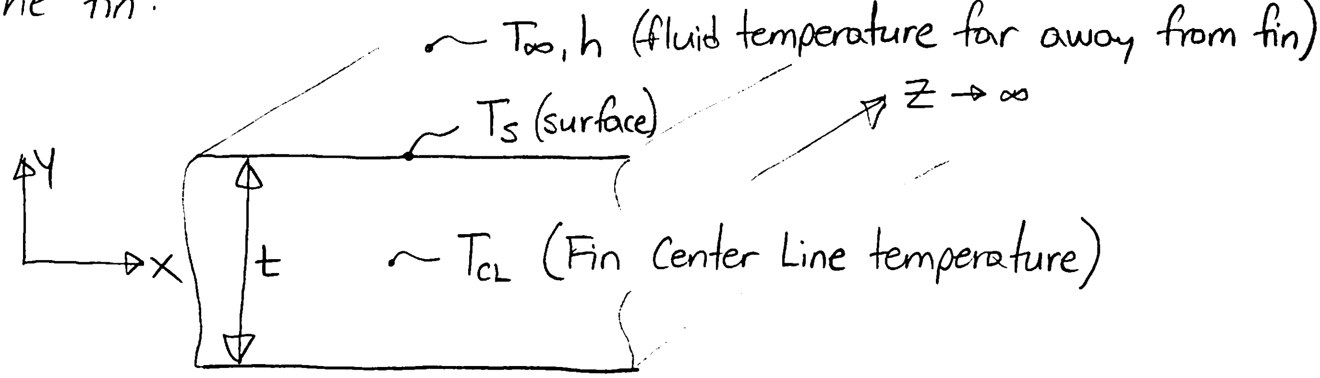
Nenad Miljkovic

So how do we quantify this? Let's look at a segment of the fin:



Energy balance across the fin (y-direction), we obtain

$$\underbrace{-k\frac{\partial T}{\partial y}(\cancel{Ldx})}_{\text{Conduction}} = \underbrace{h(\cancel{Ldx})(T_S - T_\infty)}_{\text{Convection}}$$

$$-k\frac{\partial T}{\partial y} = h(T_S - T_\infty)$$

We can approximate $\frac{\partial T}{\partial y}$ as: $\frac{T_S - T_{CL}}{t/2} = -\frac{(T_{CL} - T_S)}{t/2}$

$$+k\frac{T_{CL} - T_S}{t/2} = h(T_S - T_\infty)$$

$$\frac{T_{CL} - T_S}{T_S - T_\infty} = +\frac{1}{2}\frac{ht}{k} \approx 0.05 \quad \text{(Set it equal to this)}$$

$$\frac{1}{2}\left(\frac{ht}{k}\right) \leq \frac{1}{20} \Rightarrow \boxed{\frac{ht}{k} \leq \frac{1}{10} \equiv Bi_t} \Rightarrow \text{Biot number}$$

So how do we interpret this approximation or condition?

1) Temperature changes across the fin thickness are small compared to those external to the fin

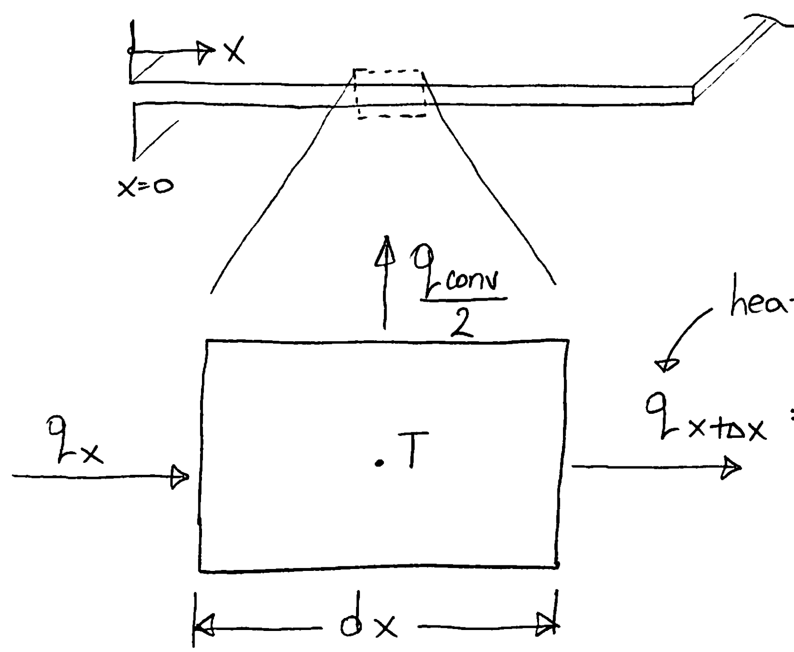
2) Internal thermal resistance to conduction heat transfer across the fin (y-direction) is small compared to the external convective heat transfer resistance. $R_{conv} \gg R_{cond}$

We can double check claim #2 from our previous results:

$$\frac{R_{cond}}{R_{conv}} = \frac{\frac{t}{k\cancel{A}}}{\frac{1}{h\cancel{A}}} = \underbrace{\frac{ht}{k}}_{Bi_t \ll 1} \ll 1$$

So now we can model the heat transfer in the x-direction as 1D $\Rightarrow$ Makes our lives a lot easier!

Let's take a differential fin element:



heat transfer at $x+\Delta x$ [W]

$$q_{x+\Delta x} = q_x + \frac{\partial}{\partial x}(q_x)\Delta x + \frac{\partial^2}{\partial x^2}(q_x)\frac{\Delta x^2}{2} + \ldots$$

$$q_{conv} = hP\,dx\,(T - T_\infty) \quad \Rightarrow \quad P = \text{perimeter of the fin} = \text{surface area/unit length}$$

$$q_x = -kA\frac{\partial T}{\partial x} \qquad P\Delta x \equiv \text{area for convective heat trasfer}$$

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Let's combine the two equations and do an energy balance on our differential element:

$$E_{in} - E_{OUT} + \underbrace{\cancel{E_{gen}}}_{Q'''=0} = \underbrace{\cancel{E_{STORED}}}_{SS}$$

$$\cancel{q_x} - \left[\cancel{q_x} + \frac{\partial q_x}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 q_x}{\partial x^2}\Delta x^2 + \ldots\right] - hP\Delta x(T-T_\infty) = 0$$

$$-\frac{\partial q_x}{\partial x}\cancel{\Delta x} - \frac{1}{2}\frac{\partial^2 q_x}{\partial x^2}\Delta x^{\cancel{2}} - \ldots = hP\cancel{\Delta x}(T-T_\infty)$$

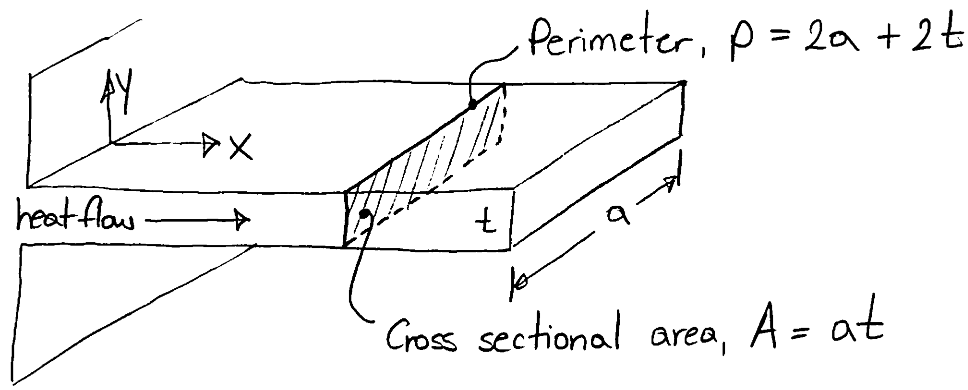
Since we are working with a differential volume: $\Delta x \to 0$

$$-\frac{\partial q_x}{\partial x} = hP(T-T_\infty) \quad (1)$$

$$q_x = -kA\frac{\partial T}{\partial x} \equiv \text{Fourier's Law} \Rightarrow \text{Back sub into } (1)$$

$$kA\frac{\partial^2 T}{\partial x^2} = hP(T-T_\infty) \quad (2)$$

Note, A is the fin cross sectional area:



To help us solve equation (2), we can assume the following:

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$$\underbrace{\theta = T - T_\infty}_{a} \; ; \quad \frac{\partial \theta}{\partial T} = 1 \Rightarrow \underbrace{\partial \theta = \partial T}_{b}$$

Convert equation ② into θ coordinates by back subbing a & b

$$kA \frac{\partial^2 \theta}{\partial x^2} - hP\theta = 0$$

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{hP}{kA}\theta = 0 \Rightarrow \text{Let } \boxed{m^2 = \frac{hP}{kA}}$$

$$\frac{\partial^2 \theta}{\partial x^2} - m^2\theta = 0 \Rightarrow \text{Linear, second order ODE}$$

Remember from calculus class, if you have an equation of the form:

n-1 derivative $\quad$ a_0 constant

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

Then you can write a characteristic equation:

$$a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1 \lambda^1 + a_0 = 0 \quad \text{(polynomial)}$$

and solve for the roots of the equation : $\lambda = \dots$

For our case:

$$\lambda^2 - m^2 = 0$$
$$\lambda^2 = m^2$$
$$\lambda = \pm m$$

So our general solution is: $\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$

We can show that this works by the following:

Assume a solution of the form:

$$\theta = C_1 e^{mx} + C_2 e^{-mx}$$
$$\theta' = mC_1 e^{mx} - mC_2 e^{-mx} \quad \left(\text{First derivative, } \frac{\partial\theta}{\partial x}\right)$$
$$\theta'' = m^2 C_1 e^{mx} + m^2 C_2 e^{-mx}$$

Back substitute into our ODE: $\theta'' - m^2\theta = 0$

$$\underbrace{m^2\left(C_1 e^{mx} + C_2 e^{-mx}\right)}_{\theta''} - m^2\underbrace{\left(C_1 e^{mx} + C_2 e^{-mx}\right)}_{\theta} = 0$$

$0 = 0$ ✓ (Works)

OK, so now back to our problem. How do we solve for C_1 & C_2 (constants). That depends on our B.C.'s

We have a linear second order ODE, so we need 2 B.C.'s

① Infinite Fin ($x \to \infty$)

Our B.C.'s become: $\theta(x=0) = T_B - T_\infty = \theta_b$

$\theta(x \to \infty) = T_\infty - T_\infty = 0$

As we move out to ∞, our fin will approach the temperature of the surrounding fluid.

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$
$$\theta(\infty) = 0 = \underbrace{C_1 e^{m(\infty)}}_{C_1 = 0 \text{ for the solution to be valid since } e^{\infty} \to \infty} + C_2 e^{-m(\infty)}$$