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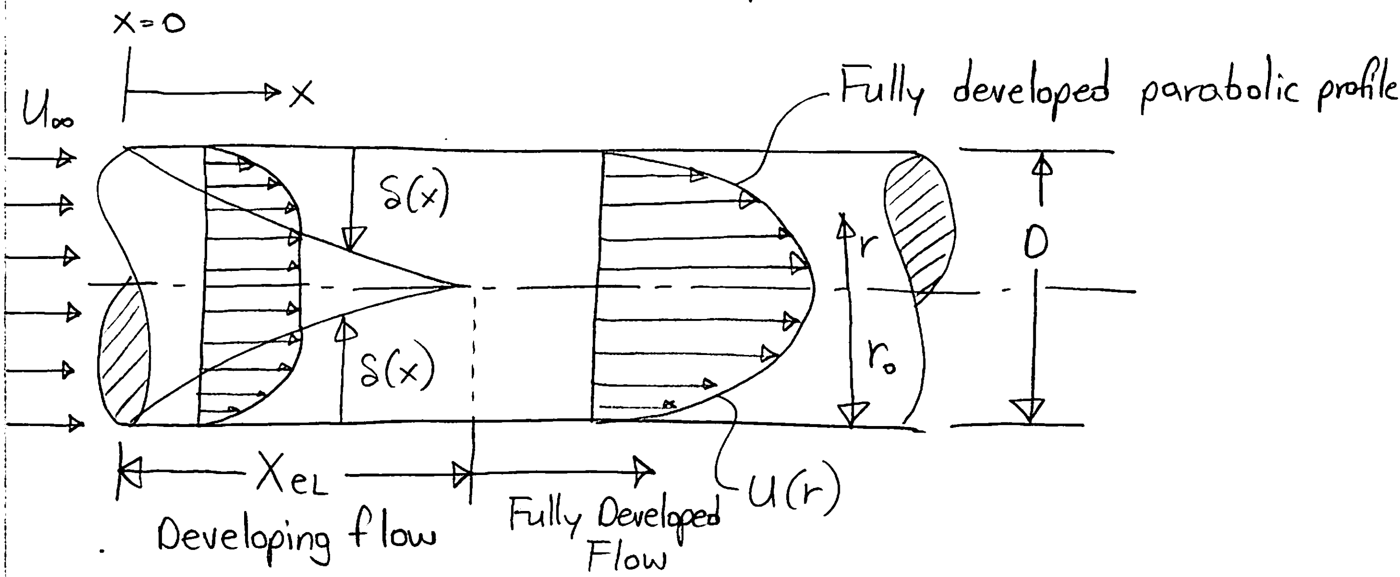
Internal Flow - Fully Developed Flow in Tubes



X_{eL} ≡ entrance length or developing length. Here, the velocity profile varies with radial position r, and axial location, x.

We can estimate the X_{eL} by using our Blasius solution to see when our boundary layers begin to overlap:

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$$

We can estimate that $\delta = \frac{D}{2}$ when b.l.'s merge

$$\frac{D}{2X_{eL}} \underset{\text{scales with}}{\sim} \frac{5.0}{\sqrt{Re_x}}$$

$$\frac{D}{X_{eL}} \sim \frac{10}{\sqrt{Re_x}} = \frac{10}{\sqrt{\frac{\rho U_\infty X_{eL}}{\mu}}} = \frac{10}{\sqrt{\frac{\rho U_\infty X_{eL}}{\mu} \cdot \frac{D}{D}}} = \frac{10}{\sqrt{\frac{\rho U_\infty D}{\mu}} \cdot \sqrt{\frac{X_{eL}}{D}}}$$

$$\sqrt{\frac{D}{X_{eL}}} \sim \frac{10}{\sqrt{Re_D}} \Rightarrow \boxed{\frac{X_{eL}}{D} \sim \frac{Re_D}{100} \sim 0.01 Re_D} \; ; \; Re_D = \frac{\rho U_\infty D}{\mu}$$

Experimentally: $\boxed{\frac{X_{eL}}{D} = 0.05 Re_D}$: If $x > X_{eL}$, flow is fully developed.

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Fully Developed Region ($x > x_{eL}$)

We need to solve for the hydrodynamic properties of the flow to help us understand the heat transfer:

Applying the axial momentum equation (x-momentum)

$$\rho\left(\frac{\partial u_x}{\partial t} + u_r\frac{\partial u_x}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_x}{\partial \phi} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial P}{\partial x}$$
$$+ \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_x}{\partial \phi^2} + \frac{\partial^2 u_x}{\partial x^2}\right] + \rho g_x = 0$$

For fully developed flow: $\frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial \phi} = 0$

$u_\phi = u_r = 0$

For flow in a horizontal tube: $g_x = 0$

Most of our terms drop out and we are left with:

$$-\frac{\partial P}{\partial x} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) = 0 \quad (1)$$

B.C.'s: $\left.\frac{\partial u_x}{\partial r}\right|_{r=0} = 0$ (Symmetry) (2)

$u_x(r=r_o) = 0$ (no slip) (3)

From here on in, I will use $u_x = u$ (dropping x subscript)

Integrating eq. (1) twice & applying (2) & (3):

$$\boxed{u(r) = \frac{r_o^2}{4\mu}\left(-\frac{\partial P}{\partial x}\right)\left(1-\frac{r^2}{r_o^2}\right)}$$

$\Rightarrow$ Fully developed velocity profile in a round pipe. (Laminar flow)

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Solving for our average velocity:

$$\bar{u} = \frac{1}{\pi r_o^2}\int_0^{r_o} u \cdot 2\pi r\,dr = \frac{r_o^2}{4\mu}\left(-\frac{\partial P}{\partial x}\right)\underbrace{\int_0^1 (1-\lambda)\,d\lambda}_{1/2} \; ; \; \lambda = \frac{r^2}{r_o^2}$$

$$\boxed{\bar{u} = \frac{r_o^2}{8\mu}\left(-\frac{\partial P}{\partial x}\right)}$$

$$\boxed{u(r) = 2\bar{u}\left(1-\frac{r^2}{r_o^2}\right)}$$

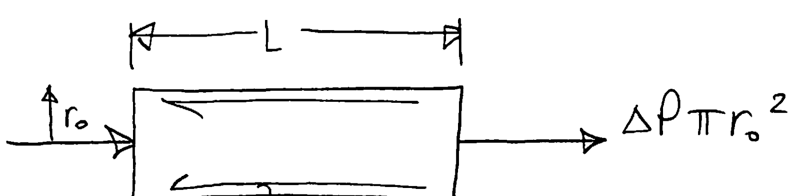
⇒ Fully developed laminar flow in a round pipe.

Typically, we need to know friction, pressure drop, or force.

$$f = \frac{\Delta P}{\left(\frac{L}{D}\right)\frac{1}{2}\rho\bar{u}^2}$$ ≡ Tube friction factor. Easy way to calculate pressure loss in tubes of length L, diameter D.

$$C_{f,x} = \frac{\tau_x}{\frac{1}{2}\rho\bar{u}^2}$$ ≡ Local tube friction coefficient. Related to the friction factor, f.

Drawing a force balance diagram on a segment of our fluid in the pipe. $\sum F_x = 0$ since $u(r) = f(x)$ ≡ fully developed



$$\tau_o(2\cancel{\pi}\cancel{r_o}L) = \Delta P\cancel{\pi}r_o^{\cancel{2}} = \frac{\Delta P D}{2}$$

$$4\tau_o = \frac{\Delta P}{\left(\frac{L}{D}\right)}$$

We can say: $\boxed{4C_f = f}$ ⇒ Let's now solve for friction factor

Laminar flow

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$$f = \frac{\Delta P}{\left(\frac{L}{D}\right)\frac{1}{2}\rho \bar{u}^2} \quad ①$$

But we've just solved for laminar flow that: $\bar{u} = \frac{r_o^2}{8\mu}\left(-\frac{\partial P}{\partial x}\right)$

$$\frac{8\mu \bar{u}}{r_o^2} = -\frac{\partial P}{\partial x} \approx \frac{\Delta P}{L} \quad ②$$

Substitute here

From ①: $\frac{\Delta P}{L} \cdot \frac{2D}{\rho \bar{u}^2} = f = \frac{8\mu \cancel{\bar{u}}}{r_o^2} \cdot \frac{2D}{\rho \bar{u}^{\cancel{2}}}$

$$\frac{16\mu D}{r_o^2 \rho \bar{u}} = f = \frac{16\mu \cancel{D}}{\left(\frac{\cancel{D}}{2}\right)\left(\frac{D}{2}\right)\rho \bar{u}} = \frac{64\mu}{\rho \bar{u} D} = \frac{64}{Re_D}$$

$\boxed{f = \frac{64}{Re_D}}$ ⇒ Pipe friction factor for laminar flow in a smooth round tube. Darcy-Weisbach equation.

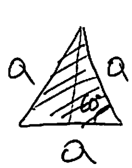
We can also say the following:

$\boxed{f \cdot Re_D = \text{constant}}$

For <u>non-circular</u> channels/pipes:

$\boxed{D_h \equiv \text{hydraulic diameter} = \frac{4A}{P}}$; $A \equiv$ cross sectional area
$P \equiv$ wetted perimeter

Example



$$A = \frac{a \cdot a \sin 60°}{2}, \quad P = 3a \quad \Bigg\} \quad D_h = \frac{4a^{\cancel{2}} \sin 60°}{3\cancel{a}} = 1.15a$$

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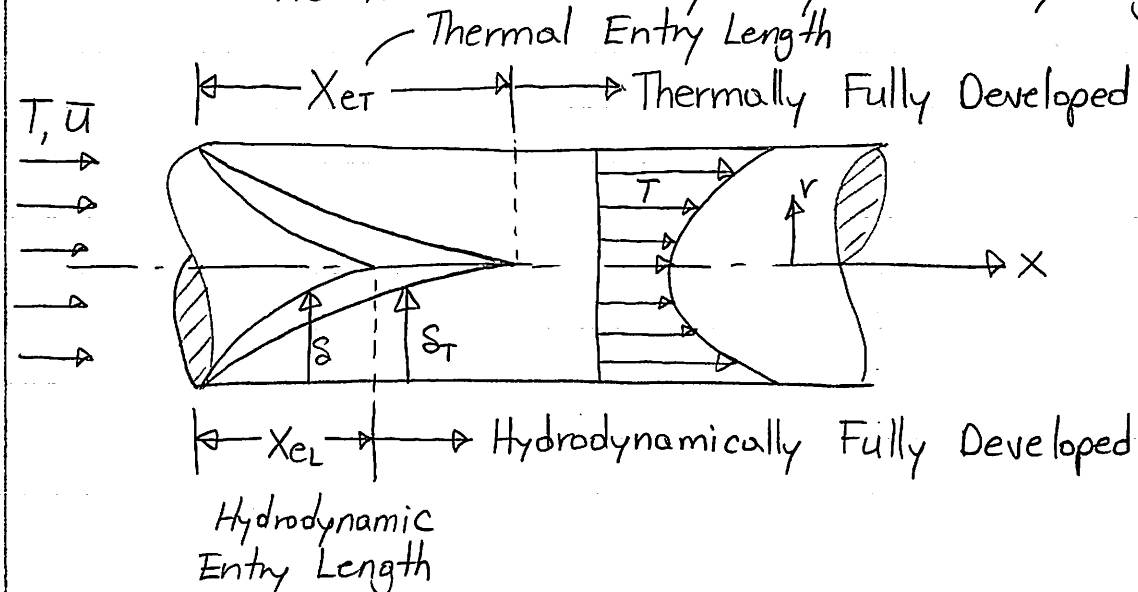
Heat Transfer in Internal Flows

How does the thermal boundary layer or entry length develop?



For fully developed flow, both the Thermal entry length (X_{eT}) and hydrodynamic entry length (X_{eL}) must be passed; $X > X_{eT} \,\&\, X_{eL}$

$$\boxed{\frac{X_{eT}}{D} = 0.017\, Re_D \cdot Pr}$$

$\Rightarrow$ Experimentally obtained Thermal Entry Length

So how about heat transfer in the fully developed region: i.e. $X > X_{eL}$ & $X > X_{eT}$

$$h = \frac{q''|_{wall}}{\Delta T}$$

From energy balance at the wall (conduction = convection)

$$h \cancel{\Delta T} = k_f \frac{\cancel{\Delta T}}{\delta_T} \Rightarrow h \sim \frac{k_f}{\delta_T}$$

For fully developed flow (overlapping thermal boundary layers):

$$\delta_T \sim \frac{D}{2}$$

$$h \sim \frac{2k_f}{D}$$

But we know that: $Nu_D = \frac{hD}{k_f} = \frac{2\cancel{k_f}}{\cancel{D}}\left(\frac{\cancel{D}}{\cancel{k_f}}\right)$

$\boxed{Nu_D \approx 2}$ ⇒ We expect a constant Nusselt #!

It turns out this scaling is pretty good. If we solve the coupled momentum and energy equations for internal smoot round tube pipe flow, we will obtain the following:

$$\boxed{\overline{Nu_D} = \frac{\bar{h}D}{k_f} = 4.364}$$

⇒ Constant wall heat flux
Laminar flow, round tube
$Re_D < 2300$; $Re_D = \frac{\rho \bar{u} D}{\mu}$

$$\boxed{\overline{Nu_D} = \frac{\bar{h}D}{k_f} = 3.66}$$

⇒ Constant wall temperature
Laminar flow, round tube
$Re_D < 2300$

Bulk Fluid Temperature

It is important to note that for internal flows, heat transfer is computed by using the bulk fluid temperature: (T_b)

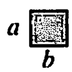
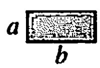
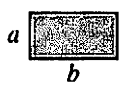
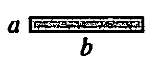
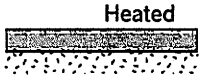
$$\bar{h} = \frac{q''|_{wall}}{T_w - T_b}$$; T_w = wall temperature
T_b = bulk fluid temperature

T_b can be thought of as if we allow the pipe fluid to uniformly mix & come to an equilibrium temperature:

$$\boxed{T_b = \frac{1}{A\bar{u}}\int_A u(r) \cdot T\, dA}$$

where A ≡ cross sectional area
$\bar{u}$ ≡ average fluid velocity

Table 8.1 Nusselt numbers and friction factors for fully developed laminar flow in tubes of differing cross section

Cross Section	$\frac{b}{a}$	$Nu_D \equiv \frac{hD_h}{k}$ (Uniform q_s'')	$Nu_D \equiv \frac{hD_h}{k}$ (Uniform T_s)	$f \, Re_{D_h}$
	—	4.36	3.66	64
	1.0	3.61	2.98	57
	1.43	3.73	3.08	59
	2.0	4.12	3.39	62
	3.0	4.79	3.96	69
	4.0	5.33	4.44	73
	8.0	6.49	5.60	82
	∞	8.23	7.54	96
Heated  Insulated	∞	5.39	4.86	96
	—	3.11	2.49	53

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