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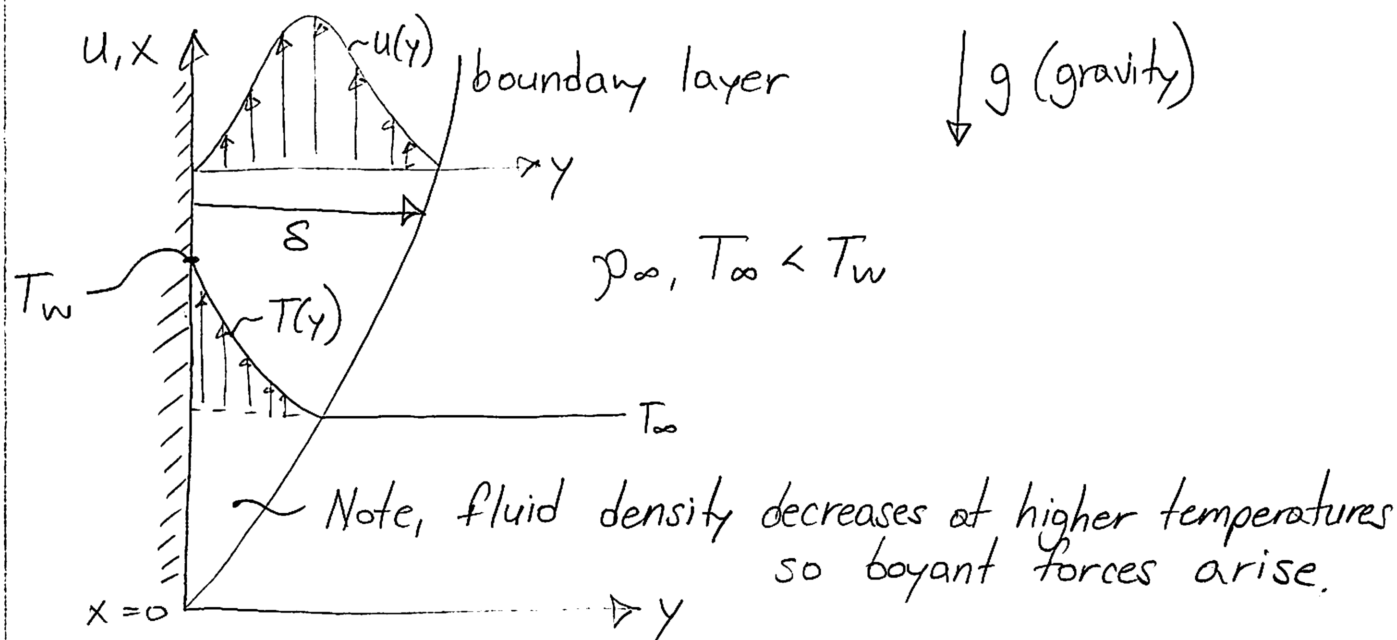
Natural Convection

Unlike forced convection, in which the fluid motion driving force is external to the fluid, natural convection is driven by body forces acting directly on the fluid as a result of heating or cooling

Simplest Case: Hot vertical plane wall

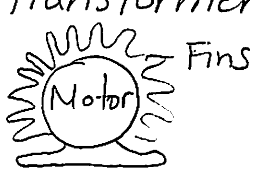


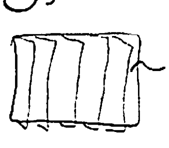
The flow is analogous to flow over a flat plate, but $u = 0$ at $y = 0$ and at $y \to \infty$.

Natural convection is all around us:

1) When humans are stationary, natural convection occurs & we lose heat to the air surrounding us.

2) Many heat sinks are designed for natural convection. i.e transformer cooling, home appliances, induction motors





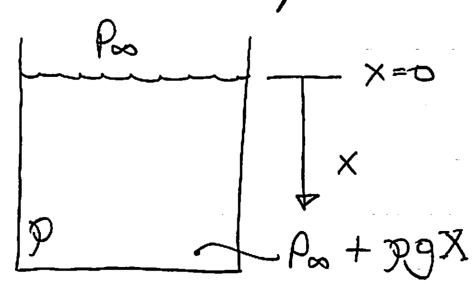
To solve, we need to use the momentum & energy equations but now taking gravity into account.

X-momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \underbrace{-\,g}_{\text{Body force (gravity in -x)}} \quad (1)$$

To help us solve, we know for a stationary fluid that:

$$\rho(x) = P_\infty + \rho g x$$
(Archemede's principle)



So we can say:

$$\frac{\partial P}{\partial x} = -\rho_\infty g$$

density of fluid far away from wall

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{\rho_\infty g}{\rho} \quad (2)$$

density of fluid close to the wall

Back substitute (2) into (1)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\left(\frac{\rho_\infty - \rho}{\rho}\right) + \nu\frac{\partial^2 u}{\partial y^2} \quad (3)$$

However we need the relation between density ρ and temperature, T. We can use the following:

$$\beta = \frac{1}{v}\frac{\partial v}{\partial T}\bigg|_p \; ;$$

$v \equiv$ fluid specific volume
$\beta \equiv$ coefficient of volumetric thermal expansion

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\bigg|_p \quad (4)$$

or $\Rightarrow \dfrac{\rho_\infty - \rho}{\rho} = \beta(T - T_\infty)$

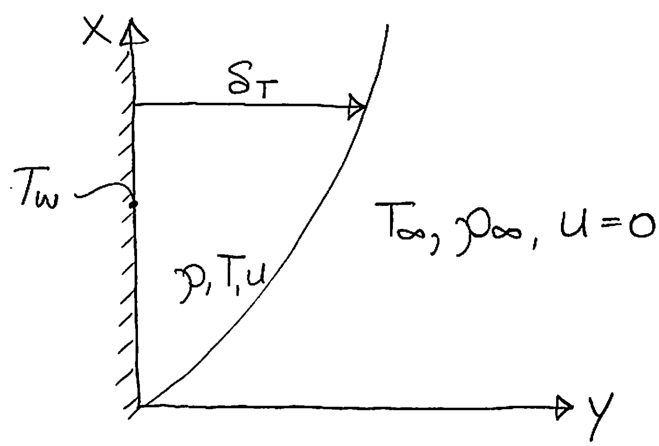
Nenad Miljkovic

So now our momentum equation becomes:

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{Inertia Forces}} = \underbrace{g\beta\left(T-T_\infty\right)}_{\text{Boyancy Forces}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\text{Viscous Forces}}$$

Let's try to solve this using scaling analysis:

Looking at the problem as a transient conduction prob.



$t \equiv$ time

$$\delta_T \sim \underbrace{\sqrt{\alpha t}}_{\text{Speed of heat conduction (page 65 of the notes)}} \; ; \quad \alpha \equiv \text{fluid thermal diffusivity of the fluid}$$

So what is t here? Well, we know $t = \frac{x}{u}$ ← location on plate / ← flow speed

$$\delta_T \sim \sqrt{\frac{\alpha x}{u}} \quad \text{so} \quad u \sim \frac{\alpha x}{\delta_T^2} \quad \text{and} \quad y \sim \delta_T$$

So let's break down the scaling of the forces

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Inertia, I	Viscosity, V	Boyancy, B
$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\sim\frac{u^2}{x}$ $\sim\frac{\alpha^2 x}{\delta_T^4}$	$\nu\frac{\partial^2 u}{\partial y^2}\sim\nu\frac{u}{y^2}$ $\sim\frac{\nu\alpha x}{\delta_T^4}$	$g\beta(T-T_\infty)$ $=g\beta\Delta T$

Now if we take ratios of our forces to see which ones dominate: (focus on boyancy)

I/B	V/B	B/B
$\frac{\alpha^2 x}{\delta_T^4 g\beta\Delta T}$	$\frac{\nu\alpha x}{\delta_T^4 g\beta\Delta T}$	1

Assuming that $V/B\sim O(1)$ (Inside the natural convection boundary layer). Think of the boundary layer definition.

$$\frac{\nu\alpha x}{\delta_T^4 g\beta\Delta T}\sim 1$$

Multiplying both sides by $\frac{x^3}{x^3}$

$$\frac{\nu\alpha}{g\beta\Delta T x^3}\left(\frac{x}{\delta_T}\right)^4\sim 1$$

$$\boxed{\frac{x}{\delta_T}\sim\left(\frac{g\beta\Delta T x^3}{\alpha\nu}\right)^{1/4}}\Rightarrow(\text{Rayleigh number})^{1/4}$$

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$$\boxed{Ra_x = \frac{g\beta \Delta T x^3}{\alpha \nu}} \; ; \; \boxed{\frac{x}{\delta_T} \sim Ra^{1/4}}$$

$$\boxed{Gr_x = \frac{g\beta\Delta T x^3}{\nu^2}} \Rightarrow \text{Grashof number}$$

$$Ra \equiv \underbrace{\frac{\text{Boyancy Force}}{\text{Viscous Force}}}_{Gr} \cdot \underbrace{\frac{\text{Momentum Diffusivity}}{\text{Thermal Diffusivity}}}_{Pr \; \left(\text{Prandtl \#} = \frac{\nu}{\alpha}\right)} = Gr \cdot Pr$$

If: $Ra < Ra_{,crit}$, conduction dominates ($Gr \ll 1, Pr \ll 1$)
$Ra > Ra_{,crit}$, convection dominates ($Gr > 1, Pr > 1$)

Think of Rayleigh # & Grashof # as an analogy to Reynolds #, but here instead of inertia, we have boyancy.

<u>Heat Transfer</u> (Natural Convection)

$$q''\Big|_{y=0} = -k_f \frac{\partial T}{\partial y}\Big|_{y=0} \sim k\frac{\Delta T}{\delta_T}$$

$$\left(\frac{q''|_{y=0}}{\Delta T}\right) \cdot \frac{x}{k_f} = \frac{h_x x}{k_f} = Nu_x \sim \frac{x}{\delta_T} \sim Ra^{1/4}$$

$$\boxed{Nu_x \sim Ra_x^{1/4}} \Rightarrow \text{Scaling tells us this is what we expect.}$$

Full analytical result:

$$\boxed{Nu_x = 0.394\, Ra_x^{1/4}} \Rightarrow T_w = \text{constant}$$

$$\boxed{\overline{Nu_L} = \frac{\bar{h}L}{k_f} = 0.525\, Ra_L^{1/4}} \Rightarrow \boxed{\overline{Nu_{L,exp}} = 0.52\, Ra_L^{1/4}}$$

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For cases where $T_w \neq$ constant & $q'' =$ constant

We can define a modified Rayleigh number, Ra_x^*

$$\boxed{Ra_x^* = \frac{g\beta(q''/k_f)x^4}{\nu\alpha}} \Rightarrow \text{Constant heat flux Rayleigh \#}$$

$$\left(\frac{q''|_{y=0}}{\Delta T}\right) \cdot \frac{x}{k_f} = \frac{h_x x}{k_f} = Nu_x$$

$$\boxed{Nu_x = 0.503\, Ra_x^{*\,1/5}}$$

$\Rightarrow$ Constant heat flux natural convection, vertical wall Analytical result.

<u>In general:</u> $Ra_L > 10^9 \Rightarrow$ Turbulent free convection

$$\boxed{\overline{Nu_L} = \frac{\bar{h}L}{k_f} = 0.1\, Ra_L^{1/3}} \;;\; 10^9 < Ra_L < 10^{13}$$

For laminar flow:

$$\boxed{\overline{Nu_L} = \frac{\bar{h}L}{k_f} = 0.59\, Ra_L^{1/4}} \;;\; 10^4 \leq Ra_L < 10^9$$

(Similar to what we derived)

Note, both correlations above are for constant vertical wall temperature (T_w = constant).

For other cases; i) Inclined walls
ii) Channels
iii) Enclosures
iv) Cylinders & Spheres

} See Chapter 9 of the textbook Incropera.

* All properties evaluated at $\bar{T} = \frac{T_w + T_\infty}{2}$; $\beta = \frac{1}{T}$ for ideal gas.

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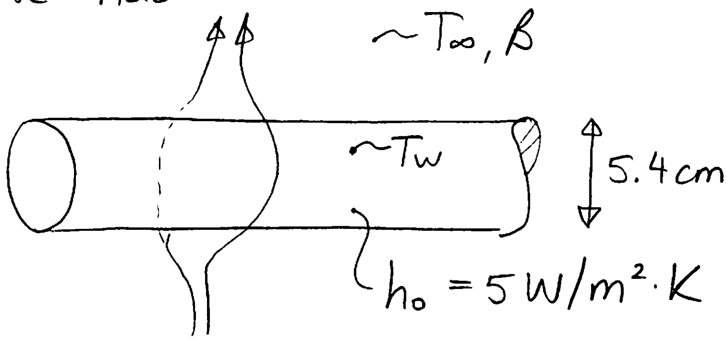
Example | Let's check the validity of our previous example with New England pipes.

We had:



$h_o = 5\ W/m^2 \cdot K$ (Let's check this)

First calculate the Rayleigh #:

$$Ra_D = \frac{g\beta \Delta T D^3}{\alpha \nu} \leftarrow \text{note, we use } D \text{ here instead of } L.$$

For air at $T = \dfrac{-10°C + 1°C}{2} = 4.5°C$ (Table: A.4)

$$\rho_{air} = 1.2\ kg/m^3$$
$$\alpha_{air} = 20 \times 10^{-6}\ m^2/s$$
$$\nu_{air} = 14 \times 10^{-6}\ m^2/s$$
$$k_{air} = 24 \times 10^{-3}\ W/m \cdot K$$

$$\beta = \frac{1}{T_\infty} = \frac{1}{263.15K} = 0.0038\ K^{-1}$$

$$Ra_D = \frac{(9.81\ N/kg)(0.0038\ K^{-1})(11°C)(0.054m)^3}{(20 \times 10^{-6} m^2/s)(14 \times 10^{-6} m^2/s)} = 2.3 \times 10^5$$

Looking at Eq. 9.33 of Textbook & corresponding Table 9.1

$$\overline{Nu_D} = \frac{\bar{h} D}{k_f} = 0.48\, Ra_D^{1/4} = 0.48 (2.3 \times 10^5)^{1/4} = 10.51$$

$$\bar{h} = \frac{\overline{Nu_D} \cdot k_f}{D} = \frac{(10.51)(0.024\ W/m \cdot K)}{0.054m} = 4.67\ W/m^2 K$$