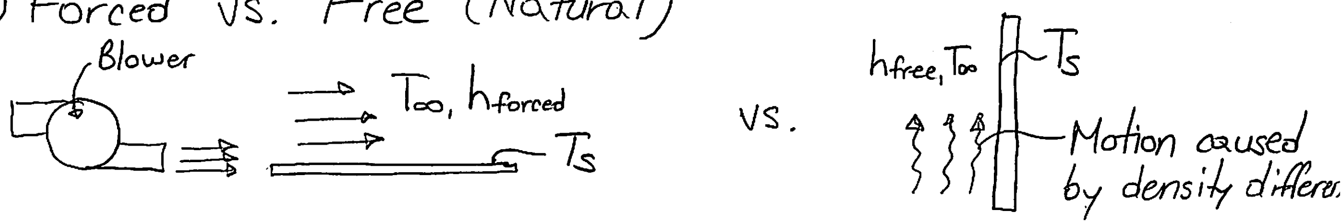
Nenad Miljkovic

Convection Heat Transfer

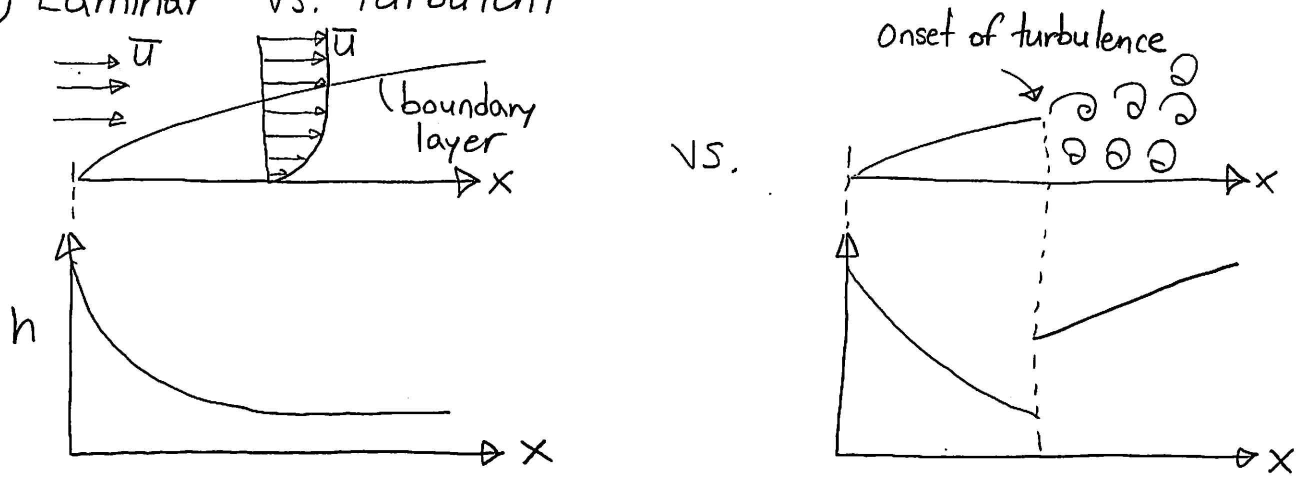
Convection is the transfer of thermal energy due to both conduction and by bulk "carrying" of the energy through the velocity of the fluid.

Four Categories

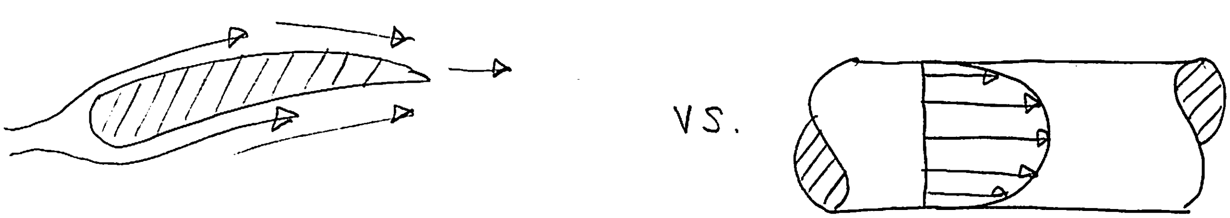
1. Forced vs. Free (Natural)



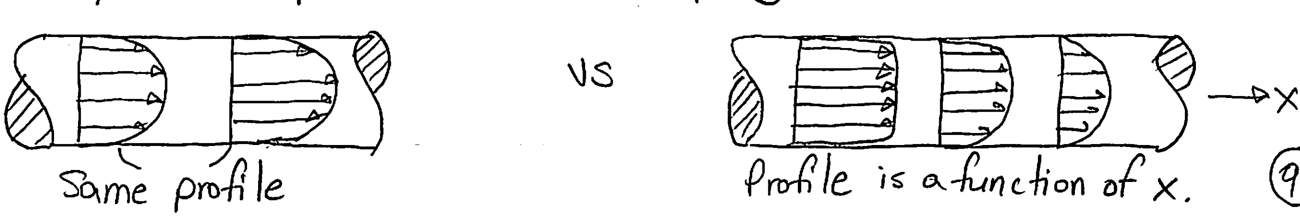
2. Laminar vs. Turbulent



3. External vs. Internal



4. Fully Developed vs. Developing



Nenad Miljkovic

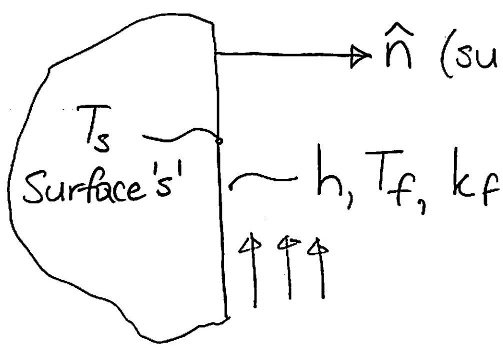
The Assumptions

1) Steady-State
2) Constant properties except where density differences cause the flow (Boussinesq approximation)
3) Incompressible

The Equations

1) Conservation of mass
2) Conservation of momentum (Navier-Stokes)
3) Conservation of energy (Heat equation for a fluid flow)

We've already seen convection before:



$T_s \equiv$ surface temperature
$T_f \equiv$ fluid temperature
$k_f \equiv$ fluid thermal cond.
$h \equiv$ heat transfer coeff.

We know that:

$$q'' = h(T_s - T_f) \quad ① \quad \text{(heat flux)}$$

We can also write that:

$$q'' = -k_f \left.\frac{\partial T_f}{\partial \hat{n}}\right|_s \quad ② \quad \text{(heat flux by Fourier's Law in fluid)}$$

Let's non dimentionalize our problem:

$$\theta = \frac{T - T_f}{T_s - T_f} \quad ; \quad n^* = \frac{\hat{n}}{L}$$

Back substituting into ① & ② & equating the two:

$$-k_f \frac{\cancel{T_s - T_f}}{L} \cdot \left.\frac{\partial \theta}{\partial n^*}\right|_s = h\left(\cancel{T_s - T_f}\right)$$

$$\boxed{-\left.\frac{\partial \theta}{\partial n^*}\right|_s = \frac{hL}{k_f} = Nu} \Rightarrow \text{Nusselt Number}$$

$$\boxed{Nu = \frac{hL}{k_f} = \frac{\text{convective heat transfer rate in fluid}}{\text{conductive heat transfer rate in fluid}}}$$

↳ Note, very similar to Biot number (B_i) but

$$B_i = \frac{hL}{k_s} \leftarrow \text{Solid thermal conductivity}$$

So Nu is a measure of the heat transfer enhancement you have due to bulk fluid motion when compared to conduction only.

Note, if $Nu = 1$, no convection effects & conduction only.

$$Nu = -\left.\frac{\partial \theta}{\partial n^*}\right|_s = 1 \Rightarrow \underbrace{-k_f \frac{T_s - T_f}{L}}_{\text{Conduction only}} = \underbrace{h(T_s - T_f)}_{\text{convection}}$$

Since $Nu \sim \left.\frac{\partial \theta}{\partial n^*}\right|_s \sim \theta$, it typically depends on:

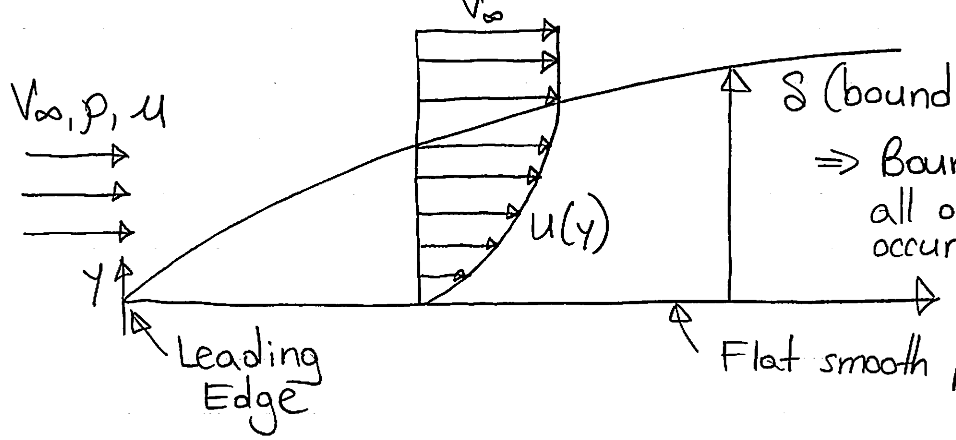
1) Flow conditions (laminar, turbulent)
2) Fluid properties
3) Geometry
4) Boundary conditions

Nenad Miljkovic

External Flow

Assuming: 1) ρ, μ, c = constant
2) Incompressible

We need to solve the coupled fluid flow (momentum) and energy equations to solve for h.



For a 2D laminar flow, flat plate, & steady

X-momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

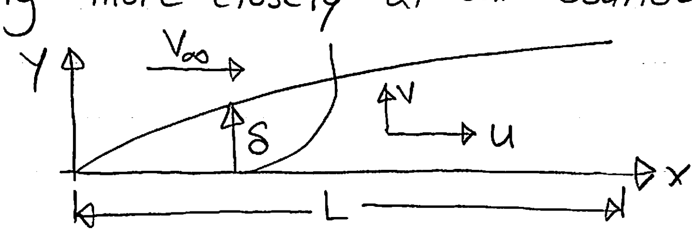
y-momentum:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Looking more closely at our boundary layer:



Let's non-dimensionalize

Nenad Miljkovic

Let: $\bar{u} = \dfrac{u}{V_\infty}$, $\bar{v} = \dfrac{v}{V_s}$, $\bar{P} = \dfrac{P}{\rho V_\infty^2}$

$\bar{x} = \dfrac{x}{L}$, $\bar{y} = \dfrac{y}{\delta}$

We know all of these parameters except for V_s.
Using continuity: $\bar{u} \sim V_\infty$, $\bar{x} \sim L$, $\bar{y} \sim \delta$ ("$\sim$" means scales as)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{V_\infty}{L} + \frac{V_s}{\delta} = 0 \Rightarrow \boxed{V_s \sim \left(\frac{\delta}{L}\right) V_\infty}$$

Note, when I use "$\sim$", it means scales as or is on the same order of magnitude as. Very powerfull method!

We know for boundary layers that $\delta \ll L$, hence $V_s \ll V_\infty$

So if we back substitute into our x-momentum

$$V_\infty \bar{u} \frac{V_\infty}{L}\frac{\partial \bar{u}}{\partial \bar{x}} + \left(\frac{\cancel{\delta}}{L}\right) V_\infty \bar{v} \frac{V_\infty}{\cancel{\delta}} \frac{\partial \bar{u}}{\partial y} = -\frac{1}{\cancel{\rho}} \frac{\cancel{\rho} V_\infty^2}{L}\frac{\partial \bar{P}}{\partial \bar{x}} + \nu\left[\frac{V_\infty}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{V_\infty}{\delta^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

$$\frac{V_\infty^2}{L}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{V_\infty^2}{L}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{V_\infty^2}{L}\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{V_\infty \nu}{L^2}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \underbrace{\left(\frac{L^2}{\delta^2}\right)}_{\gg 1,\text{ so dominates}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

Divide both sides by $\dfrac{V_\infty^2}{L}$

$$\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}}}_{\text{Note, thesa are all } \sim 1.} + \underbrace{\frac{\nu}{V_\infty L}}_{\frac{1}{Re_L}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$\dfrac{1}{Re_L} \Rightarrow \boxed{Re_L = \dfrac{V_\infty L}{\nu} = \dfrac{\rho V_\infty L}{\mu}}$

Where: $\boxed{Re_L \equiv \text{Reynolds Number} \equiv \dfrac{\text{inertial force}}{\text{viscous force}}}$

$$\boxed{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{Re_L}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}} \quad (1)$$

→ Non-dimensional x-momentum equation

Let's now look at y-momentum:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

⇒ Non-dimensionalize this

$$V_\infty \bar{u}\left(\frac{\delta}{L}\right)V_\infty \cdot \frac{1}{L}\frac{\partial \bar{v}}{\partial \bar{x}} + \left(\frac{\delta}{L}\right)V_\infty \bar{v}\left(\frac{\delta}{L}\right)V_\infty \frac{1}{\delta}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\cancel{\rho} V_\infty^2}{\cancel{\rho}\,\delta}\frac{\partial \bar{P}}{\partial \bar{y}}$$

$$+ \nu\left[\left(\frac{\delta}{L}\right)V_\infty \frac{1}{L^2}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \left(\frac{\delta}{L}\right)V_\infty \frac{1}{\delta^2}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right]$$

Collecting all terms and multiplying both sides by δ

$$V_\infty^2\left(\frac{\delta^2}{L^2}\right)\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + V_\infty^2\left(\frac{\delta^2}{L^2}\right)\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -V_\infty^2\frac{\partial \bar{P}}{\partial \bar{y}} +$$

$$+ \nu V_\infty\left[\left(\frac{\delta^2}{L^3}\right)\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{1}{L}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right]$$

Divide both sides by V_∞^2:

$$\left(\frac{\delta^2}{L^2}\right)\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \left(\frac{\delta^2}{L^2}\right)\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{y}} + \frac{\nu}{V_\infty L}\left[\left(\frac{\delta^2}{L^2}\right)\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right]$$

Since $\left(\frac{\delta}{L}\right)^2 \ll 1$ (Boundary Layer approx), many of our terms drop out.

$$\boxed{\frac{\partial \bar{P}}{\partial \bar{y}} = \frac{1}{Re_L}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}} \quad (2)$$

→ Non-dimensional y-momentum

It turns out you can use scaling analysis to show that

$$\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \ll \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \Rightarrow \frac{\partial \bar{P}}{\partial \bar{y}} \approx 0 \Rightarrow \boxed{P = f(x) \text{ only}}$$

↳ Derived in ME521 (Convective Heat Transfer)