Heat Transfer

Back from our interesting hydrodynamic excursion, we are interested in the convective heat transfer from the flat plate:

$$h = \frac{q'' | y=0}{\Delta T} = \frac{q'' | y=0}{T_s - T_\infty} = ?$$

Note, here we have the thermal boundary layer thickness, $S_T$.

$$S = \text{thickness where the velocity gradient exists} \quad \iff \quad S_T = \text{thickness where the temperature gradient exists}$$

To solve for this second boundary layer, we need to solve the energy equation for the fluid flow.

$$U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} = \kappa \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad \Rightarrow \text{Note, we always had } \nabla^2 T = 0 \text{ in conduction (R.H.S.)}$$

For the $T_s = \text{constant case}$, $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$

Boundary conditions:
1. $T(y=0) = T_s$
2. $T(y \to \infty) = T_\infty$

Non-dimensionalizing: Let $\Theta = \frac{T - T_s}{T_\infty - T_s}$

$$U \frac{\partial \Theta}{\partial x} + V \frac{\partial \Theta}{\partial y} = \kappa \frac{\partial^2 \Theta}{\partial y^2}$$
Note the similarity of the boundary layer equations

<table>
<thead>
<tr>
<th>Hydrodynamic ($\delta$)</th>
<th>Thermal ($\delta_T$)</th>
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</thead>
<tbody>
<tr>
<td>$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U \frac{\partial^2 U}{\partial y^2}$</td>
<td>$U \frac{\partial \Theta}{\partial x} + V \frac{\partial \Theta}{\partial y} = \Theta \frac{\partial^2 \Theta}{\partial y^2}$</td>
</tr>
<tr>
<td>B.C.'s: $U (y=0) = 0$</td>
<td>$\Theta (y=0) = 0$</td>
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<tr>
<td>$U (y \to \infty) = 1$</td>
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<tr>
<td>$\frac{\partial U}{\partial y} \bigg</td>
<td>_{y=\infty} = 0$</td>
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</table>

So we can find an analogy to our fluids solution.

We can first define a useful property called the Prandtl number:

$$Pr = \frac{\nu}{\alpha} \equiv \frac{\text{viscous (momentum) diffusion rate}}{\text{thermal (heat) diffusion rate}}$$

If $Pr = 1$, $\Theta = U$, $S = S_T$ \Rightarrow Hydrodynamic & thermal boundary layers overlap.

However, in real life, most fluids aren't $Pr = 1$.

$Pr \approx 0.001$ for liquid metals
$\approx 0.015$ for mercury
$\approx 0.7 - 0.8$ for air
$\approx 4 - 5$ for refrigerants
$\approx 7$ for water
$\approx 100 - 40,000$ for engine oil
$\approx 1 \times 10^{25}$ for the earth's mantle

$s < s_T$

$s > s_T$
If \( U \neq \infty \):

Note, the case drawn here is for \( Pr > 1 \).

OK, so let's solve the thermal boundary layer equation for \( \alpha \neq U \)
We will use the same similarity approach:

Assume: \( \Theta = f(\eta) \), \( \eta = \gamma \sqrt{\frac{V_{\infty}}{Ux}} \) (same similarity variable as in the hydrodynamic case)

\[
\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \\
\frac{\partial \Theta}{\partial y} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \\
\frac{\partial^2 \Theta}{\partial y^2} = \frac{\partial^2 \Theta}{\partial \eta^2} \left( \frac{\partial \eta}{\partial y} \right)^2
\]

Back substitute these into our energy equation PDE:

\[
u \frac{\partial \Theta}{\partial x} + \nu \frac{\partial \Theta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial y^2} \Rightarrow \nu \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + \nu \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial \eta^2} \left( \frac{\partial \eta}{\partial y} \right)^2
\]

Note, we also know that: \( F = \int_0^y \phi \, d\eta \), \( \bar{U} = \phi(y) = \frac{U}{V_{\infty}} = F \)
Back substituting and doing some algebra, we obtain:

\[ \frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} F \cdot Pr \frac{\partial \Theta}{\partial \eta} = 0 \]

Using a trick: \( Pr^{2/3} \cdot F(\eta) = F(\eta^*) \); \( \eta^* = \eta \cdot Pr^{1/3} \)

Now our equation becomes:

\[ \frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) Pr^{1/3} \frac{\partial \Theta}{\partial \eta^*} = 0 \]

\[ \frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) Pr^{2/3} \cdot \frac{\partial \Theta}{\partial \eta^*} = 0 \]

Multiplying through by \( Pr^{-2/3} \)

\[ \frac{1}{Pr^{2/3}} \left( \frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) Pr^{2/3} \cdot \frac{\partial \Theta}{\partial \eta^*} = 0 \right) \]

Now our POE becomes identical to our hydrodynamic ODE

\[ \frac{\partial^2 \Theta}{\partial \eta^*^2} + \frac{1}{2} F(\eta^*) \frac{\partial \Theta}{\partial \eta^*} = 0 \quad \Rightarrow \quad \Theta(\eta^*) = \bar{U}(\eta) \]

Before, we had: \( F'' + \frac{1}{2} FF'' = 0 \)

B.C.'s:

1) \( \Theta(\eta^* = 0) = 0 \)
2) \( \Theta(\eta^* \to \infty) = 1 \)

\[ \Theta = \frac{T - T_s}{T_{\infty} - T_s} \]

So we can now use the exact same Blasius solution plot that we had before.
\[
\begin{align*}
S \sqrt{\frac{V_\infty}{U_x}} &= \eta (y = \delta) = 5.0 \quad \text{(Hydrodynamic b.l.)} \\
\rho r^{\frac{1}{3}} S \sqrt{\frac{V_\infty}{U_x}} &= \eta^*(y = \delta_T) = 5.0 \quad \text{(Thermal b.l.)}
\end{align*}
\]

Taking the ratio of our two solutions above:

\[
\frac{S}{\delta_T} = \rho r^{\frac{1}{3}} \quad \Rightarrow \text{Makes sense since the only difference is} \quad \frac{U}{L_x} = \rho \frac{r^{\frac{1}{3}}}{x}
\]

Heat Transfer (\(q'' = \text{heat flux}\))

\[
q''_{|y=0} = -k \frac{\partial T}{\partial y}_{|y=0} \quad \Rightarrow \quad \Theta = \frac{T - T_3}{T_\infty - T_3}, \quad \Theta = \frac{\partial T}{\partial y}_{|y=0} \text{ Back substitution}
\]

\[
\eta^* = \eta \rho r^{\frac{1}{3}} = \gamma \sqrt{\frac{V_\infty}{U_x}} \cdot \rho r^{\frac{1}{3}}
\]

\[
\left. \frac{\partial T}{\partial y} \right|_{y=0} = \frac{1}{\Theta_{|y=0}} = \frac{1}{\frac{T - T_3}{T_\infty - T_3}} = \frac{T_\infty - T_3}{T_\infty - T_3}
\]

\[
q''_{|y=0} = -k \left( \frac{T_\infty - T_3}{T_\infty - T_3} \right) \left( \frac{V_\infty}{U_x} \right)^{\frac{1}{2}} \rho r^{\frac{1}{3}} \frac{\partial T}{\partial y}_{|y=0} \text{ substitute}
\]

\[
q''_{|y=0} = k \frac{(T_\infty - T_3)}{x} \left( \frac{V_\infty}{U_x} \right)^{\frac{1}{2}} \rho r^{\frac{1}{3}} \frac{\partial T}{\partial y}_{|y=0} \text{ substitute}
\]

\[
N_u x = 0.332 R_e x^{\frac{1}{2}} P_r^{\frac{1}{3}} \quad \text{(Local Nusselt number for a flat plate in laminar flow)}
\]

\[\frac{N_u x}{k} = \frac{q''_{|y=0} x}{T_\infty - T_3} = a_2 R_e x^{\frac{1}{2}} P_r^{\frac{1}{3}}\]