$N = M = 5$

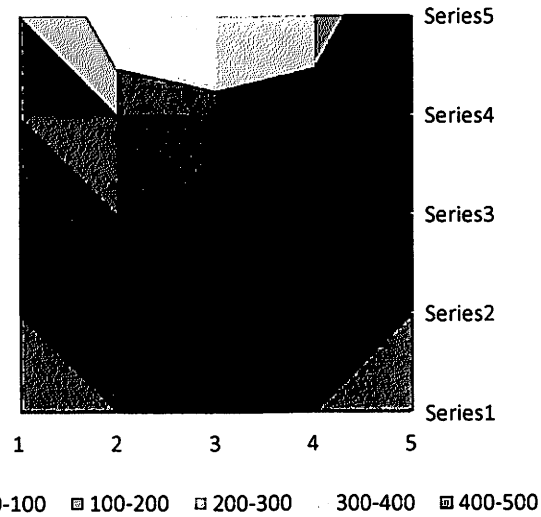
T_∞

Tinf	h	N	M
300	10	5	5
1500	9		

500	500	500	500	500
500	489.3047	485.1538	489.3047	500
500	472.0651	462.0058	472.0651	500
500	436.9498	418.7393	436.9498	500
500	356.9946	339.052	356.9946	500

$1500 \rightarrow \frac{2h\Delta y}{k} T_\infty$

$9 \rightarrow \left[2\left(\frac{h\Delta y}{k} + 2\right)\right]^{-1}$



$N = M = 20$

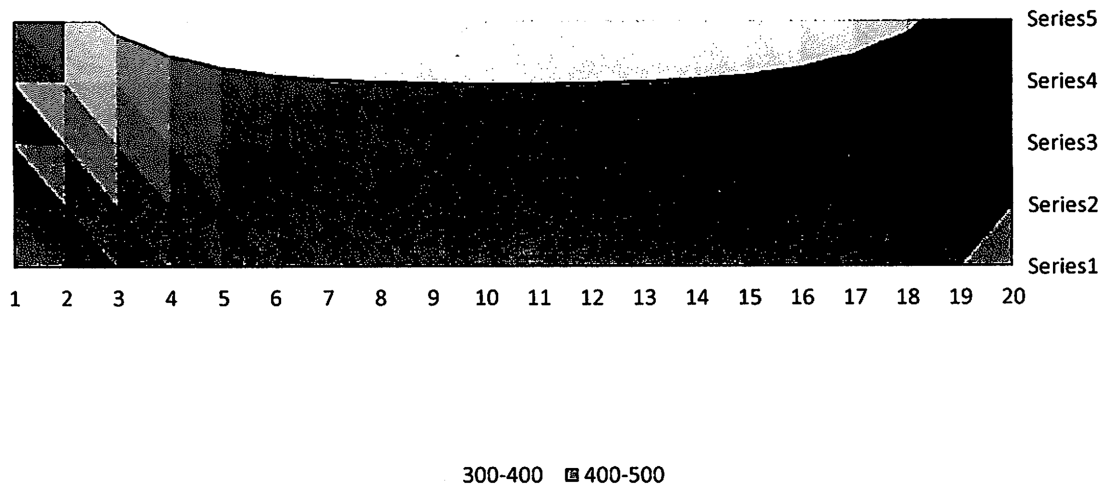
500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500
500	489.2	480.8	475	471.3	469.2	467.9	467.2	466.8	466.6	466.6	466.8	467.2	467.9	469.2	471.3	475	480.8	489.2	500
500	476.2	458.9	447.8	441.2	437.4	435.2	434	433.3	433	433	433.3	434	435.2	437.4	441.2	447.8	458.9	476.2	500
500	456.6	430.6	416.3	408.4	404	401.5	400.2	399.5	399.2	399.2	399.5	400.2	401.5	404	408.4	416.3	430.6	456.6	500
500	419.6	390.9	378.3	372	368.6	366.8	365.8	365.3	365.1	365.1	365.3	365.8	366.8	368.6	372	378.3	390.9	419.6	500

T_∞ Tinf	h	N	M
300	10	20	20

315.8 5.053

$315.8 \rightarrow \frac{2h\Delta y}{k}\cdot T_\infty$

$5.053 \rightarrow \left[2\left(\frac{h\Delta y}{k}+2\right)\right]^{-1}$



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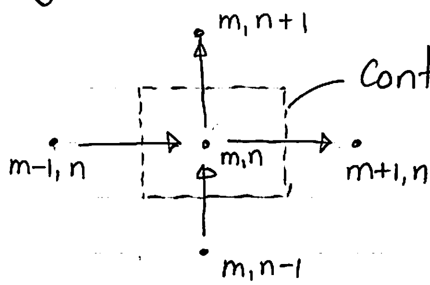
Transient Heat Conduction Numerical Solution

Looking back at our interior node:



Performing an energy balance on our control volume:

$$\dot{E}_{in} - \dot{E}_{OUT} = \dot{E}_{STored} \quad ①$$

$$\dot{E}_{STored} = \rho \forall C \frac{\partial T}{\partial t} \quad ②$$

Approximating ② as a linear function (assuming unit depth)

$$\boxed{\dot{E}_{STored} \mathrel{\hat{=}} \rho \Delta x \Delta y C \left(\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} \right)}$$

where: $\Delta t \equiv$ time step
$T_{m,n}^{p+1} \equiv$ new value
$T_{m,n}^{p} \equiv$ old value

For our other terms:

$$\boxed{\dot{E}_{in}, \dot{E}_{OUT} = -k \Delta x \,(\text{or } \Delta y) \cdot \frac{T_{m,n}^{?} - T_{m-1,n}^{?}}{\Delta y \,(\text{or } \Delta x)}}$$

where ? refers to when (time) the values are considered.

① Old values - explicit
② New values - implicit
③ Combination - Crank-Nicholson scheme

Nenad Miljkovic

The general approach is:

$$\boxed{T_{m,n}^{?} = f\, T_{m,n}^{p+1} + (1-f)\, T_{m,n}^{p}}$$

for
- $f = 0 \equiv$ explicit (old values)
- $f = 1 \equiv$ implicit (new values)
- $f = 1/2 \equiv$ Crank-Nicholson scheme (combination)

Now we can back substitute & solve:

$$\dot{E}_{in} - \dot{E}_{OUT} = \dot{E}_{STORED}$$

$$k\frac{\Delta y}{\Delta x}\left(T_{m-1,n}^{?} - T_{m,n}^{?}\right) - k\frac{\Delta y}{\Delta x}\left(T_{m,n}^{?} - T_{m+1,n}^{?}\right)$$

$$+ k\frac{\Delta x}{\Delta y}\left(T_{m,n-1}^{?} - T_{m,n}^{?}\right) - k\frac{\Delta x}{\Delta y}\left(T_{m,n}^{?} - T_{m,n+1}^{?}\right) = \rho\Delta x\Delta y\, c\left(\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t}\right)$$

Dividing both sides by $\Delta x \Delta y \rho c$; $\alpha = \frac{k}{\rho c}$

$$\boxed{\underbrace{\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t}}_{\frac{\partial T}{\partial t}} = \underbrace{\frac{T_{m+1}^{?} - 2T_{m,n}^{?} + T_{m-1,n}^{?}}{(\Delta x)^2}}_{\frac{\partial^2 T}{\partial x^2}} + \underbrace{\frac{T_{m,n+1}^{?} + 2T_{m,n}^{?} + T_{m,n-1}^{?}}{(\Delta y)^2}}_{\frac{\partial^2 T}{\partial y^2}}}$$

(Linearisation of these terms)

① Explicit ($f=0$)

$$\boxed{\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2}}$$

↳ Assuming $\Delta x = \Delta y$ and 1D (x-only)

$$T_{m,n}^{p+1} = T_{m,n}^{p} + \underbrace{\frac{\alpha \Delta t}{(\Delta x)^2}}_{} \left(\cdots \right)^{p}$$

$$\text{Fourier \#} = \frac{\alpha \Delta t}{(\Delta x)^2}$$

Expanding this expression, we obtain:

$$\boxed{T_{m,n}^{p+1} = Fo\left(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right)^{p} + (1-4Fo)T_{m,n}^{p}}$$

↳ 2D conduction

$$\boxed{T_{m}^{p+1} = Fo\left(T_{m+1}^{p} + T_{m-1}^{p}\right) + (1-2Fo)T_{m}^{p}} \rightarrow \text{1D conduction}$$

For other cases, look at Table 5.3 of text.

Stability

Although the explicit scheme is fast & nice, it suffers from instability in certain conditions. (Oscillations)

It can be shown mathematically that as long as the coefficient on the $T_{m,n}^{p}$ term is positive or zero, stability is maintained. I.e.:

1D conduction	$1-2Fo \geq 0$
2D conduction	$1-4Fo \geq 0$
3D conduction	$1-6Fo \geq 0$

$Fo = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$ (for 1D)

For small Δx, Δt must be very small. Computation expensive.

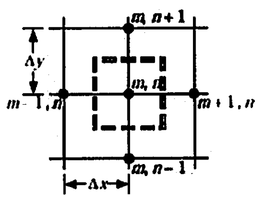
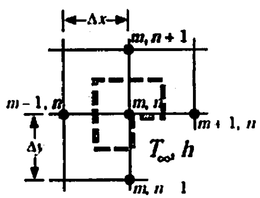
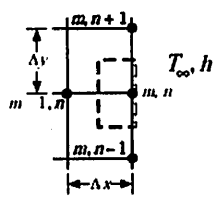
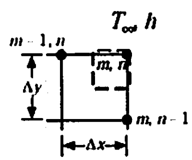
② Implicit ($f=1$)

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \left(\frac{T_{m+1,n} - 2T_{m,n} + T_{m+1,n}}{(\Delta x)^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2}\right)^{p+1}$$

For $\Delta x = \Delta y$

$$\boxed{(1+4Fo)T_{m,n}^{p+1} - Fo\left(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}\right)^{p+1} = T_{m,n}^{p}}$$

Table 5.3 of Incropera

Configuration	(a) Explicit Method: Finite-Difference Equation	(a) Explicit Method: Stability Criterion	(b) Implicit Method
	$T_{m,n}^{p+1} = Fo(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}) + (1 - 4Fo)T_{m,n}^{p}$ (5.76) 1. Interior node	$Fo \leq \frac{1}{4}$ (5.80)	$(1 + 4Fo)T_{m,n}^{p+1} - Fo(T_{m+1,n}^{p+1} + T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p}$
	$T_{m,n}^{p+1} = \frac{2}{3}Fo(T_{m+1,n}^{p} + 2T_{m-1,n}^{p} + 2T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2Bi\,T_{\infty}) + (1 - 4Fo - \frac{4}{3}Bi\,Fo)T_{m,n}^{p}$ (5.85) 2. Node at interior corner with convection	$Fo(3 + Bi) \leq \frac{3}{4}$ (5.86)	$(1 + 4Fo(1 + \frac{1}{3}Bi))T_{m,n}^{p+1} - \frac{2}{3}Fo \cdot (T_{m+1,n}^{p+1} + 2T_{m-1,n}^{p+1} + 2T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + \frac{4}{3}Bi\,Fo\,T_{\infty}$
	$T_{m,n}^{p+1} = Fo(2T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p} + 2Bi\,T_{\infty}) + (1 - 4Fo - 2Bi\,Fo)T_{m,n}^{p}$ (5.87) 3. Node at plane surface with convection[a]	$Fo(2 + Bi) \leq \frac{1}{2}$ (5.88)	$(1 + 2Fo(2 + Bi))T_{m,n}^{p+1} - Fo(2T_{m-1,n}^{p+1} + T_{m,n+1}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + 2Bi\,Fo\,T_{\infty}$
	$T_{m,n}^{p+1} = 2Fo(T_{m-1,n}^{p} + T_{m,n-1}^{p} + 2Bi\,T_{\infty}) + (1 - 4Fo - 4Bi\,Fo)T_{m,n}^{p}$ (5.89) 4. Node at exterior corner with convection	$Fo(1 + Bi) \leq \frac{1}{4}$ (5.90)	$(1 + 4Fo(1 + Bi))T_{m,n}^{p+1} - 2Fo(T_{m-1,n}^{p+1} + T_{m,n-1}^{p+1}) = T_{m,n}^{p} + 4Bi\,Fo\,T_{\infty}$

[a] To obtain the finite-difference equation and/or stability criterion for an adiabatic surface (or surface of symmetry), simply set Bi equal to zero.