In general:

- \( k_i \approx 0.1 \text{ W/m.K} \) (common insulating material)
- \( h_o \approx 5 \text{ W/m}^2\cdot\text{K} \) (natural convection)
- \( r_{3,\text{crit}} \approx 1 \text{ cm} \) ⇒ We typically design our systems (HVAC&R) to be larger than this, hence OK to insulate.

We can insulate hot water and steam pipes without worrying about increasing external heat transfer losses.

So how about electrical wires?

![Plastic insulation with radii](image)

- Plastic insulation, \( r_1 \approx 1 \text{ mm} \)
- \( r_2 \approx 2 \text{ mm} < r_{3,\text{crit}} \)

Although \( r_2 < r_{3,\text{crit}} \), we typically want to cool our electrical wires & **not** thermally insulate them, so using a bigger or thicker insulation with a lower \( R_{TOT} \) is good.

**Heat Generation**

**Assumptions:**

1. 1D
2. Steady State
3. Constant properties
4. Uniform \( Q'' \) (\( Q'' = \text{constant} \))

Note, the heat transfer rate and heat flux now need not be constant with respect to \( x \).
Writing out our heat equation: (in cartesian)

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q'''}{\kappa} = \rho \sigma \epsilon \frac{\partial T}{\partial t} \]

\[ O(10) \quad 0 (SS) \]

So our equation becomes:

\[ \frac{\partial^2 T}{\partial x^2} + \frac{Q'''}{\kappa} = 0 \quad (1) \]

Let's do the simplest case BC's (1st kind)

\[ \begin{array}{c}
T_w \\
- L \\
\uparrow \\
\downarrow \\
L \\
\end{array} \quad \begin{array}{c}
T_w \\
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
T_w \\
\end{array}
\]

\[ T(x = L) = T_w \]
\[ T(x = - L) = T_w \]
\[ \frac{\partial T}{\partial x} \bigg|_{x=0} = 0 \quad (Symmetry \ or \ insulated) \]

According to symmetry, the temperatures to the left and right of the plane of symmetry are the same, hence \( \frac{\partial T}{\partial x} = 0 \).

Integrating (1):

\[ \frac{\partial^2 T}{\partial x^2} = - \frac{Q'''}{\kappa} \]

\[ \int \frac{\partial^2 T}{\partial x^2} \, dx = \int - \frac{Q'''}{\kappa} \, dx \]

\[ \frac{\partial T}{\partial x} = - \frac{Q'''}{\kappa} x + C_1 \]

But we know \( \frac{\partial T}{\partial x} \bigg|_{x=0} = 0 \Rightarrow C_1 = 0 \)

Integrating once more:

\[ T(x) = - \frac{Q'''}{2\kappa} x^2 + C_2 \]

We know \( T(x = \pm L) = T_w \)
\[ T(x=L) = T_w = -\frac{Q''L^2}{2k} + C_2 \Rightarrow C_2 = T_w + \frac{Q''L^2}{2k} \]

\[ T(x) = T_w + \frac{Q''L^2}{2k} \left(1 - \left(\frac{x}{L}\right)^2\right) \Rightarrow \text{Temperature profile inside the slab.} \]

So where is \( T_{\text{max}} \)?
\[ \frac{dT}{dx} = -\frac{Q''x}{k} = 0 \Rightarrow x=0 \text{ is the solution} \]

Let's check if it is a maximum at \( x=0 \):
\[ \left. \frac{d^2T}{dx^2} \right|_{x=0} = -\frac{Q''}{k} < 0 \text{ (So } T \text{ is a max at } x=0) \]

\[ T_{\text{max}} = T_w + \frac{Q''L^2}{2k} \]

How about heat flux \( q'' \):
\[ q'' = -k \frac{dT}{dx} = Q''x = \begin{cases} q'' = Q''x \end{cases} \]

At the boundaries: \( (x=\pm L) \)
\[ q'' \big|_{x=L} = Q''L \Rightarrow \text{makes sense.} \]

**Other Cases:**
If we had different wall temperatures: \( T(x=-L) = T_{s1} \)
We would solve the same problem but \( T(x=L) = T_{s2} \)
with the new B.C.'s.

\[ T(x) = \frac{Q''L^2}{2k} \left(1 - \left(\frac{x}{L}\right)^2\right) + \frac{T_{s2} - T_{s1}}{2} \cdot \frac{x}{L} + \frac{T_{s1} + T_{s2}}{2} \]
Example: Pipe flow & freezing water

At what outside temperature $T_\infty$, will the water in the pipe begin to freeze?

We can solve this with our thermal resistance approach:

$$
\frac{1}{h_i A_i} \frac{\ln \left( \frac{r_i}{r_m} \right)}{2\pi k_m L} + \frac{\ln \left( \frac{r_m}{r_i} \right)}{2\pi h_i L} + \frac{1}{h_\infty A_\infty}
$$

We know when freezing starts that $T_{s,1} = 0^\circ C \Rightarrow$ freezing point of water.

Let's solve the first leg of the resistance diagram:

$$
Q = \frac{\Delta T}{R} = \frac{T_i - T_{s,1}}{\frac{1}{h_i A_i}} = \frac{10^\circ C - 0^\circ C}{\frac{1}{100 W/m^2\cdot K}(2\pi (0.02 m)(1 m))} = 125.6 W
$$

Since we don't have any heat generation, we know that $Q$ is constant in our thermal circuit. So now we can solve for $T_\infty$. 

$T_i = 10^\circ C$, $h_i = 100 W/m^2\cdot K$

$r_i = 2 cm$

$r_m = 2.5 cm$, $k_m = 400 W/m\cdot K$

$r_i = 5 cm$, $h_i = 2 W/m\cdot K$

$T_\infty = \ ?$, $h_\infty = 50 W/m^2\cdot K$
\[ R_{\text{TOT}} = \frac{1}{h_i A_i} + \frac{\ln \left( \frac{r_m}{r_i} \right)}{2\pi k_m L} + \frac{\ln \left( \frac{r_i}{r_m} \right)}{2\pi k_i L} + \frac{1}{h_\infty A_\infty} \]

\[ = \frac{1}{(100)(2\pi (0.02)(1))} + \frac{\ln \left( \frac{2.5}{2} \right)}{2\pi (400)(1)} + \frac{\ln \left( \frac{5/2.5}{2} \right)}{2\pi (0.05)(1)} + \frac{1}{(50)(2\pi (0.05)(1))} \]

\[ = 0.0796 \text{ K/W} + 8.88 \times 10^{-5} \text{ K/W} + 0.055 \text{ K/W} + 0.064 \text{ K/W} \]

\[ R_{\text{TOT}} = 0.20 \text{ K/W} \]

Now we can solve for \( T_\infty \):

\[ Q = \frac{\Delta T_{\text{tot}}}{R_{\text{TOT}}} = 125.6 \text{ W} = \frac{10^\circ \text{C} - T_\infty}{0.2 \text{ K/W}} \]

\[ 10^\circ \text{C} - T_\infty = 24.98^\circ \text{C} \]

\[ T_\infty = -15^\circ \text{C} \]

Therefore, we don't need to worry about our pipe freezing until the outside air reaches -15°C. In reality we probably should add some more insulation just in case since on occasion, temperatures in Illinois can reach below -15°C.
Circular Cylindrical with Heat Generation

Many times the constant boundary temperature condition is not valid. Here, we will explore this case in a radial geometry:

![Diagram of a circular cylinder with temperature profile and heat generation]

**Assumptions:**
1. SS and 10
2. Constant properties
3. Uniform heat gen: \( Q^\infty \)

Writing out our heat equation in radial form:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial T}{\partial r} \right] + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + Q'' = \frac{\partial Q^\infty}{\partial t}
\]

\[
\frac{k}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + Q'' = 0
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = - \frac{Q''}{k}
\]

\[
\frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = - \frac{Q'' r}{k} \quad \text{(Integrate wrt. } r)\]

\[
\int \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) = - \frac{Q''}{k} \int r \, dr
\]

\[
r \frac{\partial T}{\partial r} = - \frac{Q'' r^2}{2k} + C \quad \text{(Integrate again)}
\]
\[ \int 2T = - \frac{Q''''}{2k} \int r \, dr + \int \frac{G}{r} \, dr \]
\[ T(r) = - \frac{Q''''r^2}{4k} + C_1 \ln r + C_2 \]

Our boundary conditions are a little different now:

1. \(-k \frac{dT}{dr} \bigg|_{r=R} = h \left( T \bigg|_{r=R} - T_\infty \right) \) \[\text{[Energy balance at the surface]}\]

2. \( \frac{dT}{dr} \bigg|_{r=0} = 0 \) \[\text{[Symmetry of the problem]}\]

Right away we can solve for one of our constants. Since \( \ln r = \infty \), then \( C_1 = 0 \). Since we know we have a finite temperature.

The other way to see this is to apply B.C. \#2 to our solution:

\[ \frac{dT}{dr} \bigg|_{r=0} = - \frac{Q''''(0)}{2k} + \frac{C_1}{r} \]
\[ \bigg| \frac{Q'''}{2k} \bigg|_{r=0} = \frac{C_1}{r} \]

\( G(0) = \infty \), hence the only way this works is if \( C_1 = 0 \).

Now we apply B.C. \#1 to our equation:

\[ T(r) = - \frac{Q''''r^2}{4k} + C_2 \]
\[ -k \frac{dT}{dr} \bigg|_{r=R} = +k \frac{Q'''}{2k} R = h \left( - \frac{Q''''r^2}{4k} + C_2 - T_\infty \right) \] \[\text{[B.C. \#1]}\]

\[ \frac{1}{h} \left[ \frac{kQ'''}{2k} \left( 1 + \frac{hR^2}{2k} \right) + hT_\infty \right] = C_2 \]
\[ \frac{1}{h} \left[ \frac{Q'''}{2} + hT_\infty \right] = C_2 \Rightarrow \text{Done.} \]