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If we look back at our derivation of the heat equation, we can make things more general by:

$d\forall = ds_1\, ds_2\, ds_3$ where s_1, s_2, s_3 are the coordinates in consideration

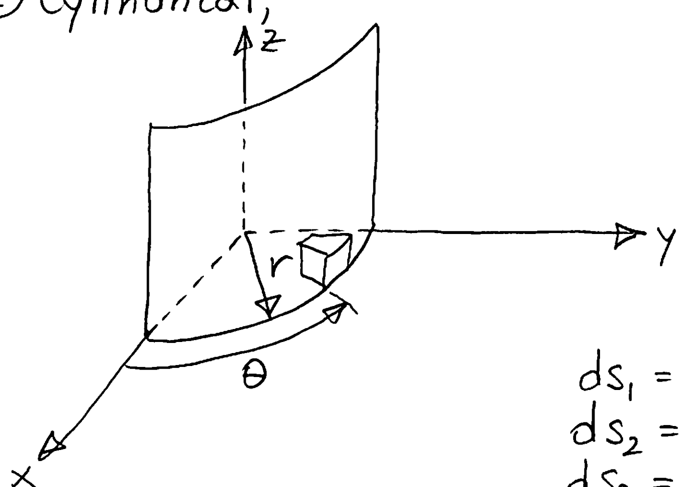
$$\boxed{\frac{1}{ds_1\, ds_2\, ds_3}\left\{\frac{\partial}{\partial s_1}\left(k\, \partial s_2\, \partial s_3 \frac{\partial T}{\partial s_1}\right)\partial s_1 + \frac{\partial}{\partial s_2}\left(k\, \partial s_1\, \partial s_3 \frac{\partial T}{\partial s_2}\right)\partial s_2 + \frac{\partial}{\partial s_3}\left(k\, \partial s_1\, \partial s_2 \frac{\partial T}{\partial s_3}\right)\partial s_3\right\} + Q''' = \frac{\partial}{\partial t}(\rho c_p T)}$$

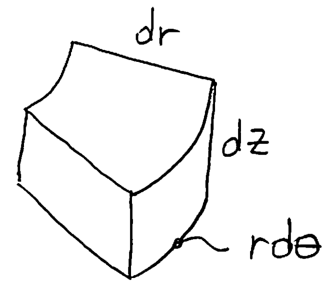
Specific Cases:

1. Cartesian,

$$\left.\begin{aligned} ds_1 &= dx \\ ds_2 &= dy \\ ds_3 &= dz \end{aligned}\right\}$$ We'll get what we already solved

2. Cylindrical,





$$\begin{aligned} ds_1 &= dr \\ ds_2 &= dz \\ ds_3 &= r\,d\theta \end{aligned}$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(k r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + Q''' = \rho c_p \frac{\partial T}{\partial t}}$$

↳ Heat equation in cylindrical coordinates.

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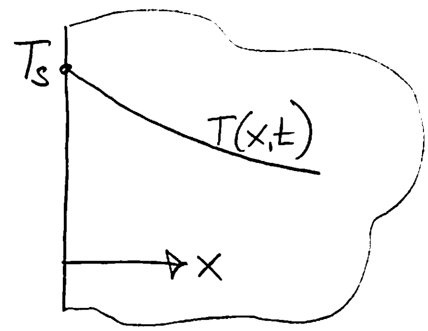
Boundary Conditions

① Dirichlet (1'st kind)

$T =$ specified on the boundary

or

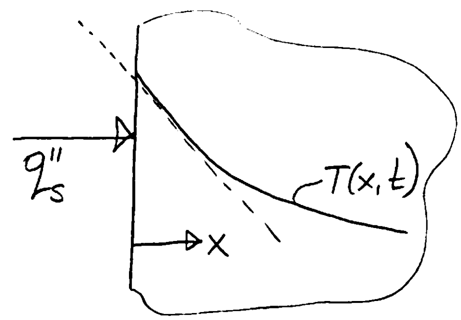
$$T(0,t) = T_s$$



② Neumann (2'nd kind) or Constant surface heat flux

(a) Finite Heat Flux

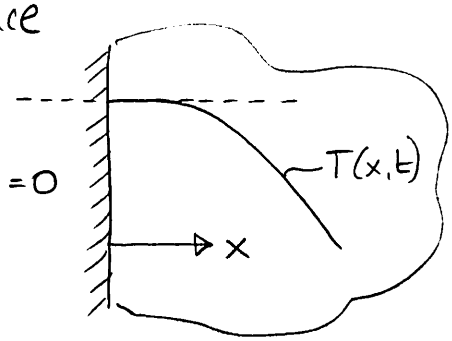
$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q''_s$$



or

(b) Adiabatic or Insulated Surface

$$\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$



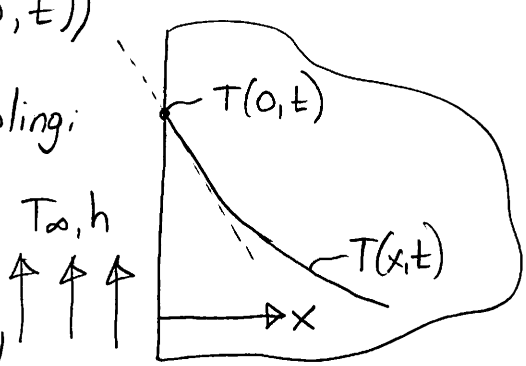
③ Robin (3'rd kind) or Convection surface condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = h\left(T_\infty - T(0,t)\right)$$

Stems from Newton's Law of cooling:

$$q'' = h(T_s - T_\infty)$$

↳ Says heat flux is proportional to some constant h, and temperature difference ΔT.



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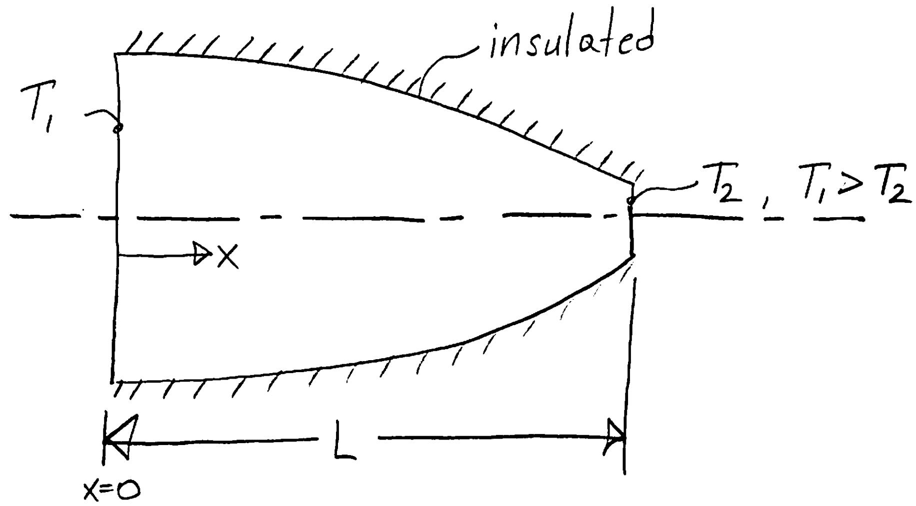
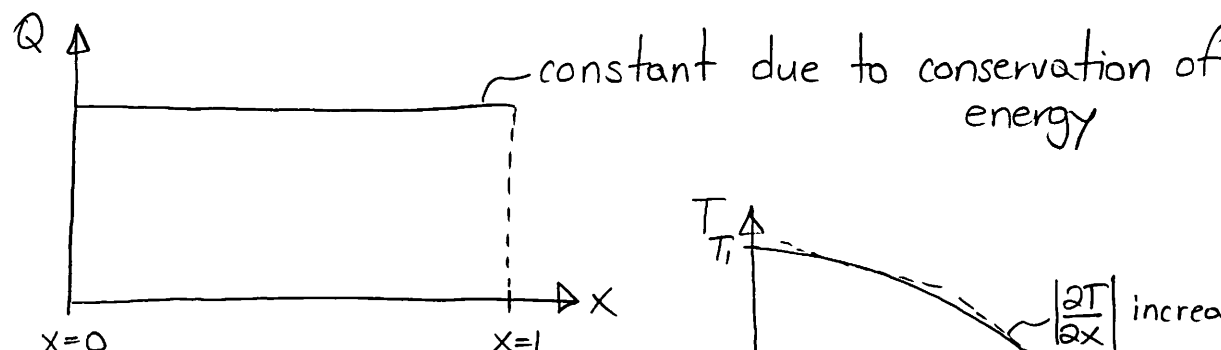
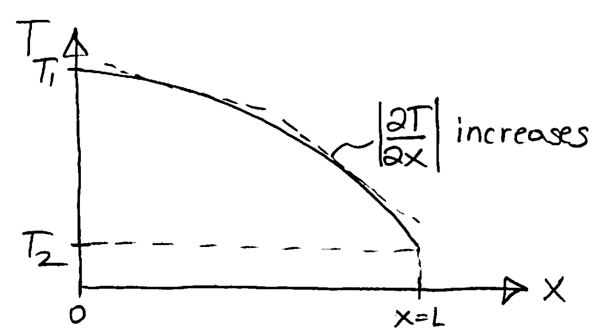
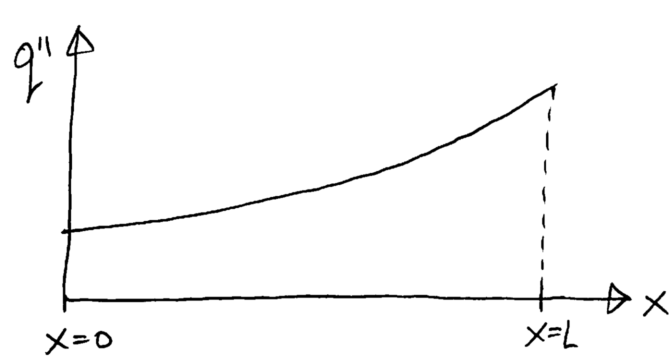
Example Assume 1-D, steady-state, heat conduction through the axisymmetric shape below. Assume no heat generation & that it is well insulated on the outsides. Assume constant properties.



Sketch the temperature distribution, heat flux distribution, and heat transfer distribution as a function of x. Sketch only, no need for detailed calculations.







$$\Rightarrow Q = -k A_x \frac{\partial T}{\partial x} = \text{constant}$$

A_x is decreasing so $\frac{\partial T}{\partial x} \uparrow$

$$q'' = \frac{Q}{A_x} \equiv \text{increasing w/ } x$$

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Steady-State 1D Conduction (Chapter 3 of textbook)

So far we've developed the governing equation of heat conduction in a stationary medium, and the associated boundary conditions. Let's apply them to some problems.

Assumptions:

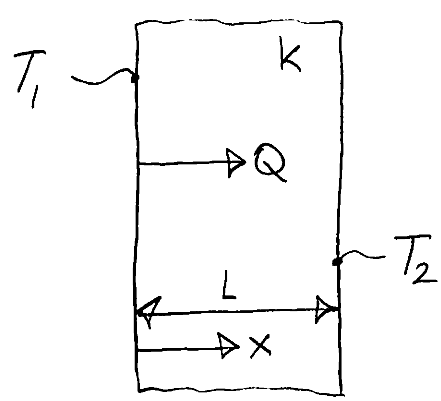
1) $\vec{V} = 0 \Rightarrow$ Medium is stationary
2) Steady state $\Rightarrow$ not a function of time
3) $\dot{Q}''' = 0 \Rightarrow$ no heat generation
4) k = constant $\Rightarrow$ isotropic medium

Let's write down our heat equation:

$$\frac{\partial^2 T}{\partial x^2} + \underbrace{\cancelto{0}{\frac{\partial^2 T}{\partial y^2}} + \cancelto{0}{\frac{\partial^2 T}{\partial z^2}}}_{\text{1-D conduction}} + \underbrace{\cancelto{0}{\frac{\dot{Q}'''}{k}}}_{\dot{Q}'''=0} = \underbrace{\cancelto{0}{\frac{1}{\alpha}\frac{\partial T}{\partial t}}}_{\text{Transient term}}$$

So we are left with: $\dfrac{\partial^2 T}{\partial x^2} = 0 \Rightarrow$ We can solve this easily

① Slab



Note; $Q \equiv$ heat transfer rate [W]

We already know our heat equation governs the heat transfer in the slab, so let's solve it.

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad \text{(integrate once)}$$

$$\int \frac{\partial^2 T}{\partial x^2}\, dx = \int 0\, dx$$

$$\frac{\partial T}{\partial x} = C_1 \quad \text{(Integrate once more)}$$

$$\int \frac{\partial T}{\partial x}\,\partial x = \int C_1\,dx$$

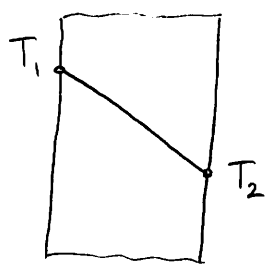
$T = C_1 x + C_2$ $\Rightarrow$ Now we apply our boundary conditions:

$$T\Big|_{x=0} = T_1 = C_1(0) + C_2 \Rightarrow \boxed{C_2 = T_1}$$

$$T\Big|_{x=L} = T_2 = C_1 L + C_2 = C_1 L + T_1 \Rightarrow \boxed{C_1 = \frac{T_2 - T_1}{L}}$$

$$\boxed{T = \frac{T_2 - T_1}{L}x + T_1}$$

↳ Temperature profile in the solid: linear and decreasing from hot to cold.

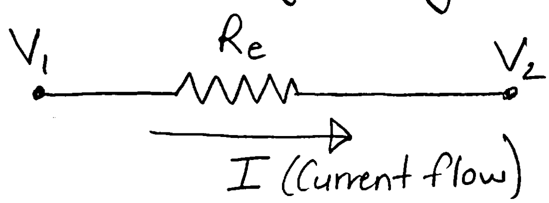


Heat transfer: Fourier's Law

$$Q = -kA\frac{\partial T}{\partial x} = -kA\left(\frac{T_2 - T_1}{L}\right)$$

Heat flux: $q'' = -k\left(\frac{T_2 - T_1}{L}\right)$

Note here we can see an interesting analogy. In electrical engineering:

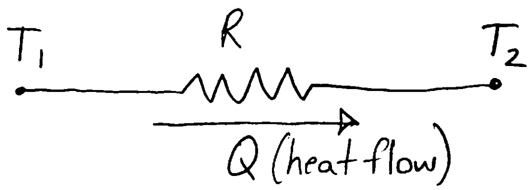


Voltage differen.

$$I = \frac{V_1 - V_2}{R_e} = \frac{\Delta V}{R_e}$$

Electrical resistance

Can we do the same for heat transfer:



$$Q = \frac{T_1 - T_2}{R} = \frac{\Delta T}{R}$$