Nenad Miljkovic

Let's see what our thermal resistance would be:

$$Q = -kA\left(\frac{T_2 - T_1}{L}\right) = kA\left(\frac{T_1 - T_2}{L}\right) \Rightarrow \Delta T = T_1 - T_2$$

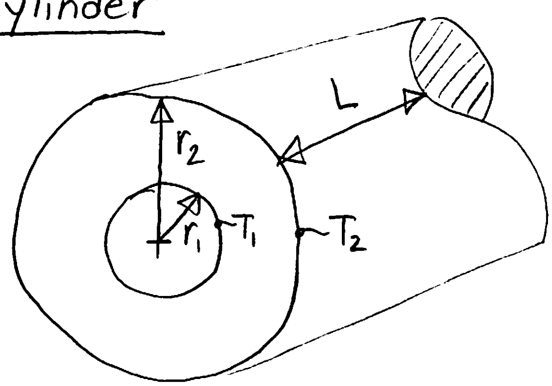
$$Q = kA\frac{\Delta T}{L}$$

But we know we want the form: $Q = \frac{\Delta T}{R}$

$$Q = kA\frac{\cancel{\Delta T}}{L} = \frac{\cancel{\Delta T}}{R}$$

$$\boxed{R = \frac{L}{kA}} \Rightarrow \text{Thermal resistance of a 1D slab}$$

2 Cylinder



Writing out our heat equation in radial coordinates: (page. 14)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\cancel{\frac{\partial T}{\partial \theta}}\right)^{0} + \frac{\partial}{\partial z}\left(k\cancel{\frac{\partial T}{\partial z}}\right)^{0} + \cancel{\dot{Q}'''}^{0} = \rho c_p \cancel{\frac{\partial T}{\partial t}}^{0}$$

$T = f(r)$ only $\quad T = f(r)$ only $\quad \dot{Q}''' = 0 \quad$ Steady-state

$$\cancel{\frac{1}{r}}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0 \quad \text{(integrate once)}$$

$$\int \frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)\partial r = \int 0\,\partial r$$

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$$r\frac{\partial T}{\partial r} = C_1 \quad \text{(Integrate once more)}$$

$$\int \frac{\partial T}{\partial r}\partial r = \int \frac{C_1}{r}\partial r \qquad \text{* Remember } \int \frac{dr}{r} = \ln(r)$$

$$T = C_1 \ln r + C_2$$ $\Rightarrow$ Now we apply our boundary conditions

$$T\Big|_{r=r_1} = T_1 = C_1 \ln r_1 + C_2 \quad (1)$$

$$T\Big|_{r=r_2} = T_2 = C_1 \ln r_2 + C_2 \quad (2)$$

Subtract (2) from (1)

$$T_1 - T_2 = C_1 \ln r_1 - C_1 \ln r_2 + \cancel{C_2} - \cancel{C_2}$$

$$T_1 - T_2 = C_1(\ln r_1 - \ln r_2)$$ $\Rightarrow$ Remember ln rules;

(1) $\ln(a) - \ln(b) = \ln(a/b)$

(2) $\ln(a) + \ln(b) = \ln(ab)$

(3) $\ln(a^x) = x\ln(a)$

$$C_1 = \frac{T_1 - T_2}{\ln(r_1/r_2)} \quad (3)$$

Back substitute (3) into (1) & solve for C_2

$$C_2 = T_1 - \frac{(T_1 - T_2)}{\ln(r_1/r_2)}\ln(r_1) \quad (4)$$

Back substitute (3) & (4) into our solution for T above

$$T = \frac{T_1 - T_2}{\ln(r_1/r_2)}\ln r + T_1 - \frac{(T_1 - T_2)\ln r_1}{\ln(r_1/r_2)}$$

$$T - T_1 = (T_1 - T_2)\left[\frac{\ln r}{\ln(r_1/r_2)} - \frac{\ln(r_1)}{\ln(r_1/r_2)}\right]$$ $\Rightarrow$ Remember the ln rules

$$\frac{T - T_1}{T_1 - T_2} = \frac{\ln(r/r_1)}{\ln(r_1/r_2)} \quad \text{or} \quad \boxed{\frac{T - T_1}{T_2 - T_1} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}}$$ $\Rightarrow$ Radial tempera. profile.

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Can we do the thermal resistance concept here too?

$$Q = \frac{\Delta T}{R} = -kA \left.\frac{\partial T}{\partial r}\right|_r = -kA \frac{\partial}{\partial r}\left[\frac{T_1 - T_2}{\ln(r_1/r_2)} \ln r + T_1 - \frac{(T_1 - T_2)\ln r_1}{\ln(r_1/r_2)}\right]$$

$\frac{\partial}{\partial r}(T_1) = 0$

$\frac{\partial}{\partial r}(\dots) = 0$

We know $\frac{\partial}{\partial r}[\ln r] = \frac{1}{r}$

$$Q = -kA \frac{\partial}{\partial r}\left[\frac{T_1 - T_2}{\ln(r_1/r_2)} \ln r\right] = -kA \frac{(T_1 - T_2)}{\ln(r_1/r_2)} \cdot \frac{1}{r}$$

But $\ln(r_1/r_2) = -\ln(r_2/r_1)$ ⇒ Back substitute into above

$$Q = kA \frac{T_1 - T_2}{\ln(r_2/r_1)} \cdot \frac{1}{r} \Rightarrow A = 2\pi r L$$

← Length of tube (axial)

$$Q = 2\pi \cancel{r} L k \frac{T_1 - T_2}{\ln(r_2/r_1)} \cdot \frac{1}{\cancel{r}} = \boxed{2\pi L k \frac{T_1 - T_2}{\ln(r_2/r_1)} = Q}$$

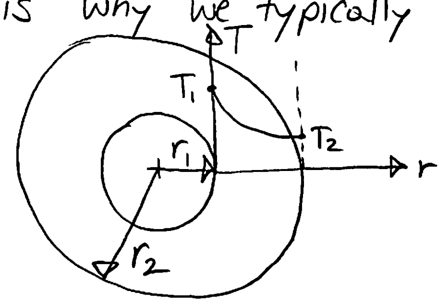
So for resistance, we want $R = \frac{\Delta T}{Q}$

$$R_{cyl} = \frac{T_1 - T_2}{2\pi L k \frac{T_1 - T_2}{\ln(r_2/r_1)}} = \frac{\ln(r_2/r_1)}{2\pi L k}$$

$$\boxed{R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi L k}}$$ ⇒ Cylindrical thermal resistance

Note, if $r_2 < r_1$, expression is the same since then $\Delta T = T_2 - T_1$ so negatives would cancel. This is why we typically write:

$$\boxed{R_{cyl} = \frac{|\ln(r_2/r_1)|}{2\pi L k}}$$



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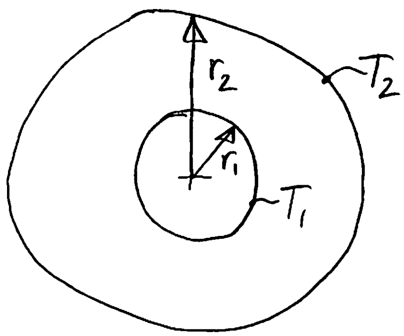
3. Sphere



Writing out our spherical coordinate system heat equation:

$$\cancel{\frac{1}{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0$$

Integrate once

$$\int \partial\left(r^2\frac{\partial T}{\partial r}\right) = \int 0\,\partial r$$

$$r^2\frac{\partial T}{\partial r} = C_1$$

$$\frac{\partial T}{\partial r} = \frac{C_1}{r^2} \quad \text{(Integrate again)}$$

$$\int \partial T = \int \frac{C_1}{r^2}\,\partial r$$

$$T = -\frac{C_1}{r} + C_2 \Rightarrow \text{Apply boundary cond.}$$

$$T\big|_{r_1} = T_1 = -\frac{C_1}{r_1} + C_2 \quad (1)$$

$$T\big|_{r_2} = T_2 = -\frac{C_1}{r_2} + C_2 \quad (2)$$

Subtract (2) from (1) $\Rightarrow C_1 = \dfrac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$ $\Rightarrow$ Back substitute into (2)

$$C_2 = T_2 + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\cdot\frac{1}{r_2}$$

$$T = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}\cdot\frac{1}{r} + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\cdot\frac{1}{r_2} + T_2$$

$$\boxed{\frac{T - T_2}{T_1 - T_2} = \frac{\left(\frac{1}{r} - \frac{1}{r_2}\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}} \Rightarrow \text{Spherical temperature profile}$$

Let's try calculating the thermal resistance for the sphere:

$$Q = -kA\left.\frac{\partial T}{\partial r}\right|_r \Rightarrow \frac{\partial T}{\partial r} = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}\cdot\left(-\frac{1}{r^2}\right)$$

$$A = 4\pi r^2$$

$$Q = \cancel{-}k\left(4\pi \cancel{r^2}\right)\frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}\left(\cancel{-}\frac{1}{\cancel{r^2}}\right) = \boxed{\frac{4\pi k\left(T_1 - T_2\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} = Q}$$

But we need $R = \dfrac{\Delta T}{Q}$

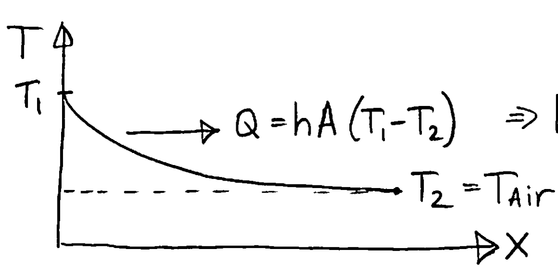
$$\boxed{R_{sph} = \frac{\left|\frac{1}{r_1} - \frac{1}{r_2}\right|}{4\pi k}}$$

$\Rightarrow$ Spherical thermal resistance
Note, same thing as before with the absolute value rule

(4.) Convection thermal resistance (while we're on the topic)

It is often very usefull to handle convection the same way.
We learned before that: (pg. 15)

$$Q = hA\Delta T \; ; \quad h \equiv \text{heat transfer coefficient } [W/m^2\cdot K]$$



$\Rightarrow$ hot surface exchanging heat w/ fluid

$$Q = hA\Delta T \Rightarrow R = \frac{\Delta T}{Q} = \frac{\cancel{\Delta T}}{hA\cancel{\Delta T}} \Rightarrow \boxed{R_{conv} = \frac{1}{hA}}$$

$\hookrightarrow$ Thermal resistance associated with convection