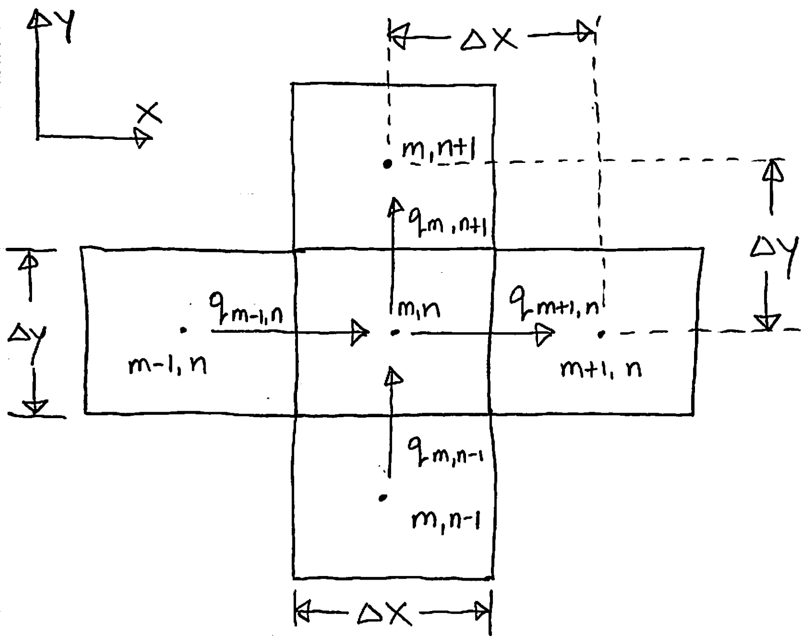
Applying an energy balance on our control volume:



$$\dot{E}_{in} - \dot{E}_{OUT} = 0 \quad \text{(Assuming SS, unit depth (2D), } Q''' = 0\text{)}$$

Applying Fourier's Law:

$$q = -kA\frac{\partial T}{\partial x}$$

Note, negative sign dropped because I flipped ΔT.

$$q_{m-1,n \to m,n} = k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{m,n \to m+1,n} = k\Delta y \frac{T_{m,n} - T_{m+1,n}}{\Delta x}$$

We can do the same for the y-direction & substitute into our energy balance:

$$k\frac{\Delta y}{\Delta x}\left(T_{m-1,n} - T_{m,n}\right) - k\frac{\Delta y}{\Delta x}\left(T_{m,n} - T_{m+1,n}\right)$$
$$+ k\frac{\Delta x}{\Delta y}\left(T_{m,n-1} - T_{m,n}\right) - k\frac{\Delta x}{\Delta y}\left(T_{m,n} - T_{m,n+1}\right) = 0$$

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Assuming $\Delta x = \Delta y$, we obtain:

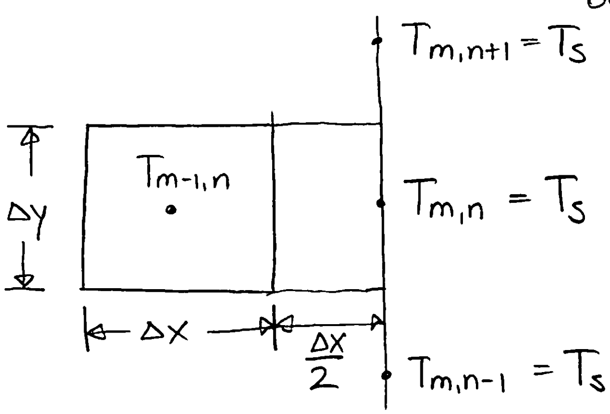
$$\boxed{\frac{T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1}}{4} = T_{m,n}}$$

⇒ The nodal temp. is the average of its neighbours.

We would apply this equation for all interior nodes.

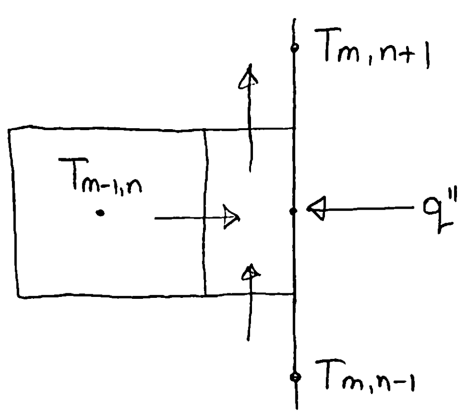
<u>Boundary Conditions</u> (Applied to all surface or boundary nodes)

① Fixed Temperature – set node temperature to equal the boundary condition.



For all other B.C.'s see Table 4.2, pg. 246 of Textbook.

② Uniform Heat Flux (q'')



Applying an energy balance:

$$\dot{E}_{in} - \dot{E}_{OUT} = 0$$

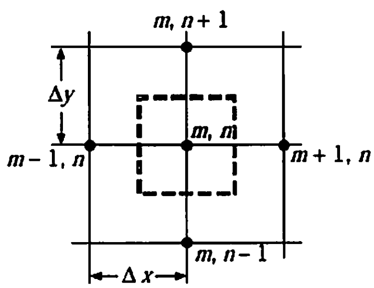
$$k\frac{\Delta y}{\Delta x}\left(T_{m-1,n} - T_{m,n}\right) + \frac{k\Delta x}{2\Delta y}\left(T_{m,n-1} - T_{m,n}\right) - k\frac{\Delta x}{2\Delta y}\left(T_{m,n} - T_{m,n+1}\right) + q''\Delta y = 0$$

If $\Delta x = \Delta y$: $\boxed{2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} + \frac{2q''\Delta x}{k} - 4T_{m,n} = 0}$

Special case ($q''=0$): $\boxed{T_{m,n} = \frac{1}{4}\left(2T_{m-1,n} + T_{m,n-1} + T_{m,n+1}\right)}$

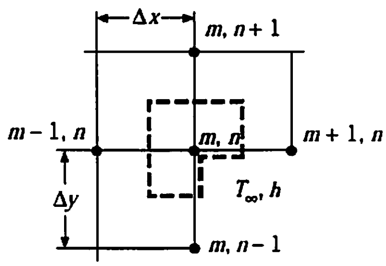
Table 4.2 of Incropera

Configuration	Finite-Difference Equation for $\Delta x = \Delta y$



$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$

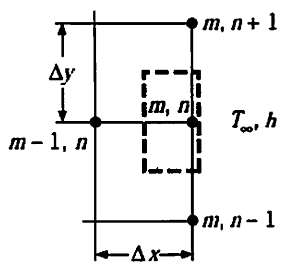
Case 1. Interior node



$$2(T_{m-1,n} + T_{m,n+1}) + (T_{m+1,n} + T_{m,n-1})$$

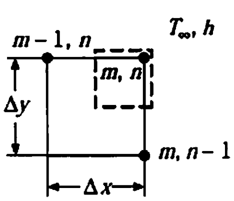
$$+ 2\frac{h\Delta x}{k}T_\infty - 2\left(3 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

Case 2. Node at an internal corner with convection



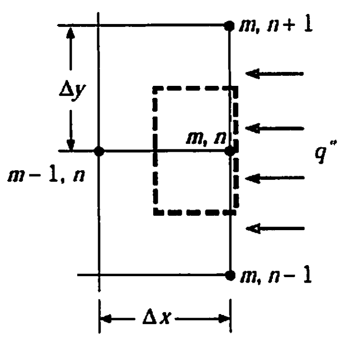
$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 2\right)T_{m,n} = 0$$

Case 3. Node at a plane surface with convection



$$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right)T_{m,n} = 0$$

Case 4. Node at an external corner with convection



$$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2q''\Delta x}{k} - 4T_{m,n} = 0$$

Case 5. Node at a plane surface with uniform heat flux

[a,b] To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set h or q'' equal to zero.

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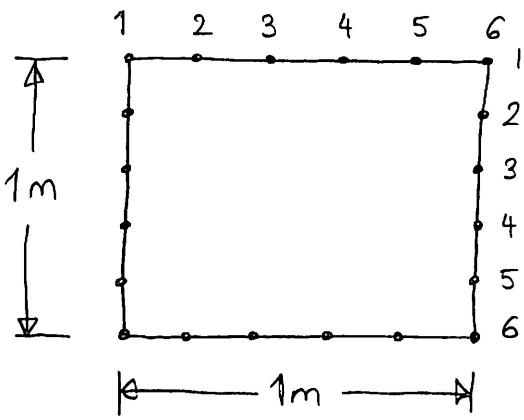
Solution Methodology

① Discretize the solution domain



$m = 6, \ n = 6$

M = # of nodes in x-direction
N = # of nodes in y-direction

$$\Delta x = \frac{1}{M-1} = 0.2\,m$$

$$\Delta y = \frac{1}{N-1} = 0.2\,m$$

# of interior nodes = 16
# of exterior nodes = 20

} 36 total nodes

② Form finite volume equations for all internal nodes

$$2 \le m \le M-1, \quad 2 \le n \le N-1$$

③ Form finite volume equations for all boundary nodes

④ Solve M × N equations and unknowns using:
 i) Gaussian ellimination (direct)
 ii) Gauss-Seidel method (indirect and iterative)
 iii) Software (i.e. Excel, Matlab, Maple, etc...)

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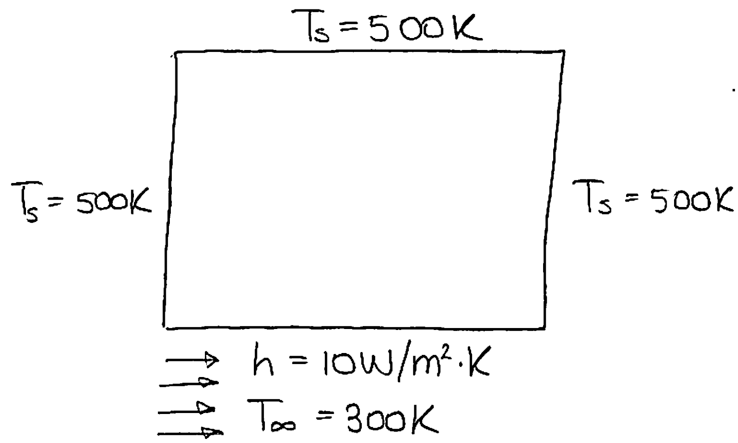
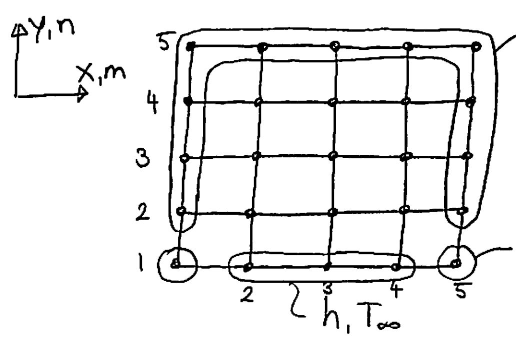
Example A 1×1 m square concrete column, $k = 1\ W/m \cdot K$.
Find the temperature distribution in the column.



① Discretize the solution domain:



$T_s = 500K$

$M = 5,\ N = 5,\ \Delta x = \Delta y = 0.25m$

Ambiguous (what do we assign here?)
we'll discuss this later.

② Interior nodes: $2 \le m \le 4,\ 2 \le n \le 4$

$$\boxed{T_{m,n} = \frac{1}{4}\left(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right)}$$

↳ 9 equations

③ Exterior nodes:

For: $m = 1,\ 2 \le n \le 5$
$n = 5,\ 1 \le m \le 5$
$m = 5,\ 2 \le n \le 5$

$\boxed{T_{m,n} = T_s = 500K}$ → 11 equations

$n = 1,\ 2 \le m \le 4 \Rightarrow$ Convective boundary condition

For the convective boundary condition, look at Table 4.2, eq. 4.42a

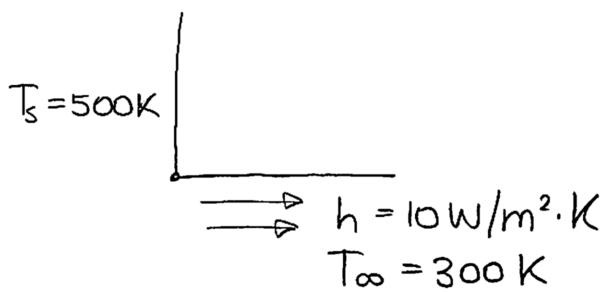
$$\left(2T_{m,n+1} + T_{m-1,n} + T_{m+1,n}\right) + \frac{2h\Delta y}{k}T_\infty - 2\left(\frac{h\Delta y}{k} + 2\right)T_{m,n} = 0$$

Back substituting h & T_∞, & solving for $T_{m,n}$:

$$\boxed{T_{m,n} = \frac{1}{9}\left(2T_{m,n+1} + T_{m-1,n} + T_{m+1,n} + 1500\right)}$$

↳ 3 equations

But what about the corners?



⇒ Assume:

$$\boxed{\begin{aligned} T_{1,1} &= T_s \\ T_{5,1} &= T_s \end{aligned}}$$

→ 2 equations

4. Assemble the equation set: 25 equations, 25 unknowns

5. Solve the set of equations (we will use Excel)