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Writing out the first law of thermodynamics:

$$\underbrace{Q_{in} - Q_{OUT}}_{\text{(Energy conducted into the } C\forall)} + \underbrace{Q_{gen}}_{\text{(Energy Generated within the } C\forall)} = \underbrace{Q_{ST}}_{\text{(The increase in energy in the } C\forall \text{ per unit time)}}$$

(Energy conducted into the $C\forall$) + (Energy Generated within the $C\forall$) = (The increase in energy in the $C\forall$ per unit time)

Entering & leaving the surfaces we have:

$$Q_{in} = q_x A_x + q_y A_y + q_z A_z$$

$$Q_{OUT} = q_{x+\Delta x} A_{x+\Delta x} + q_{y+\Delta y} A_{y+\Delta y} + q_{z+\Delta z} A_{z+\Delta z}$$

For our cartesian coordinate system:

$$A_x = A_{x+\Delta x} = \Delta y \Delta z$$
$$A_y = A_{y+\Delta y} = \Delta x \Delta z$$
$$A_z = A_{z+\Delta z} = \Delta x \Delta y$$

Within our $C\forall$, we have energy generation, Q'''

$$\boxed{Q_{gen} = Q''' \forall = Q''' \Delta x \Delta y \Delta z}$$

Within the control volume, the change in stored energy is:

$$Q_{ST} = \frac{dU}{dt} = \frac{d(Mu)}{dt} = \frac{d(\rho \forall u)}{dt}$$

U = internal energy
u = specific internal energy
M = total mass of $C\forall$
ρ = density

Assuming $\rho \equiv$ constant

$$Q_{ST} = \rho \frac{du}{dt} \Delta x \Delta y \Delta z$$

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But we know from thermodynamics that:

$$U = Mc_p(T - T_{ref}) \Rightarrow \text{Equation of state for a simple incompressible substance with constant specific heat.}$$

or

$$u = c_p(T - T_{ref})$$

Back substituting:

$$Q_{ST} = \rho \frac{d(c_p(T-T_{ref}))}{dt} \Delta x \Delta y \Delta z$$

T_{ref} = constant & c_p = constant, hence

$$\boxed{Q_{ST} = \rho c_p \frac{dT}{dt} \Delta x \Delta y \Delta z}$$

Putting everything together, we obtain:

$$Q_{in} - Q_{OUT} = \Delta y \Delta z\left(q_x - q_{x+\Delta x}\right) + \Delta x \Delta z\left(q_y - q_{y+\Delta y}\right) + \Delta x \Delta y\left(q_z - q_{z+\Delta z}\right)$$

Using Tailor series expansion:

$$q_{x+\Delta x} = q_x + \frac{dq_x}{dx}\Delta x + \frac{1}{2}\frac{d^2 q_x}{dx^2}\Delta x^2 + \dots + \frac{1}{n!}\frac{d^n q_x}{dx^n}\Delta x^n$$

So

$$\left(q_x - q_{x+\Delta x}\right) = -\frac{dq_x}{dx}\Delta x - \frac{1}{2}\frac{d^2 q_x}{dx^2}\Delta x^2 - \dots - \frac{1}{n!}\frac{d^n q_x}{dx^n}\Delta x^n$$

$$\left(q_y - q_{y+\Delta y}\right) = -\frac{dq_y}{dy}\Delta y - \frac{1}{2}\frac{d^2 q_y}{dy^2}\Delta y^2 - \dots - \frac{1}{n!}\frac{d^n q_y}{dy^n}\Delta y^n$$

Same for $\left(q_z - q_{z+\Delta z}\right)$.

Now let's put everything together

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$$\boxed{\begin{aligned} Q_{in} - Q_{OUT} = &-\Delta y \Delta z\left(\frac{dq_x}{dx}\Delta x + \frac{d^2 q_x}{2\,dx^2}\Delta x^2 + \dots + \frac{1}{n!}\frac{d^n q_x}{dx^n}\Delta x^n\right) \\ &-\Delta x \Delta z\left(\frac{dq_y}{dy}\Delta y + \frac{d^2 q_y}{2dy^2}\Delta y^2 + \dots + \frac{1}{n!}\frac{d^n q_y}{dy^n}\Delta y^n\right) \\ &-\Delta x \Delta y\left(\frac{dq_z}{dz}\Delta z + \frac{d^2 q_z}{2dz^2}\Delta z^2 + \dots + \frac{1}{n!}\frac{d^2 q_z}{dz^n}\Delta z^n\right) \end{aligned}}$$

↳ Don't be scared, this will simplify a lot!

$$Q_{in} - Q_{OUT} + Q_{gen} = Q_{ST}$$

$$-\Delta x \Delta y \Delta z\left[\left(\frac{dq_x}{dx} + \frac{1}{2}\frac{d^2 q_x}{dx^2}\Delta x + \dots\right) + \left(\frac{dq_y}{dy} + \frac{1}{2}\frac{d^2 q_y}{dy}\Delta y + \dots\right) + \left(\frac{dq_z}{dz} + \frac{1}{2}\frac{d^2 q_z}{dz}\Delta z + \dots\right)\right] + Q'''\Delta x \Delta y \Delta z = \rho C_p \frac{dT}{dt}\Delta x \Delta y \Delta z$$

Note, we can divide both sides by $\Delta x \Delta y \Delta z$ to simplify

$$\left(\frac{dq_x}{dx} + \frac{1}{2}\frac{d^2 q_x}{dx^2}\Delta x + \dots\right) + \left(\frac{dq_y}{dy} + \frac{1}{2}\frac{d^2 q_y}{dy^2}\Delta y + \dots\right) + \left(\frac{dq_z}{dz} + \frac{1}{2}\frac{d^2 q_z}{dz^2}\Delta z\right) + Q''' = \rho C_p \frac{dT}{dt}$$

Taking the limit of $\Delta x, \Delta y, \Delta z \longrightarrow 0$ (differential volume) many of our terms drop out, & we are left with only:

$$-\left(\frac{dq_x}{dx} + \frac{dq_y}{dy} + \frac{dq_z}{dz}\right) + Q''' = \rho C_p \frac{dT}{dt}$$

But we have already shown that:

$$q_x = -k\frac{dT}{dx} \quad ; \quad q_y = -k\frac{dT}{dy} \quad ; \quad q_z = -k\frac{dT}{dz}$$

Back substituting, we obtain:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\frac{d}{dy}\left(k\frac{dT}{dy}\right)+\frac{d}{dz}\left(k\frac{dT}{dz}\right)+Q''' = \rho c_p \frac{dT}{dt}$$

Note though, more rigorously written, we should have $d = \partial$ since $T = f(x, y, z, t)$

$$\frac{dT}{dx} \Rightarrow \frac{\partial T}{\partial x} \text{ (partial derivative)}$$

↳ Would be valid if $T = f(x)$ only.

Rewriting, we obtain:

$$\boxed{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+Q''' = \rho c_p \frac{\partial T}{\partial t}}$$

↳ Second order partial differential equation (PDE). Represents the conservation of thermal energy for an isotropic, incompressible substance, with density & specific heat independent of time.

A well posed problem requires two boundary conditions in each coordinate & an initial condition.

If thermal conductivity is isotropic (doesn't depend on location or direction):

$$\boxed{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{Q'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

where $\boxed{\alpha = \frac{k}{\rho c_p}}$ ⇒ Thermal diffusivity material property.

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For constant properties with 1D conduction:

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{Q'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

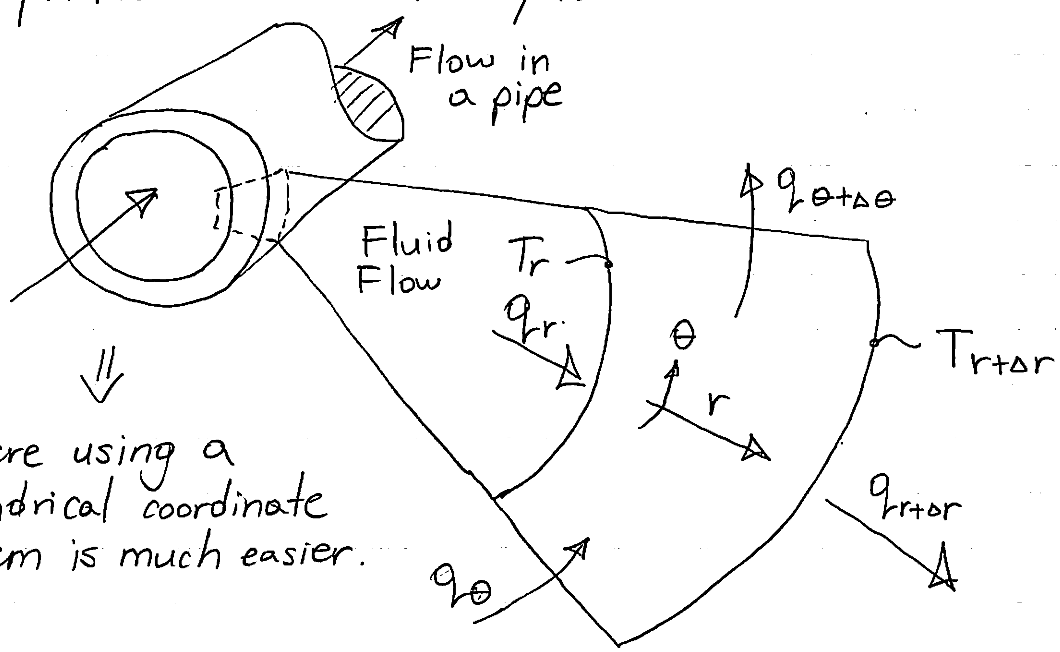
For no heat generation, 1-D conduction:

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

For steady state, 1D, & no heat generation:

$$\boxed{\frac{\partial^2 T}{\partial x^2} = 0}$$ $\Rightarrow$ This is probably the most useful form to solve problems.

So far, we've only used cartesian coordinates, but many heat transfer problems are easier to solve with cylindrical or spherical coordinate systems:



$\Downarrow$

Here using a cylindrical coordinate system is much easier.