Internal Flow - Fully Developed Flow in Tubes

\[ U_\infty \rightarrow x \rightarrow U(r) \]

\[ X_{el} = \text{entrance length or developing length. Here, the velocity profile varies with radial position } r, \text{ and axial location, } x. \]

We can estimate the \( X_{el} \) by using our Blasius solution to see when our boundary layers begin to overlap:

\[ \frac{S}{X} = \frac{5.0}{\sqrt{Re_x}} \]

We can estimate that \( S = \frac{D}{2} \) when b.l.'s merge:

\[ \frac{D}{2X_{el}} \sim \frac{5.0}{\sqrt{Re_x}} \]

\[ \frac{D}{X_{el}} \sim \frac{10}{\sqrt{Re_x}} = \frac{10}{\sqrt{pU_\infty X_{el}} U} \Rightarrow \frac{X_{el}}{D} \sim \frac{Re_0}{100} \sim 0.01 Re_0 \]

\[ \frac{D}{\sqrt{X_{el}}} \sim \frac{10}{\sqrt{Re_0}} \]

Experimentally: \[ \frac{X_{el}}{D} = 0.05 Re_0 \] If \( x > X_{el} \), flow is fully developed.
Fully Developed Region \((x > x_e)\)

We need to solve for the hydrodynamic properties of the flow to help us understand the heat transfer.

Applying the axial momentum equation (\(x\)-momentum)

\[
\rho \left( \frac{2u_x}{2t} + u_r \frac{2u_x}{2r} + u_\phi \frac{2u_x}{2\phi} + u_x \frac{2u_x}{2x} \right) = - \frac{2p}{2x} \\
+ u \left[ \frac{1}{r} \frac{2}{2r} (\frac{2u_x}{2r}) + \frac{1}{r^2} \frac{2^2 u_x}{2\phi^2} + \frac{2^2 u_x}{2x^2} \right] + \rho g_x = 0
\]

For fully developed flow:

\[
\frac{2u_x}{2x} = \frac{2u_x}{2t} = \frac{2u_x}{2\phi} = 0
\]

\[
u_\phi = u_r = 0
\]

For flow in a horizontal tube: \(g_x = 0\)

Most of our terms drop out and we are left with:

\[
- \frac{2p}{2x} + u \frac{1}{r} \frac{2}{2r} (r \frac{2u_x}{2r}) = 0 \quad (1)
\]

B.C.'s:

\[
\frac{2u_x}{2r} \bigg|_{r=0} = 0 \quad \text{(Symmetry)} \quad (2)
\]

\[
u_x (r=r_o) = 0 \quad \text{(no slip)} \quad (3)
\]

From here on in, I will use \(u_x = U\) (dropping \(x\) subscript)

Integrating eq. (1) twice & applying (2) & (3):

\[
U(r) = \frac{r_o^2}{4U} \left( \frac{2p}{2x} \right) \left( 1 - \frac{r^2}{r_o^2} \right) \quad \Rightarrow \text{Fully developed velocity profile in a round pipe. (Laminar flow)}
\]
Solving for our average velocity:
\[
\overline{U} = \frac{1}{2\pi r_o^2} \int_0^{r_o} U \cdot 2\pi r dr = \frac{r_o^2}{4\mu} \left(-\frac{2\rho}{2\pi} \right) \int_0^1 (1 - \lambda) d\lambda ; \lambda = \frac{r^2}{r_o^2}
\]
\[
\overline{U} = \frac{r_o^2}{8\mu} \left(-\frac{\partial p}{\partial x} \right)
\]
\[
\Rightarrow \text{Fully developed laminar flow in a round pipe.}
\]

Typically, we need to know friction, pressure drop, or force.
\[
\int = \frac{\Delta P}{\left(\frac{L}{D}\right) \frac{1}{2} \rho \overline{U}^2} = \text{Tube friction factor. Easy way to calculate pressure loss in tubes of length } L, \text{ diameter } D.
\]
\[
C_{f,x} = \frac{C_x}{\frac{1}{2} \rho \overline{U}^2} = \text{Local tube friction coefficient. Related to the friction factor, } f.
\]

Drawing a force balance diagram on a segment of our fluid in the pipe. \( \sum F_x = 0 \) since \( U(r) = f(x) = \text{fully developed} \)

\[
\Delta P \pi r_o^2
\]
\[
\tau_o (2\pi r_o L) = \Delta P \pi r_o^2 = \frac{\Delta P D}{2}
\]
\[
4 \tau_o = \frac{\Delta P}{\left(\frac{L}{D}\right)}
\]

We can say: \( 4C_f = f \Rightarrow \text{Let's now solve for friction factor} \)
\[ f = \frac{\Delta P}{(L) \frac{1}{2} \rho U^2} \quad (1) \]

But we've just solved for laminar flow that:

\[ \frac{8 \mu U}{r_0^2} = -\frac{2P}{\delta x} \approx \frac{\Delta P}{L} \quad (2) \]

Substitute here.

From (1):

\[ \frac{\Delta P}{L} \cdot \frac{20}{\rho U^2} = f = \frac{8 \mu U}{r_0^2} \cdot \frac{20}{\rho U^2} \]

\[ \frac{16 \mu D}{r_0^2 \rho U} = f = \frac{16 \mu D}{(D/2)(D/2)} \rho U = \frac{64 \mu U}{\rho U 0} = \frac{64}{Re_0} \]

\[ f = \frac{64}{Re_0} \Rightarrow \text{Pipe friction factor for laminar flow in a smooth round tube, Darcy-Weisbach equation.} \]

We can also say the following:

\[ f \cdot Re_0 = \text{constant} \]

For non-circular channels/pipes:

\[ D_h = \text{hydraulic diameter} = \frac{4A}{\rho} \]

\[ A = \rho \cdot a \cdot \sin 60^\circ \]

\[ \rho = 3a \]

\[ D_h = \frac{4a^2 \sin 60^\circ}{3a} = 1.15a \]
Heat Transfer in Internal Flows

How does the thermal boundary layer or entry length develop?

\[ \frac{X_{et}}{D} = 0.017 \text{Re}_D \cdot \text{Pr} \quad \Rightarrow \text{Experimentally obtained Thermal Entry Length} \]

For fully developed flow, both the Thermal entry length \((X_{et})\) and hydrodynamic entry length \((X_{el})\) must be passed: \(X > X_{et} \& X_{el}\)

So how about heat transfer in the fully developed region: i.e. \(X > X_{el} \& X > X_{et}\)

\[ h = \frac{q_{\text{wall}}}{\Delta T} \]

From energy balance at the wall (conduction = convection)

\[ h \Delta t = k_f \frac{\Delta T}{S_T} \Rightarrow h \sim \frac{k_f}{S_T} \]

For fully developed flow (overlapping thermal boundary layers): \(S_T \sim \frac{D}{2}\)
\[ h \sim \frac{2\kappa_f}{D} \]

But we know that:

\[ \text{Nu}_0 = \frac{\dot{h}D}{\kappa_f} = \frac{2\kappa_f}{D} \left( \frac{\partial}{\partial} \right) \frac{\partial}{\kappa_f} \]

\[ \text{Nu}_0 \approx 2 \]

So we expect a constant Nusselt number.

It turns out this scaling is pretty good. If we solve the coupled momentum and energy equations for internal smooth round tube pipe flow, we will obtain the following:

\[ \text{Nu}_0 = \frac{\dot{h}D}{\kappa_f} = 4.364 \quad \Rightarrow \text{Constant wall heat flux} \]

Laminar flow, round tube

\( \text{Re}_0 < 2300; \quad \text{Re}_0 = \frac{\rho D_0 u_0}{\mu} \)

\[ \text{Nu}_0 = \frac{\dot{h}D}{\kappa_f} = 3.66 \quad \Rightarrow \text{Constant wall temperature} \]

Laminar flow, round tube

\( \text{Re}_0 < 2300 \)

**Bulk Fluid Temperature**

It is important to note that for internal flows, heat transfer is computed by using the bulk fluid temperature: \( T_b \)

\[ \dot{h} = \frac{\dot{Q}_{\text{wall}}}{T_w - T_b} \quad ; \quad T_w = \text{wall temperature} \]

\[ T_b = \text{bulk fluid temperature} \]

\( T_b \) can be thought of as if we allow the pipe fluid to uniformly mix & come to an equilibrium temperature:

\[ T_b = \frac{1}{A \overline{u}} \int_A \int \dot{u}(r) \cdot T \, dA \]

where \( A = \text{cross sectional area} \)

\( \overline{u} = \text{average fluid velocity} \)
<table>
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<tr>
<th>Cross Section</th>
<th>$\frac{b}{a}$</th>
<th>$\frac{hD_h}{k}$ (Uniform $q''_w$)</th>
<th>$\frac{hD_h}{k}$ (Uniform $T_v$)</th>
<th>$Re_{D_h}$</th>
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