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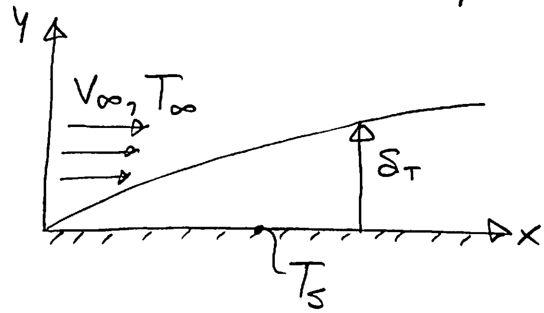
## Heat Transfer

Back from our interesting hydrodynamic excursion, we are interested in the convective heat transfer from the flat plate:

$$h = \frac{q''|_{y=0}}{\Delta T} = \frac{q''|_{y=0}}{T_s - T_\infty} = ?$$




Note, here we have the thermal boundary layer thickness, $\delta_T$.

$\delta \equiv$ thickness where the velocity gradient exists $\Longleftrightarrow$ $\delta_T \equiv$ thickness where the temperature gradient exists.

To solve for this second boundary layer, we need to solve the energy equation for the fluid flow.

$$\underbrace{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}}_{\text{Convection of energy}} = \underbrace{\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{\text{Conduction of energy}}$$

$\Rightarrow$ Note, we always had $\nabla^2 T = 0$ in conduction (R.H.S.)

For the $T_s$ = constant case, $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$

Boundary conditions: 1) $T(y=0) = T_s$
2) $T(y\to\infty) = T_\infty$

Non-dimensionalizing: Let $\theta = \frac{T - T_s}{T_\infty - T_s}$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}$$

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Note the similarity of the boundary layer equations

| Hydrodynamic ($\delta$) | Thermal ($\delta_T$) |
|---|---|
| $u\dfrac{\partial \bar{u}}{\partial x} + v\dfrac{\partial \bar{u}}{\partial y} = \nu \dfrac{\partial^2 \bar{u}}{\partial y^2}$ | $u\dfrac{\partial \theta}{\partial x} + v\dfrac{\partial \theta}{\partial y} = \alpha \dfrac{\partial^2 \theta}{\partial y^2}$ |
| B.C.'s: $\bar{u}(y=0) = 0$ | $\theta(y=0) = 0$ |
| $\bar{u}(y\to\infty) = 1$ | $\theta(y\to\infty) = 1$ |
| $\left.\dfrac{\partial \bar{u}}{\partial y}\right\|_{y\to\infty} = 0$ | $\left.\dfrac{\partial \theta}{\partial y}\right\|_{y\to\infty} = 0$ |

So we can find an analogy to our fluids solution

We can first define a usefull property called the Prandtl number

$$\boxed{Pr = \frac{\nu}{\alpha} \equiv \frac{\text{viscous (momentum) diffusion rate}}{\text{thermal (heat) diffusion rate}}}$$

If $Pr = 1$, $\boxed{\theta = \bar{u}, \quad \delta = \delta_T}$ $\Rightarrow$ Hydrodynamic & thermal boundary layers overlap.

However, in real life, most fluids aren't $Pr = 1$.

$$\left.\begin{aligned} Pr &\cong 0.001 \text{ for liquid metals} \\ &\cong 0.015 \text{ for mercury} \\ &\cong 0.7 - 0.8 \text{ for air} \end{aligned}\right\} \delta < \delta_T$$

$$\left.\begin{aligned} &\cong 4 - 5 \text{ for refrigerants} \\ &\cong 7 \text{ for water} \\ &\cong 100 - 40{,}000 \text{ for engine oil} \\ &\cong 1\times10^{25} \text{ for the earths mantle} \end{aligned}\right\} \delta > \delta_T$$

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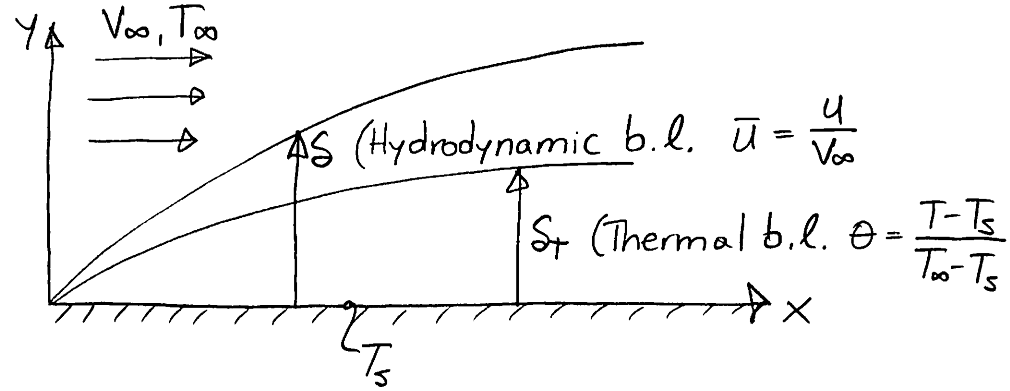
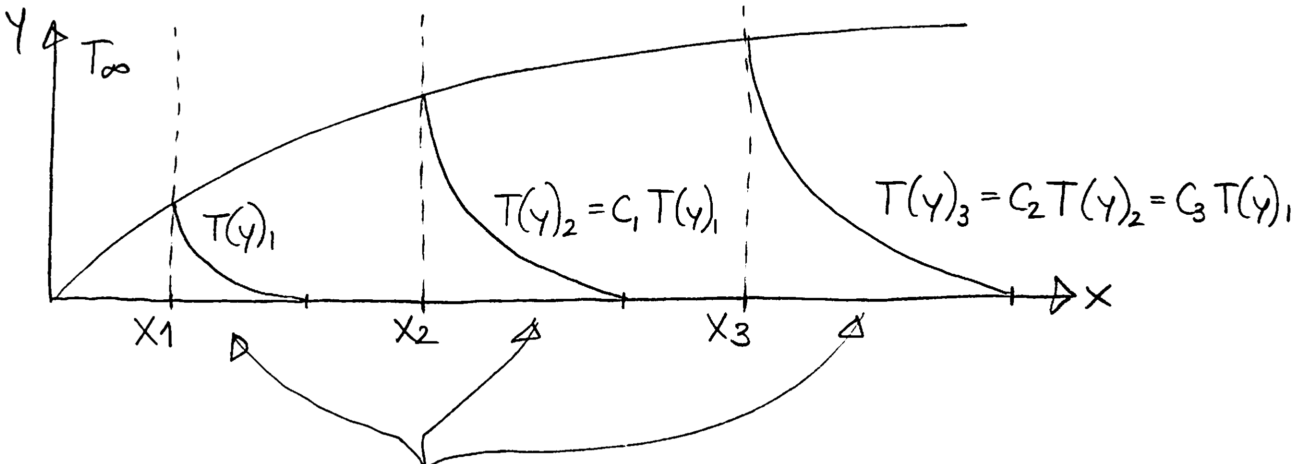
If $\nu \neq \alpha$:

Note, the case I drew here is for $Pr > 1$. $\Leftarrow$




OK, so lets solve the thermal b.l. equation for $\alpha \neq \nu$
We will use the same similarity approach:




All three profiles look self-similar in nature.

Assume: $\theta = f(\eta)$ , $\eta = y\sqrt{\frac{V_\infty}{\nu x}}$ (same similarity variable as in the hydrodynamic case)

$$\left.\begin{aligned} \frac{\partial\theta}{\partial x} &= \frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} \\ \frac{\partial\theta}{\partial y} &= \frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} \\ \frac{\partial^2\theta}{\partial y^2} &= \frac{\partial^2\theta}{\partial\eta^2}\cdot\left(\frac{\partial\eta}{\partial y}\right)^2 \end{aligned}\right\}$$ Back substitute these into our energy equation PDE:

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = \alpha\frac{\partial^2\theta}{\partial y^2} \Rightarrow u\frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} + v\frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} = \alpha\frac{\partial^2\theta}{\partial\eta^2}\cdot\left(\frac{\partial\eta}{\partial y}\right)^2$$

Note, we also know that: $F = \int_0^\eta \phi\, d\eta$, $\bar{u} = \phi(\eta) = \frac{u}{V_\infty} = F'$

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Back substituting and doing some algebra, we obtain:

$$\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2}F\cdot Pr\frac{\partial\theta}{\partial\eta} = 0$$

Using a trick: $Pr^{2/3}\cdot F(\eta) = F(\eta^*)$ ; $\eta^* = \eta Pr^{1/3}$, $d\eta^* = Pr^{1/3}d\eta$

Note, this works because: $F = \frac{a_2\eta^2}{2} - \dots$ (H.O.T.)
If we only use the lead term:
$$F(\eta) = \frac{a_2\eta^2}{2}$$
$$F(\eta^*) = \frac{a_2 Pr^{2/3}\eta^{*2}}{2}$$
$$F(\eta^*) = Pr^{2/3}F(\eta)$$

Now our equation becomes:

F at $\eta^*$, not F times $\eta^*$!

$$\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2}F(\eta^*)Pr^{1/3}\frac{\partial\theta}{\partial\eta} = 0$$

$$\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2}F(\eta^*)Pr^{2/3}\cdot\frac{\partial\theta}{\underbrace{\partial\eta\, Pr^{1/3}}_{\partial\eta^*}} = 0$$

Multiplying through by $Pr^{-2/3}$

$$\underbrace{\frac{1}{Pr^{2/3}}\left(\frac{\partial^2\theta}{\partial\eta^2}\right.}_{\partial\eta^{*2}} + \frac{1}{2}F(\eta^*)\cancel{Pr^{2/3}}\cdot\frac{\partial\theta}{\partial\eta^*} = 0\Bigg)$$

Now our PDE becomes identical to our hydrodynamic ODE

$$\boxed{\frac{\partial^2\theta}{\partial\eta^{*2}} + \frac{1}{2}F(\eta^*)\frac{\partial\theta}{\partial\eta^*} = 0} \Rightarrow \theta(\eta^*) \equiv \bar{u}(\eta)$$

Before, we had: $F''' + \frac{1}{2}FF'' = 0$

B.C.'s : 1) $\theta(\eta^* = 0) = 0$
2) $\theta(\eta^* \to \infty) = 1$

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

So we can now use the exact same Blasius solution plot that we had before.

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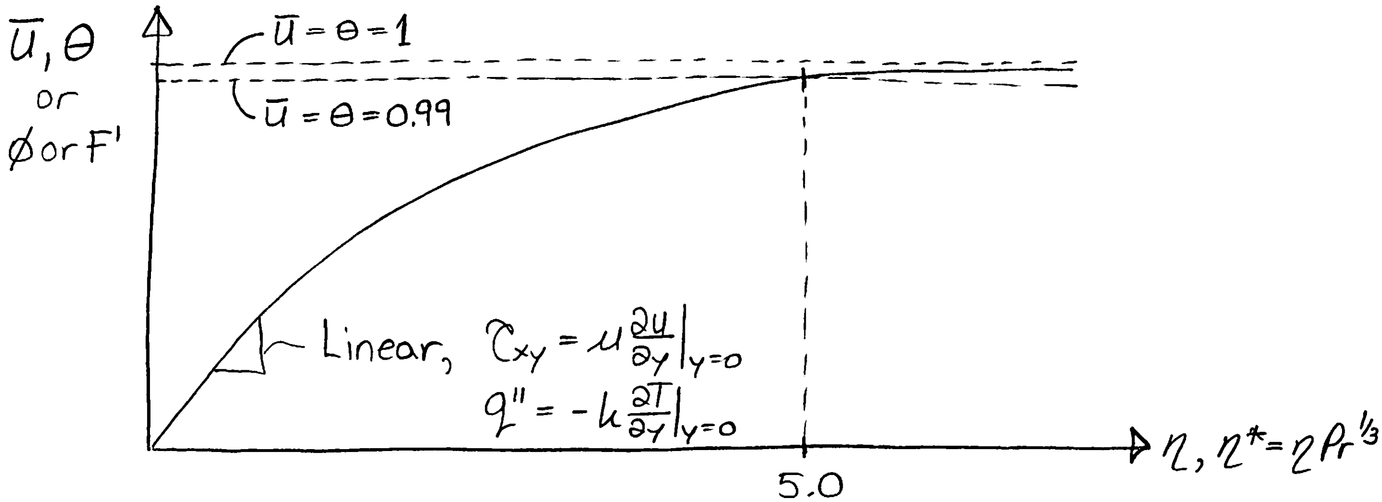


$$\delta \sqrt{\frac{V_\infty}{\nu x}} = \eta(y=\delta) = 5.0 \quad \text{(Hydrodynamic b.l.)}$$

$$Pr^{1/3} \delta_T \sqrt{\frac{V_\infty}{\nu x}} = \eta^*(y=\delta_T) = 5.0 \quad \text{(Thermal b.l.)}$$

Taking the ratio of our two solutions above:

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}}$$

$\Rightarrow$ Makes sense since the only difference is $\nu$ & $\alpha$ $\Rightarrow$ $Pr = \frac{\nu}{\alpha}$

<u>Heat Transfer</u> ($q''$ = heat flux)

$$q''\Big|_{y=0} = -k \frac{\partial T}{\partial y}\Big|_{y=0} \Rightarrow \theta = \frac{T - T_s}{T_\infty - T_s}, \quad \partial\theta = \frac{\partial T}{T_\infty - T_s}$$

$$\eta^* = \eta Pr^{1/3} = y\sqrt{\frac{V_\infty}{\nu x}} \cdot Pr^{1/3}$$

$$\partial\eta^* = \partial y \left(\frac{V_\infty}{\nu x}\right)^{1/2} Pr^{1/3}$$

Back substitute

$$q''\Big|_{y=0} = -k(T_\infty - T_s)\left(\frac{V_\infty}{\nu x}\right)^{1/2} Pr^{1/3} \frac{\partial\theta}{\partial\eta^*}\Big|_{\eta^*=0}$$

$$= k\frac{(T_s - T_\infty)}{x}\underbrace{\left(\frac{V_\infty x}{\nu}\right)^{1/2}}_{Re_x^{1/2}} Pr^{1/3} \underbrace{F''(0)}_{a_2 = 0.332}$$

$$\boxed{Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}}$$

$\hookrightarrow$ Local Nusselt number for a flat plate in laminar flow.

$$q''\Big|_{y=0} = \frac{k\Delta T}{x} Re_x^{1/2} Pr^{1/3} a_2, \quad Nu_x = \frac{hx}{k} = \frac{q''\big|_{y=0}}{\Delta T} \cdot \frac{x}{k} = a_2 Re^{1/2} Pr^{1/3}$$