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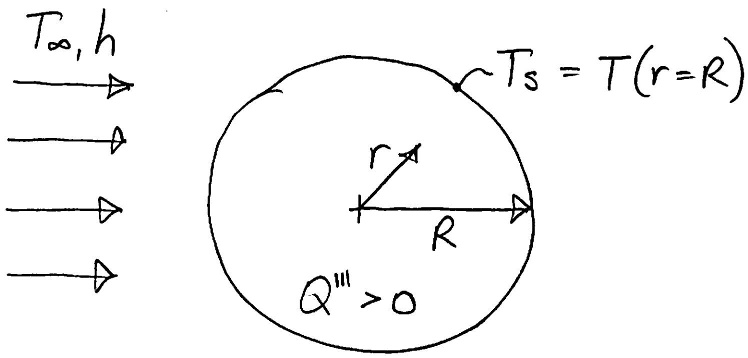
Sphere with Heat Generation

Here we'll do the same approach as with the cylinder but with a different flavor:



Assumptions:
1) SS & 1D
2) Constant properties
3) Uniform heat generation

Writing out our spherical coordinate heat equation with terms already canceled:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{Q'''}{k} = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{Q''' r^2}{k} \quad \text{(Integrate once)}$$

$$\int \partial\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{Q'''}{k}\int r^2 \partial r$$

$$r^2\frac{\partial T}{\partial r} = -\frac{Q'''}{k}\frac{r^3}{3} + C_1$$

$$\frac{\partial T}{\partial r} = -\frac{Q''' r}{3k} + \frac{C_1}{r^2} \quad \text{(Integrate again)}$$

$$\int \partial T = -\int \frac{Q''' r}{3k}\partial r + \int \frac{C_1}{r^2}\partial r$$

$$T(r) = -\frac{Q''' r^2}{6k} - \frac{C_1}{r} + C_2$$

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Now we can apply our boundary conditions:

1) $T(r=R) = T_s$ [Surface temperature]

2) $\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$ [Symmetry of the problem]

B.C #2 is handeled just like in the radial solution:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = -\frac{Q'''\cancel{(0)}^{0}}{\cancel{3k}} + \underbrace{\frac{C_1}{0^2}}_{\infty} = 0$$

$\Rightarrow$ Only way this goes to 0 is if $C_1 = 0$

$$\therefore \boxed{C_1 = 0}$$

Applying B.C #1:

$$T(r=R) = -\frac{Q'''R^2}{6k} + C_2 = T_s$$

$$\boxed{C_2 = T_s + \frac{Q'''R^2}{6k}}$$

So our solution becomes:

$$\boxed{T(r) = \frac{Q'''R^2}{6k}\left(1 - \frac{r^2}{R^2}\right) + T_s}$$

$\Rightarrow$ But note, we didn't specify T_s, we specified T_∞, & h. So how do we reconcile this.

We can solve for T_s with an energy balance at the surface:

Total energy generated in the sphere $= \frac{4}{3}\pi R^3 Q'''$

Total heat transfer rate at the surface $= 4\pi R^2 h(T_s - T_\infty)$

$$\frac{4}{3}\cancel{\pi} R^{\cancel{3}} Q''' = \cancel{4\pi} \cancel{R^2} h(T_s - T_\infty) \Rightarrow \boxed{T_s = T_\infty + \frac{RQ'''}{3h}}$$

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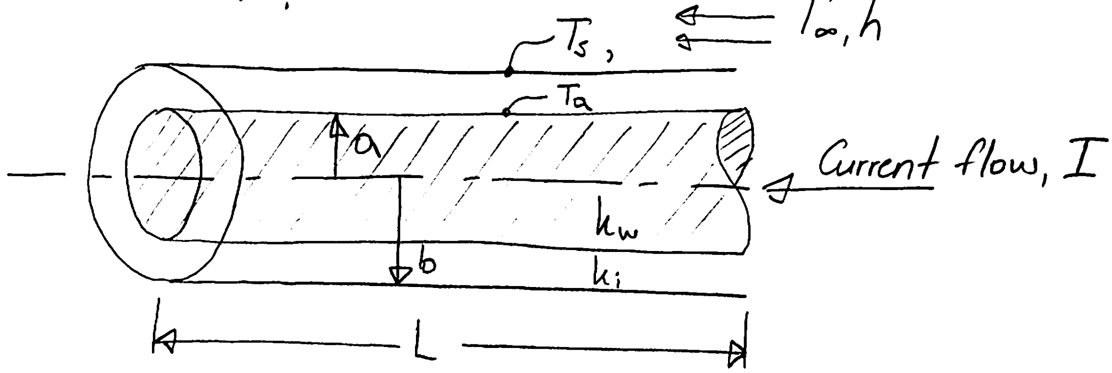
Example / Wire with electrical resistance R_e and current I, what is T_{max}?

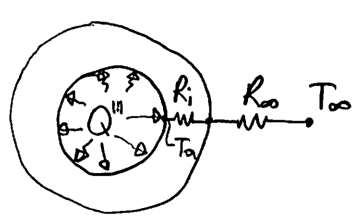


Assumptions:
1) 1D & SS
2) Constant properties: $k_i \equiv$ insulation thermal conductivity
$k_w \equiv$ wire thermal conductivity
$h =$ heat transfer coefficient on outside of wire.

This problem seems complicated as it involves a heat generation problem with a thermal resistance problem. Let's put the two together:

$$\text{Energy generation in the wire} = \frac{I^2 R_e}{\pi a^2 L} = Q''' \quad [\text{heat generation term}]$$

Now that we have the total heat generation, we can use our thermal resistance approach:



$$\Rightarrow R_i = \frac{\ln(b/a)}{2\pi L k_i} \quad ; \quad R_\infty = \frac{1}{h 2\pi b L}$$

$$Q_{TOT} = \pi a^2 L Q''' = \frac{\Delta T}{R_{TOT}} = \frac{T_a - T_\infty}{\frac{\ln(b/a)}{2\pi L k_i} + \frac{1}{2\pi b L h}} \quad \text{(Solve for } T_a\text{)}$$

$$T_a = T_\infty + \frac{a^2 Q'''}{2}\left[\frac{\ln(b/a)}{k_i} + \frac{1}{bh}\right] \quad (1)$$

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So we've solved for T_a but is that the maximum temperature?

Looking back at our notes to the cylindrical solution with heat generation (pg. 34)

$$T(r) = -\frac{Q''' r^2}{4k_w} + C_2$$

Applying our boundary condition here: $T(r=a) = T_a$

$$T_a = -\frac{Q''' a^2}{4k_w} + C_2 \Rightarrow \boxed{C_2 = T_a + \frac{Q''' a^2}{4k_w}}$$

$$T(r) = \frac{Q''' a^2}{4k_w}\left(1 - \frac{r^2}{a^2}\right) + T_a$$

To obtain the maximum temperature in the wire, we can differentiate (note: we already know it is at $r=0$, however its good to be rigorous)

$$\frac{\partial T}{\partial r} = -\frac{Q''' 2 \cancel{a^2} r}{4k_w \cancel{a^2}} = \frac{2r}{4k_w} = 0 \Rightarrow r = 0 \text{ is the maximum}$$

$$\left.\frac{\partial^2 T}{\partial r^2}\right|_{r=0} = -\frac{2Q'''}{4k_w} < 0 \text{ hence always a maximum for } Q''' > 0$$

Hence: $\boxed{T_{max} = T(r=0) = T_a + \frac{Q''' a^2}{4k_w}}$ (2) Back sub. into (1)

$$\boxed{T_{max} = T_\infty + \frac{a^2 Q'''}{2}\left[\frac{1}{2k_w} + \frac{\ln(b/a)}{k_i} + \frac{1}{bh}\right]}$$

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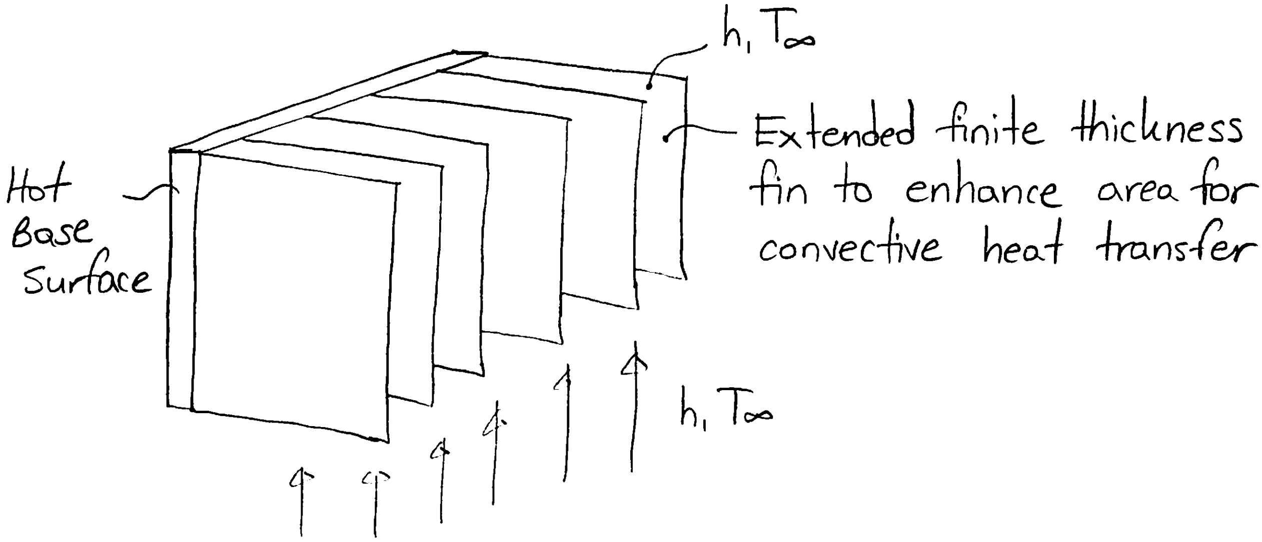
Quasi - 1D Conduction : Fins

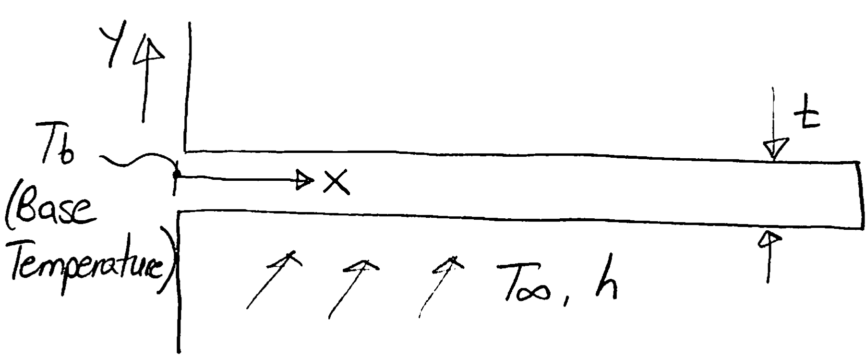
Fins are generaly used when the heat transfer coefficient on the surface is not large and we want to augment heat transfer.



The primary function of the fin is to add surface area for convection heat transfer.

Assumptions:
1) SS
2) No heat generation
3) Fin cross-sectional area & perimeter are constant
4) 1D heat transfer

What does assumption ④ mean & how do we ensure it is valid



For 1D heat transfer, our fin temperature should be a function of x-only.

$$T = f(x) \neq f(y)$$