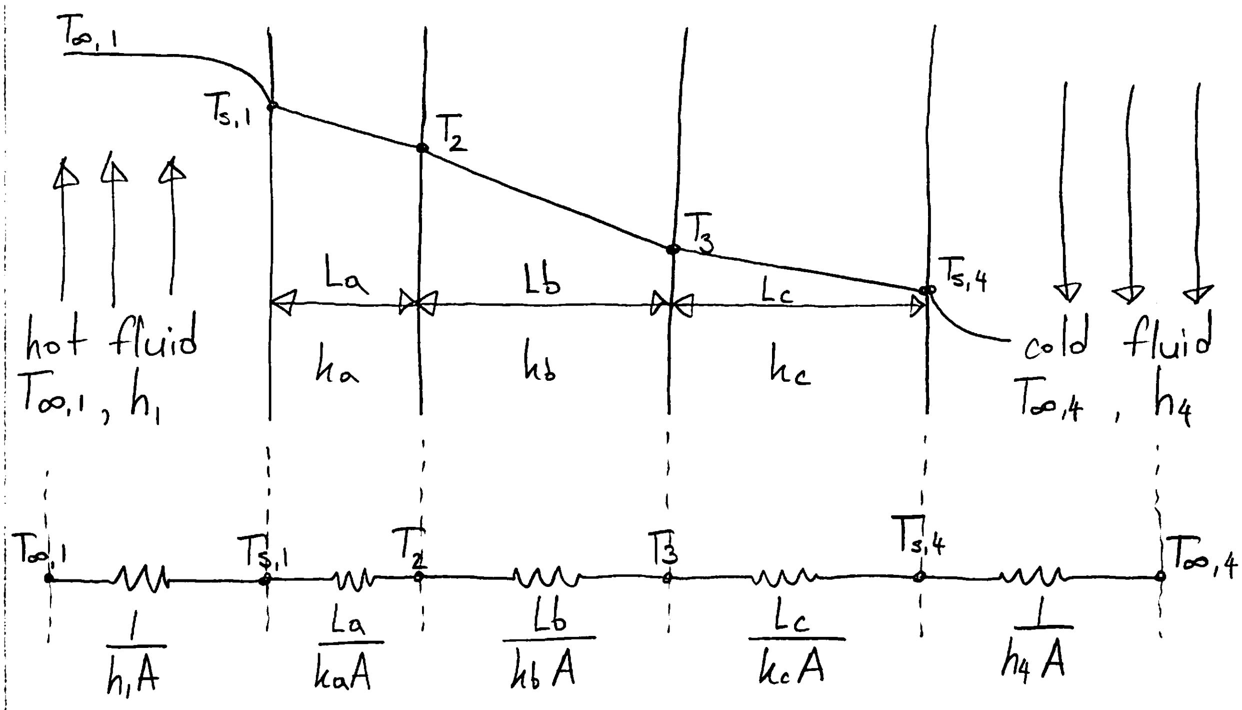
Nenad Miljkovic

Composite Problems

Just like in circuits, we simply add up our thermal resistances to solve for heat transfer and temperature at any node.

For a composite wall:



So if we want the overall heat transfer, we know $T_{\infty,1}$ & $T_{\infty,4}$ and all material properties, we can sum the resistances in series:

$$Q_{TOT} = \frac{\Delta T_{TOT}}{R_{TOT}} = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{h_1 A} + \frac{L_a}{k_a A} + \frac{L_b}{k_b A} + \frac{L_c}{k_c A} + \frac{1}{h_4 A}}$$

Also, we can write:

$$Q_{TOT} = \frac{\Delta T}{R} = \frac{T_{\infty,1} - T_{s,1}}{\frac{1}{h_1 A}} = \frac{T_{s,1} - T_2}{\frac{L_a}{k_a A}} = \frac{T_2 - T_3}{\frac{L_b}{k_b A}} = \dots$$

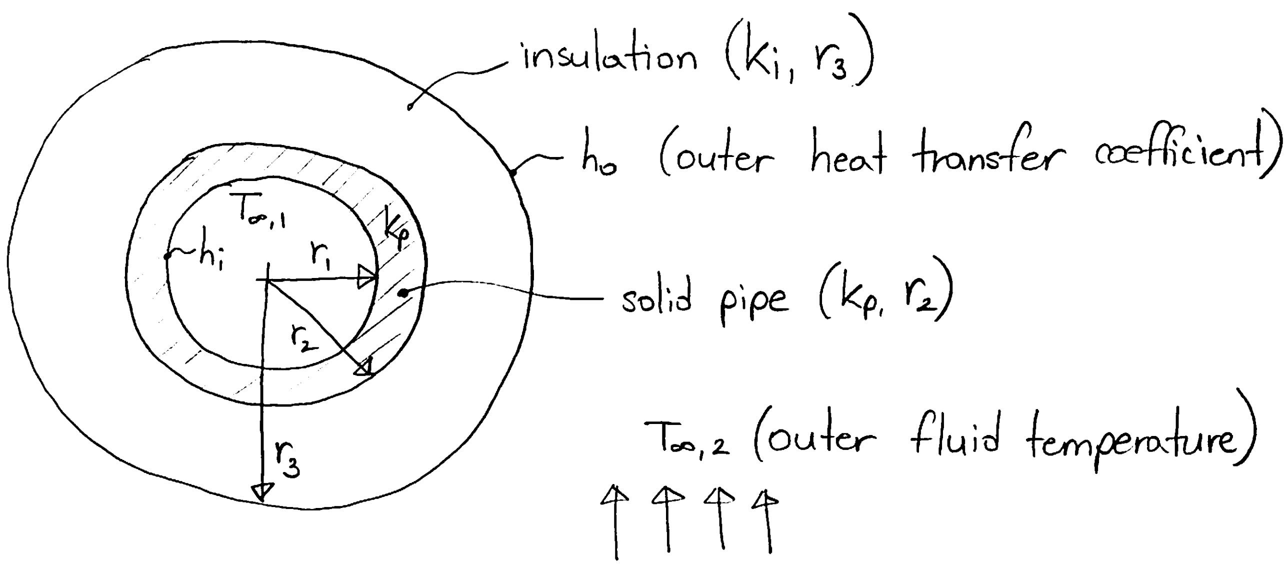
By doing the individual segment approach, we can then solve for intermediate temperatures $T_{s,1}, T_2, T_3, T_{s,4}$ which are typically unknowns in a design problem.

So in general, we can write: (for a wall with n layers)

$$\boxed{Q = \frac{A\left(T_{\infty,1} - T_{\infty,n}\right)}{\frac{1}{h_1} + \frac{1}{h_n} + \sum_{j=1}^{n} \frac{L_j}{k_j}}}$$

$$\boxed{R_{TOT} = \sum_{j=1}^{n} R_j}$$

Critical Thickness of Insulation



What should r_3 be in order to minimize heat transfer from the fluid flowing within the tube to the outside fluid. Let's write out our total thermal resistance:

$$R_{TOT} = \underbrace{\frac{1}{2\pi r_1 h_i L}}_{\text{inner convection}} + \underbrace{\frac{\ln(r_2/r_1)}{2\pi k_p L}}_{\text{pipe conduction}} + \underbrace{\frac{\ln(r_3/r_2)}{2\pi k_i L}}_{\text{insulation conduction}} + \underbrace{\frac{1}{2\pi r_3 h_o L}}_{\text{outer convection}}$$

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We can see from our equation that as we increase r_3, the insulation conduction resistance increases, but the outer convection resistance decreases. So there must be an optimum!

We want to maximize R_{TOT} with respect to r_3.

Remember, partial derivative

$$\frac{\partial R_{TOT}}{\partial r_3} = 0$$

$$\frac{\partial R_{TOT}}{\partial r_3} = \underbrace{\frac{\partial}{\partial r_3}\left(\frac{1}{2\pi r_1 h_i L}\right)}_{=0 \text{ since } \neq f(r_3)} + \underbrace{\frac{\partial}{\partial r_3}\left(\frac{\ln(r_2/r_1)}{2\pi k_p L}\right)}_{=0 \text{ since } \neq f(r_3)} + \frac{\partial}{\partial r_3}\left(\frac{\ln(r_3/r_2)}{2\pi k_i L}\right) + \frac{\partial}{\partial r_3}\left(\frac{1}{2\pi r_3 h_o L}\right)$$

$$\frac{\partial R_{TOT}}{\partial r_3} = 0 = \frac{1}{\cancel{2\pi} k_i \cancel{L}}\frac{\partial}{\partial r_3}\left[\ln(r_3/r_2)\right] + \frac{1}{\cancel{2\pi} h_o \cancel{L}}\frac{\partial}{\partial r_3}\left[\frac{1}{r_3}\right]$$

$$\frac{1}{k_i}\frac{\partial}{\partial r_3}\ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_o}\frac{\partial}{\partial r_3}\left(\frac{1}{r_3}\right) = 0$$

$$\frac{1}{k_i}\cdot\frac{\cancel{r_2}}{\cancel{r_3}\cdot\cancel{r_2}} + \frac{1}{h_o}\left(-\frac{1}{r_3^{\cancel{2}}}\right) = 0$$

$\Rightarrow$ Remember: $\frac{\partial}{\partial x}\ln(x) = \frac{1}{x}$

$\frac{\partial}{\partial x}\ln(ax) = \frac{1}{x}\cdot a$

$$\frac{1}{k_i} - \frac{1}{h_o r_3} = 0$$

$$\boxed{r_{3,crit} = \frac{k_i}{h_o}}$$

$\Rightarrow$ Critical thickness of insulation for a pipe.

So is this a maximum or a minimum?
We can check with a second derivative.

$$\frac{\partial^2 R_{TOT}}{\partial r_3^2} = \left[\frac{\partial}{\partial r_3}\left(\frac{1}{k_i r_3}\right) - \frac{\partial}{\partial r_3}\left(\frac{1}{h_o r_3^2}\right)\right]\frac{1}{2\pi L}$$

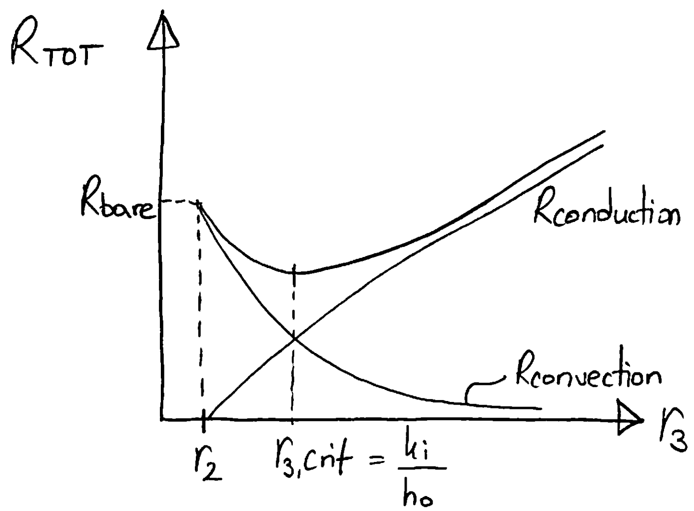
$$\frac{\partial^2 R_{TOT}}{\partial r_3^2} = \left(-\frac{1}{k_i r_3^2} + \frac{2}{h_o r_3^3}\right)\cdot\frac{1}{2\pi L} \qquad (1)$$

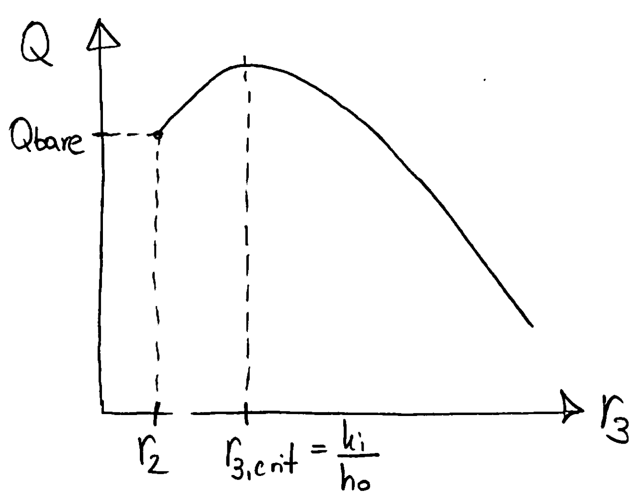
Back substituting $r_{3,crit}$ into equation (1)

$$\left.\frac{\partial^2 R_{TOT}}{\partial r_3^2}\right|_{r_{3,crit}} = \left(-\frac{1}{k_i\left(\frac{k_i^2}{h_o^2}\right)} + \frac{2}{\cancel{h_o}\left(\frac{k_i^3}{h_o^{\cancel{3}2}}\right)}\right)\frac{1}{2\pi L} = \left(-\frac{h_o^2}{k_i^3} + \frac{2h_o^2}{k_i^3}\right)\cdot\frac{1}{2\pi L}$$

$$\left.\frac{\partial^2 R_{TOT}}{\partial r_3^2}\right|_{r_{3,crit}} = +\frac{h_o^2}{2\pi k_i^3} > 0 \quad \text{(Always)}$$

So our calculated $r_{3,crit}$ is a global minimum. No optimum thickness exists, only a critical insulation thickness.





This result tells us when it is ok to add insulation. For example, for a hot water pipe, $r_2 = 10\,cm$

$$\left.\begin{aligned} k_i &= 0.1\,W/m\cdot K \\ h_o &= 5\,W/m^2\cdot K \end{aligned}\right\} \quad r_{3,crit} = \frac{0.1\,W/m\cdot K}{5\,W/m^2\cdot K} = 0.02m = 2cm$$

Here $r_{3,crit} < r_2 \Rightarrow$ means it is OK to insulate!