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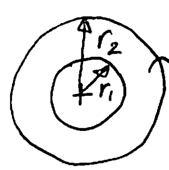
In general: $k_i \cong 0.1\ W/m\cdot K$ (common insulating material)
$h_o \cong 5\ W/m^2\cdot K$ (natural convection)
$r_{3,crit} \cong 1cm \Rightarrow$ We typically design our systems (HVAC&R) to be larger than this, hence OK to insulate.

$\therefore$ We can insulate hot water and steam pipes without worrying about increasing external heat transfer losses

So how about <u>electrical wires</u>?

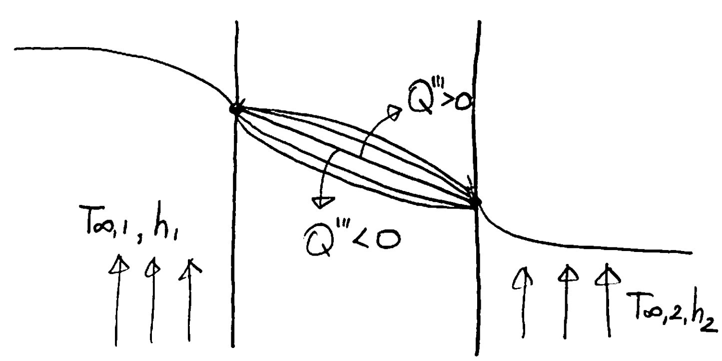


plastic insulation, $r_1 \cong 1mm$
$r_2 \cong 2mm < r_{3,crit}$

Although $r_2 < r_{3,crit}$, we typically <u>want</u> to cool our electrical wires & <u>NOT</u> thermally insulate them, so using a bigger or thicker insulation with a lower R_{TOT} is good!

<u>Heat Generation</u>

<u>Assumptions:</u>
1) 1D
2) Steady State
3) Constant properties
4) Uniform $\dot{Q}'''$ ($\dot{Q}'''$ = constant)



Note, the heat transfer rate and heat flux now need not be constant with respect to x.

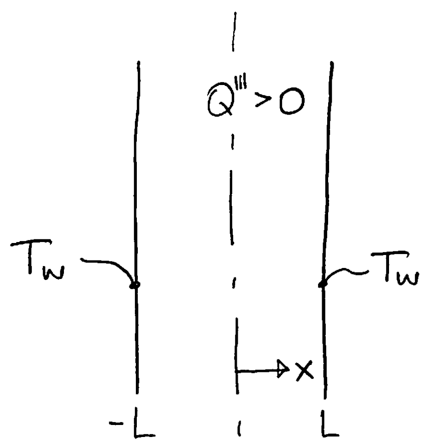
Writing out our heat equation: (in cartesian)

$$\frac{\partial^2 T}{\partial x^2} + \underbrace{\cancel{\frac{\partial^2 T}{\partial y^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}}_{0\ (1D)} + \frac{Q'''}{k} = \underbrace{\cancel{\frac{\rho C_p}{k}\frac{\partial T}{\partial t}}}_{0\ (SS)}$$

So our equation becomes:

$$\frac{\partial^2 T}{\partial x^2} + \frac{Q'''}{k} = 0 \quad (1)$$

Let's do the simplest case BC's (1'st kind)



$T(x = L) = T_w$

$T(x = -L) = T_w$

$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ (Symmetry or insulated)

↳ According to symmetry, the temperatures to the left and right of the plane of symmet. are the same, hence $\partial T/\partial x = 0$.

Integrating (1):

$$\frac{\partial^2 T}{\partial x^2} = -\frac{Q'''}{k}$$

$$\int \frac{\partial^2 T}{\partial x^2}\, dx = \int -\frac{Q'''}{k}\, dx$$

$$\frac{\partial T}{\partial x} = -\frac{Q'''}{k} x + C_1$$

But we know $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \Rightarrow C_1 = 0$

Integrating once more: $T(x) = -\frac{Q''' x^2}{2k} + C_2$

We know $T(x = \pm L) = T_w$

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$$T(x=L) = T_w = -\frac{Q'''L^2}{2k} + C_2 \Rightarrow C_2 = T_w + \frac{Q'''L^2}{2k}$$

$$\boxed{T(x) = T_w + \frac{Q'''L^2}{2k}\left(1-\left(\frac{x}{L}\right)^2\right)} \Rightarrow$$ Temperature profile inside the slab.

So where is T_{max}?

$$\frac{\partial T}{\partial x} = -\frac{Q'''x}{k} = 0 \Rightarrow x=0 \text{ is the solution}$$

Let's check if it is a maximum at $x=0$:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=0} = -\frac{Q'''}{k} < 0 \quad (\text{So } T \text{ is a max at } x=0)$$

$$\boxed{T_{max} = T_w + \frac{Q'''L^2}{2k}}$$

How about heat flux q'':

$$q'' = -k\frac{\partial T}{\partial x} = Q'''x = \boxed{q'' = Q'''x}$$

At the boundaries: $(x=\pm L)$ $\boxed{q''|_{x=L} = Q'''L}$ $\Rightarrow$ makes sense.

<u>Other Cases:</u>

If we had different wall temperatures: $T(x=-L) = T_{s1}$, $T(x=L) = T_{s2}$
We would solve the same problem but with the new B.C.'s.

$$\boxed{T(x) = \frac{Q'''L^2}{2k}\left(1-\left(\frac{x}{L}\right)^2\right) + \frac{T_{s2}-T_{s1}}{2}\cdot\frac{x}{L} + \frac{T_{s1}+T_{s2}}{2}}$$

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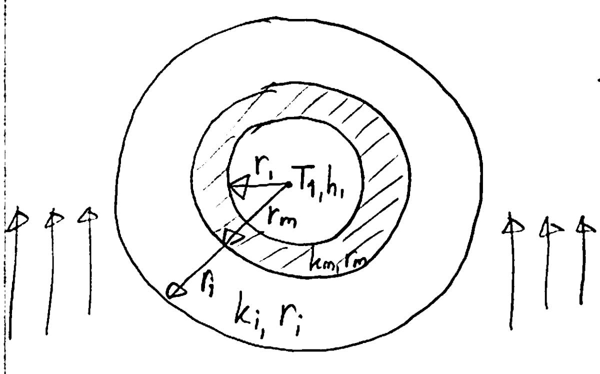
Example | Pipe flow & freezing water



$T_1 = 10°C$, $h_1 = 100 W/m^2 \cdot K$
$r_1 = 2cm$
$r_m = 2.5cm$, $k_m = 400 W/m \cdot K$
$r_i = 5 cm$, $k_i = 2 W/m \cdot K$
$T_\infty = ?$, $h_\infty = 50 W/m^2 \cdot K$

At what outside temperature T_∞, will the water in the pipe begin to frezze?

We can solve this with our thermal resistance approach

$$T_1 \xrightarrow{\frac{1}{h_i A_i}} T_{s,1} \xrightarrow{\frac{\ln(r_m/r_1)}{2\pi k_m L}} T_{s,m} \xrightarrow{\frac{\ln(r_i/r_m)}{2\pi k_i L}} T_{s,i} \xrightarrow{\frac{1}{h_\infty A_\infty}} T_\infty$$

We know when freezing starts that $T_{s,1} = 0°C \Rightarrow$ freezing point of water.

Let's solve the first leg of the resistance diagram:

$$Q = \frac{\Delta T}{R} = \frac{T_1 - T_{s1}}{\frac{1}{h_i A_i}} = \frac{10°C - 0°C}{\frac{1}{\underbrace{(100 W/m^2 \cdot K)}_{h_i}\underbrace{(2\pi(0.02m)(1m))}_{A_i \text{ (assuming 1m long tube, doesn't matter in the end)}}}}$$

$Q = 125.6W$

Since we don't have any heat generation, we know that Q is constant in our thermal circuit. So now we can solve for T_∞.

$$R_{TOT} = \frac{1}{h_i A_i} + \frac{\ln(r_m/r_1)}{2\pi k_m L} + \frac{\ln(r_i/r_m)}{2\pi k_i L} + \frac{1}{h_\infty A_\infty}$$

$$= \frac{1}{(100)(2\pi(0.02)(1))} + \frac{\ln(2.5/2)}{2\pi(400)(1)} + \frac{\ln(5/2.5)}{2\pi(.2)(1)} + \frac{1}{(50)(2\pi(0.05)(1))}$$

$$= 0.0796\,K/W + \underbrace{8.88\times10^{-5}\,K/W}_{\text{Thermal resistance of metal}} \ll \underbrace{0.055\,K/W}_{\text{Thermal resistance of insulation}} + 0.064\,K/W$$

$$R_{TOT} = 0.20\,K/W$$

Now we can solve for T_∞:

$$Q = \frac{\Delta T_{TOT}}{R_{TOT}} = 125.6\,W = \frac{10°C - T_\infty}{0.2\,K/W}$$

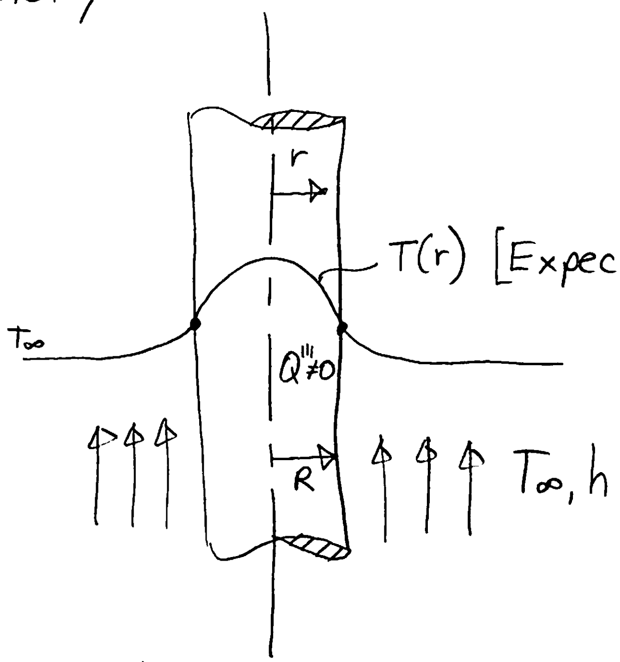
$$10°C - T_\infty = 24.98\,°C$$

$$\boxed{T_\infty = -15°C}$$

Therefore, we don't need to worry about our pipe freezing until the outside air reaches -15°C. In reality, we probably should add some more isulation just in case since on occasion, temperatures in Illinois can reach below -15°C

Circular Cylinder with Heat Generation

Many times the constant boundary temperature condition is not valid. Here, we will explore this case in a radial geometry:



Assumptions:
1) SS and 1D
2) Constant properties
3) Uniform heat gen: $Q''' > 0$

Writing out our heat equation in radial form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k\cancel{\frac{\partial T}{\partial \theta}}^{\,0}\right)}_{1D} + \underbrace{\frac{\partial}{\partial z}\left(k\cancel{\frac{\partial T}{\partial z}}^{\,0}\right)}_{1D} + Q''' = \rho c_p \underbrace{\cancel{\frac{\partial T}{\partial t}}^{\,0}}_{SS}$$

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + Q''' = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{Q'''}{k}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{Q'''r}{k} \quad \text{(Integrate wrt. r)}$$

$$\int \partial\left(r\frac{\partial T}{\partial r}\right) = -\frac{Q'''}{k}\int r\,\partial r$$

$$\cancel{r}\frac{\partial T}{\partial r} = -\frac{Q'''r^{\cancel{2}}}{2k} + \frac{C_1}{r} \quad \text{(Integrate again)}$$

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$$\int \partial T = -\frac{Q'''}{2k}\int r\,\partial r + \int \frac{C_1}{r}\,\partial r$$

$$T(r) = -\frac{Q''' r^2}{4k} + C_1 \ln r + C_2$$

Our boundary conditions are a little different now:

1) $-k\frac{\partial T}{\partial r}\Big|_{r=R} = h\left(T\Big|_{r=R} - T_\infty\right)$ [Energy balance at the surface]

2) $\frac{\partial T}{\partial r}\Big|_{r=0} = 0$ [Symmetry of the problem]

Right away we can solve for one of our constants. Since $\ln r \underset{r\to 0}{} = \infty$, then $C_1 = 0$. since we know we have a finite temperature.

The other way to see this is to apply B.C. #2 to our solution

$$\frac{\partial T}{\partial r}\Big|_{r=0} = -\frac{Q'''(\cancel{0})^{0}}{\cancel{2}k} + \underbrace{\frac{C_1}{0}}_{\infty}$$

, hence the only way this works is if $C_1 = 0$

Now we apply BC#1 to our equation:

$$T(r) = -\frac{Q''' r^2}{4k} + C_2$$

$$-k\frac{\partial T}{\partial r}\Big|_{r=R} = +\frac{k Q''' R}{2k} = h\left(-\frac{Q''' R^2}{4k} + C_2 - T_\infty\right) \quad [BC\#1]$$

$$\frac{1}{h}\left[\frac{\cancel{k} Q''' R}{2\cancel{k}} + \frac{h Q''' R^2}{4k} + h T_\infty\right] = C_2$$

$$\frac{1}{h}\left[\frac{Q''' R}{2}\left(1 + \frac{hR}{2k}\right) + h T_\infty\right] = C_2 \Rightarrow \text{Done.}$$