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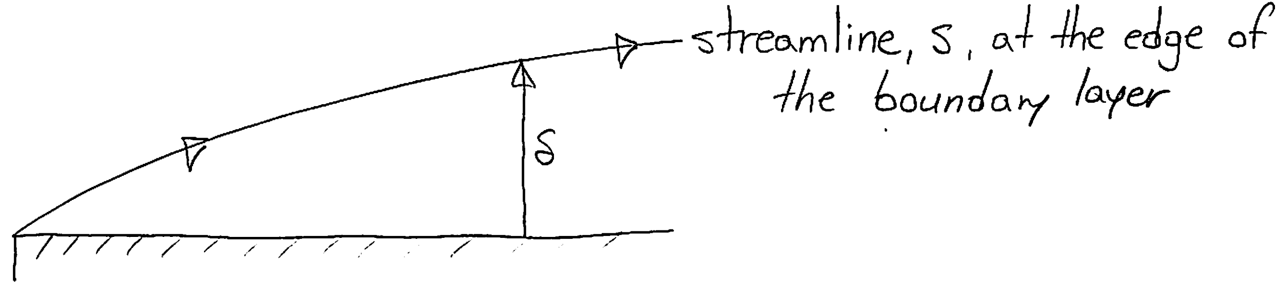
Now our life becomes much easier. We can use one more trick up our sleves:



At the streamline, our x-momentum equation becomes:

$$u\frac{\partial u}{\partial x} + \underbrace{v\frac{\partial u}{\partial y}}_{0 \text{ at } s \text{ (no shear)}} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \underbrace{\nu\left(\frac{\partial^2 u}{\partial y^2}\right)}_{0 \text{ at } s \text{ (no shear)}}$$

At s, $u = V_\infty$

$$V_\infty \frac{\partial V_\infty}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0 \overset{\text{integrate}}{\Rightarrow} \boxed{P + \frac{1}{2}\rho V_\infty^2 = \text{constant}}$$

↳ Bernoulli equation

And since we know $P \neq f(y)$, this is also true inside the boundary layer.

We also know that; $V_\infty = \text{constant}$, so $\frac{\partial V_\infty}{\partial x} = 0$

$$\therefore \frac{1}{\rho}\frac{\partial P}{\partial x} = 0 \Rightarrow \boxed{P(x) = \text{constant}} \text{ or } \boxed{\frac{\partial P}{\partial x} = \frac{\partial \bar{P}}{\partial x} = 0}$$

Note, this is only true for a flat plate. Not true for cylinders or spheres where $V_\infty \neq \text{constant}$.

Now we can re-cast our x-momentum equation:

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}}$$

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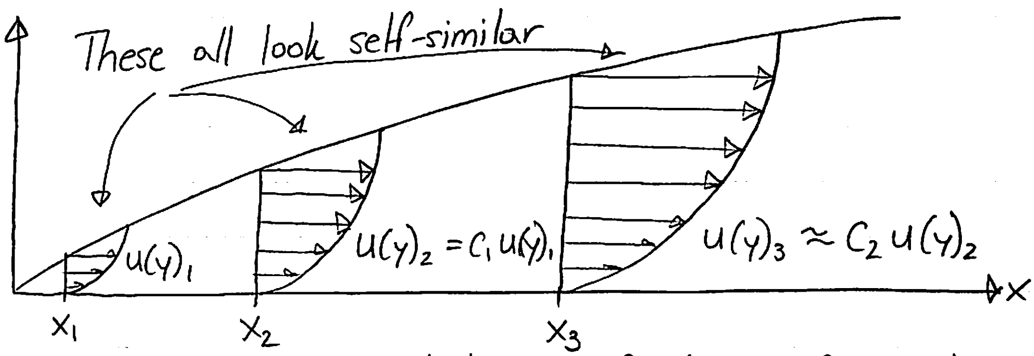
Our boundary conditions are: 1) $u = v = 0$ at $y = 0$
2) $u = V_\infty$ at $y \to \infty$

Just like in transient conduction, we want to convert our simpler but still complex PDE into an ODE.

Cannot use separation of variables (infinite domain, $y \to \infty$)
Let's try the similarity solution:



A function ϕ that is a function of η only.

$$\frac{u}{V_\infty} = \phi(\eta)\ ; \quad \text{where} \quad \eta = \frac{y}{\delta}$$

$\frac{u}{V_\infty}$ → Scales from 0 to 1 $\qquad$ η → Scales from 0 to 1

But how do we define δ?

Let's look back at our dimensionless momentum equation:

$$\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}}_{\text{We know these are both on the order 1.}} = \underbrace{\frac{1}{Re_L}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}_{\text{This must also be on the order 1.}}$$

Also: $\bar{u} \sim V_\infty$

$$\bar{y} = \frac{y}{L} \sim \frac{\delta}{L} \Rightarrow \partial\bar{y} \sim \frac{\delta}{L}$$

$$\frac{\nu}{V_\infty L} \cdot V_\infty \cdot \left(\frac{L}{\delta}\right)^2 \sim 1 \Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}} \quad \text{or} \quad \frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}}$$

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So we end up with: $\underbrace{\delta \sim \sqrt{\frac{\nu x}{V_\infty}}}$ or $\boxed{\eta = y\sqrt{\frac{V_\infty}{\nu x}}}$

We'll see a much easier way to get this later.

So what do we do about v?
Using continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \Rightarrow \bar{u} = \phi(\eta) = \frac{u}{V_\infty}$$

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \bar{u}}{\partial x} = -\frac{1}{2x}\eta\phi' \Rightarrow \text{Back substitute into continuity}$$

$$\frac{\partial \bar{v}}{\partial y} = \frac{1}{2x}\eta\phi' \qquad d\eta = dy\sqrt{\frac{V_\infty}{\nu x}}$$

$$\bar{v} = \frac{1}{2x}\int_0^\eta \phi'\eta\, dy = \underbrace{\frac{1}{2x}\sqrt{\frac{x\nu}{V_\infty}}}_{\frac{1}{2}\frac{1}{\sqrt{Re_x}}}\int_0^\eta \phi'\eta\, d\eta$$

We need to solve this integral:

$$\int_0^\eta \eta\phi'\, d\eta = uv - \int v\, du \quad \text{(Integration by parts)}$$

$$\int u\, dv = uv - \int v\, du$$

$$u = \eta, \quad v = \phi$$

$$= \eta\phi - \int_0^\eta \phi\, d\eta$$

$$\text{let } \int_0^\eta \phi\, d\eta = F(\eta)$$

Now we have: $\boxed{\bar{v} = \frac{1}{2}\frac{1}{\sqrt{Re_x}}(\eta F' - F)}$

Back substituting into our initial PDE

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{Re_L}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\bar{u} = F' \qquad \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = F''\cdot\frac{\partial \eta}{\partial x}$$ ... (Back substitute the rest)

Simplifying, we will obtain:

$$\boxed{F''' + \frac{1}{2}FF'' = 0} \quad \text{where} \quad F = \int_0^{\eta}\phi\, d\eta\,, \quad \eta = y\sqrt{\frac{V_\infty}{x\nu}}$$

↳ ODE!

Our boundary conditions now become:

1) At the wall $(y=0,\ \eta=0)$, $F' = F = 0$

2) Outside the boundary layer $(y\to\infty,\ \eta\to\infty)$, $F' = 1$

Rewriting our equation in terms of $\bar{u}$, F, and η:

$$\boxed{\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{1}{2}F\frac{\partial \bar{u}}{\partial \eta} = 0} \quad (1)$$

To solve this ODE, we can assume an infinite series sol.:

$$F = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots$$
$$F' = a_1 + 2a_2\eta + 3a_3\eta^2 + \cdots$$
$$F'' = 2a_2 + 6a_3\eta + \cdots$$
$$F''' = 6a_3 + \cdots$$

Back substitute into (1) and solve for the coefficients

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$$\underbrace{(\cdots)}_{0}\eta^0 + \underbrace{(\cdots)}_{0}\eta^1 + \underbrace{(\cdots)}_{0}\eta^2 + \underbrace{(\cdots)}_{0}\eta^3 + \cdots + \underbrace{(\cdots)}_{0}\eta^n = 0$$

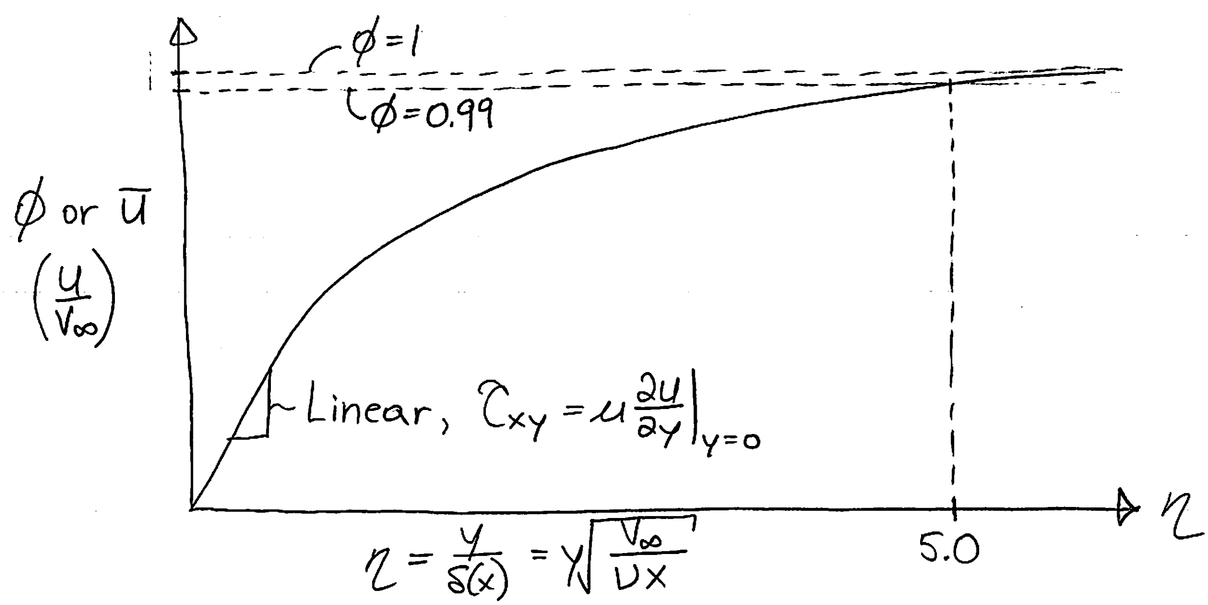
You obtain a recursion formula that relates your constants:

$$\boxed{F = \frac{a_2\eta^2}{2!} - \frac{a_2\eta^5}{2\cdot 5!} + \frac{11}{4}\frac{a_2^3\eta^8}{8!} + \cdots}$$

$\boxed{a_2 = 0.332}$ $\Rightarrow$ Heinrich Blasius solved this in 1911 for his PhD work with Ludwig Prandtl.

So $F = \int_0^\eta \phi \, d\eta$, $F' = \phi = \bar{u} = \dfrac{u}{V_\infty}$

We can now plot our result:



From our solution, we can solve for the hydrodynamic b.l. thick.

$$5.0 = \delta\sqrt{\frac{V_\infty}{x\nu}} = \frac{\delta}{x}\sqrt{Re_x} \Rightarrow \boxed{\delta(x) = \frac{5x}{\sqrt{Re_x}}}$$

↳ Boundary layer thickness