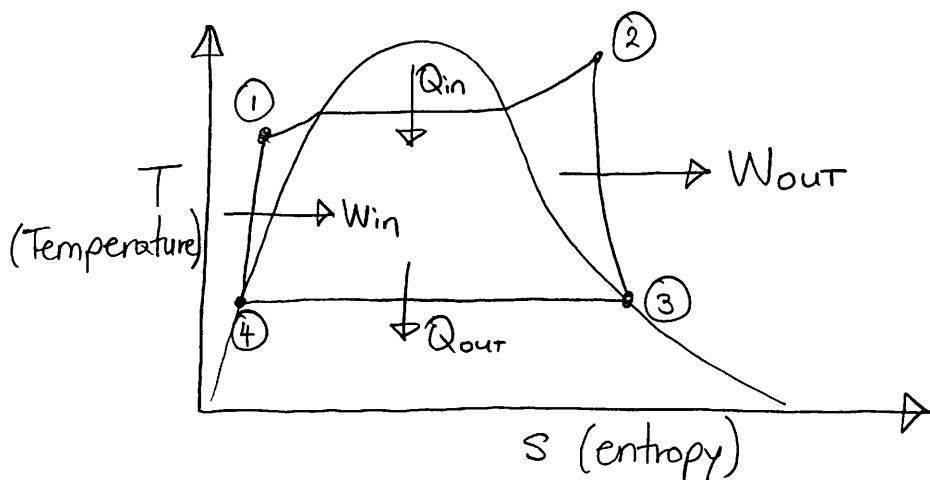
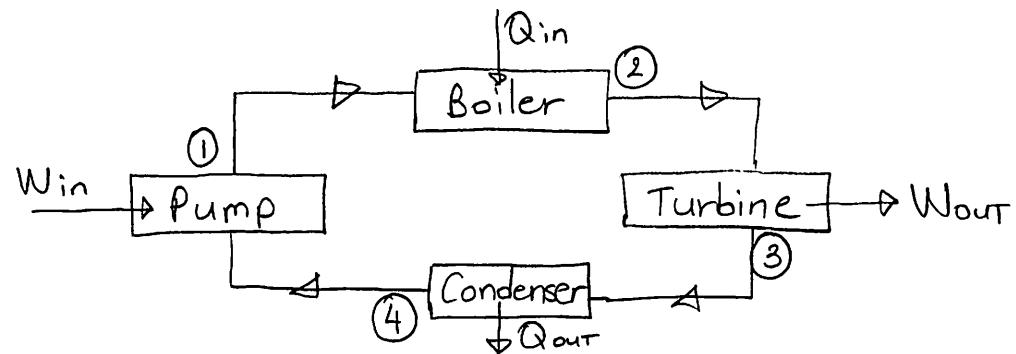


ME 320 - Heat Transfer

Why is heat transfer important and what is it?

A good example is power generation (from thermo)



What parts of this thermodynamic cycle (Rankine) involve heat transfer? \Rightarrow ALL of Them!

Pump: Friction Losses generate heat \Rightarrow needs to be dissipated to keep pump cool.

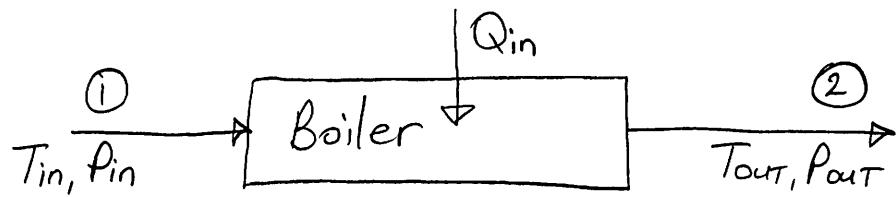
Boiler: Boiling process adds heat to the working fluid

Turbine: Friction losses & conduction losses result in heat loss to the surroundings \Rightarrow S increases

Condenser: Condensation process removes heat from working fluid.

In general, thermodynamics is a state approach. Heat transfer tells you the details of what is in between each state & how it happens. Allows us to develop tools to help design real life thermodynamic components.

For example:

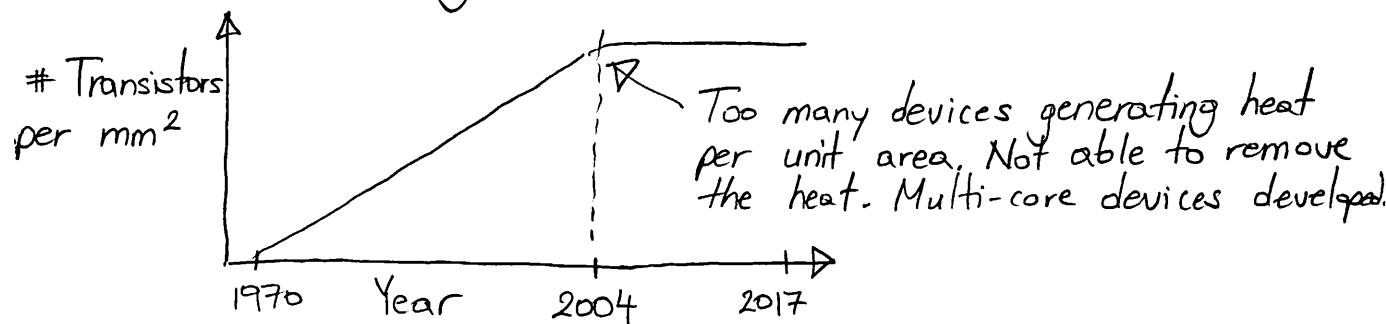


From enthalpy tables, you can determine T_{out}, P_{out} if given T_{in}, P_{in} & Q_{in} . However, as an engineer, what size should your boiler be to obtain the Q_{in} required? How is the fluid obtaining the heat in the boiler? How do we make this process most efficient?

These are all questions you will be able to answer by the end of ME320.

Other important applications:

- Electronics Cooling \Rightarrow Moore's Law



- Biology \Rightarrow Human Body \Rightarrow Sweating, Blood flow

Tools we Need:

- ① Conservation of mass
- ② Conservation of momentum
- ③ Conservation of energy * (most important one)
- ④ Entropy \Rightarrow Heat always flows from T_{high} to T_{low} (naturally)

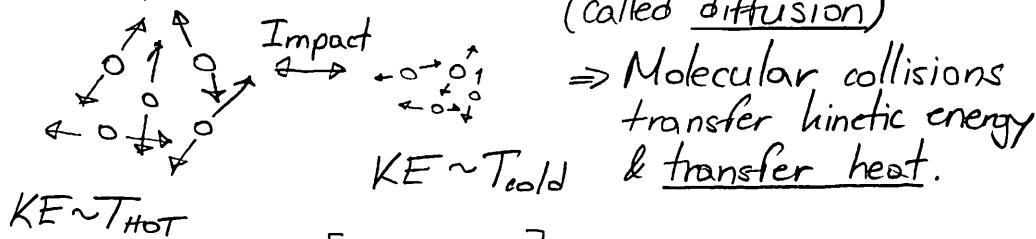
Some Basics:

What are the important parameters in heat transfer:

- $T \Rightarrow [{}^\circ\text{C} \text{ or } \text{K}] \Rightarrow \text{Temperature} = \text{average kinetic energy}$
- $Q \Rightarrow [\text{W} \text{ or } \text{J/s}] \Rightarrow \text{Heat transfer rate}$ in random directions.
- $q'' \Rightarrow [\text{W/m}^2] \Rightarrow \text{Heat flux}$
- $C_p \Rightarrow [\text{J/kg} \cdot \text{K}] \Rightarrow \text{Specific heat} \Rightarrow \text{how much energy it takes to raise the temperature of 1 kg of a material by } 1^\circ\text{C or 1 K.}$
- $k \Rightarrow [\text{W/m} \cdot \text{K}] \Rightarrow \text{thermal conductivity} \Rightarrow \text{more on this later}$

Modes of Heat Transfer

Conduction: \Rightarrow Energy transfer via direct molecular contact (called diffusion)



$$KE_{Avg} = \left[\frac{1}{2} mv^2 \right] = \frac{3}{2} k_B T \Rightarrow \text{ME 420}$$

Examples: 1) Heat transfer in a solid object heated on one end
2) When you touch something hot

Convection: \Rightarrow Conduction with fluid flow

Examples: 1) Boiling, pipe flow, ocean currents, blood flow

Radiation: \Rightarrow Heat transfer via electromagnetic waves

No need for a medium \Rightarrow not diffusion

Examples: 1) Sun heating the earth

2) Radiative heaters in store entrances & theaters
3) IR (infrared) cameras & detection

Fourier's Law of Heat Conduction

In 17th century, a guy named Joseph Fourier discovered an empirical correlation that governs heat transfer in a solid:

$$q''_x = -k \frac{dT}{dx}$$

Heat flux Thermal conductivity of the material transferring heat
 in the x direction Temperature gradient in the x-direction. Ordinary derivative

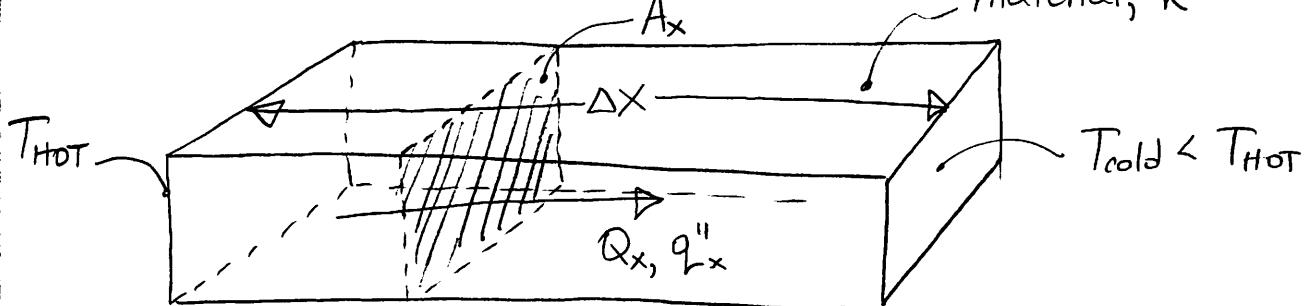
Sometimes written as:

$$Q_x = -k A_x \frac{dT}{dx}$$

Cross sectional area perpendicular to heat transfer dir.

$$\frac{dT}{dx} = \frac{T_{cold} - T_{hot}}{x}$$

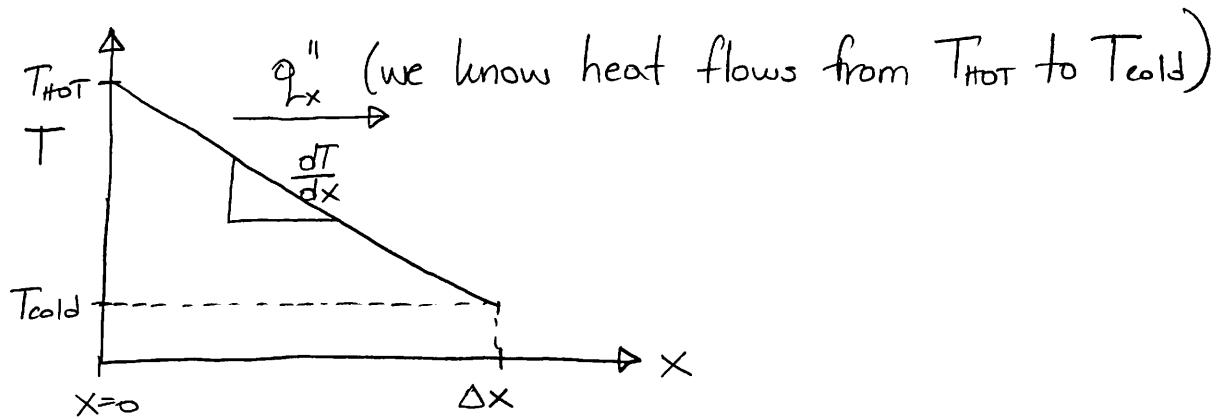
material, k



Fourier did this experiment for many metals to obtain his relation. (4)

Let's take a closer look at his equation:

$$\dot{q}_x'' = -k \frac{dT}{dx}$$

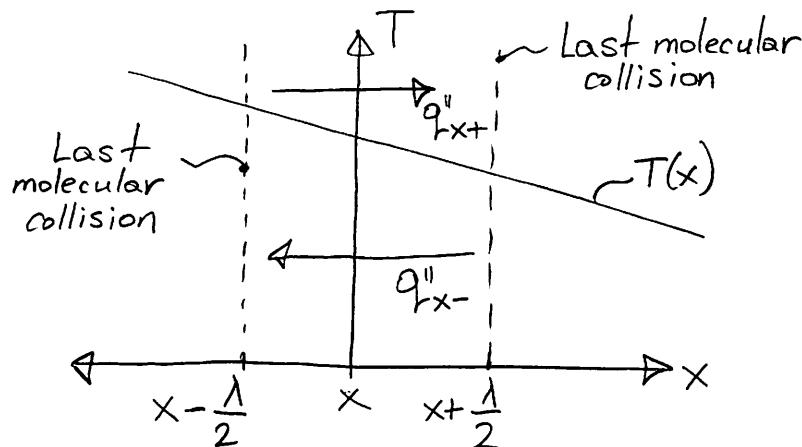


$$\frac{dT}{dx} = \frac{T_{cold} - T_{hot}}{\Delta x} < 0 \quad (\text{negative})$$

But \dot{q}'' must be positive since heat flows in the positive x-direction?

→ Hence the negative sign in the equation.
 k is always a positive quantity, hence things make sense. Heat flows down the temperature gradient.

Derivation of Fourier's Law



$n = \# \text{ of particles/m}^3$
 $m = \text{mass per particle}$
 $C_p = \text{specific heat capacity}$
 $\bar{c} = \text{average speed of particles}$

$C_p [\text{J/kg} \cdot \text{K}]$; $n [\#/m^3]$, $m \left[\frac{\text{kg}}{\#} \right]$

Let's look at the energy exchange at our plane of interest (x)

$$q''_{x+} = n \cdot m \cdot c_p \bar{C} \cdot T \left(x - \frac{\lambda}{2} \right)$$

Temperature at $x = x - \frac{\lambda}{2}$

$\underbrace{[\frac{\#}{m^3}] \cdot [\frac{kg}{\#}] \cdot [J/kg \cdot K] \cdot [m/s] \cdot [K]}_{\left[\frac{kg}{m^2 \cdot s} \right]} \quad \underbrace{[J/kg]}_{\begin{array}{l} \text{mass flow rate} \\ \text{of particles per unit area.} \end{array}}$

$\underbrace{[J/kg]}_{\text{energy flow per unit mass of particles.}}$

We can notice that: $n \cdot m = \rho [kg/m^3] \Rightarrow \text{Density}$

$$q''_{x+} = + \rho c_p \bar{C} T \left(x - \frac{\lambda}{2} \right)$$

$$q''_{x-} = + \rho c_p \bar{C} T \left(x + \frac{\lambda}{2} \right)$$

$$\begin{aligned} q''_x &= q''_{x+} - q''_{x-} = \rho c_p \bar{C} \left[T \left(x - \frac{\lambda}{2} \right) - T \left(x + \frac{\lambda}{2} \right) \right] \\ &= - \rho c_p \bar{C} \left[T \left(x + \frac{\lambda}{2} \right) - T \left(x - \frac{\lambda}{2} \right) \right] \end{aligned}$$

Multiply the right hand side by (λ/λ) :

$$q''_x = - \rho c_p \bar{C} \lambda \left[\frac{T \left(x + \frac{\lambda}{2} \right) - T \left(x - \frac{\lambda}{2} \right)}{\lambda} \right]$$

$\underbrace{\frac{\partial T}{\partial x}}_{W}$

$$\boxed{q''_x = - \rho c_p \bar{C} \lambda \frac{\partial T}{\partial x}}$$

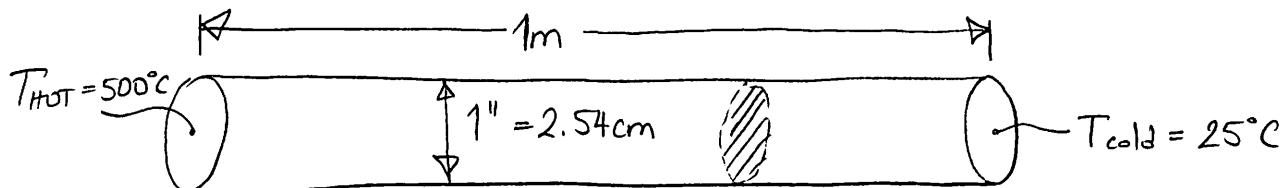
$$k = \rho c_p \bar{C} \lambda \Rightarrow \left[\frac{kg}{m^3} \right] \cdot \left[\frac{J}{kg \cdot K} \right] \left[\frac{kg}{s} \right] \left[m \right] = \left[\frac{J}{ms \cdot K} \right] = \left[\frac{W}{m \cdot K} \right]$$

Typical Values

- $k_{\text{air}} \approx 0.1 \text{ W/m}\cdot\text{K}$
- $k_{\text{water}} = 0.6$
- $k_{\text{polymer}} \approx 0.1$
- $k_{\text{cu}} = 400$
- $k_{\text{Al}} = 300$

Example | Conduction through a bar:

A bar of copper ($k_{Cu} = 400 \text{ W/m}\cdot\text{K}$) contacts a hot source at 500°C on one end. The opposite end of the bar is cooled to ambient ($T_{cold} = 25^\circ\text{C}$) by cooling water. The length of the bar is 1m, and it is cylindrical with a diameter of 1". What is the heat transfer rate? What is the heat flux?



$$q'' = -k \frac{dT}{dx} = - (400 \text{ W/m}\cdot\text{K}) \cdot \left(\frac{25^\circ\text{C} - 500^\circ\text{C}}{1\text{m}} \right)$$

$$= 190\,000 \text{ W/m}^2$$

$q'' = 19 \text{ W/cm}^2$

$$Q = -kA \frac{\partial T}{\partial x} = q'' A = (19 \text{ W/cm}^2) \left(\pi \frac{(2.54\text{cm})^2}{4} \right)$$

$Q = 96.2 \text{ W}$

So how large is this?

$$q''_{boiling} \approx 100 \text{ W/cm}^2$$

$$q''_{nat. convection} \approx 0.01 \text{ W/cm}^2$$

Think about power plants: $\dot{W} = 1000 \text{ MW}$ electrical output
Assuming efficiency of 50% ($\eta = 0.5$)

$$\eta = \frac{\dot{W}}{Q_{in}} \Rightarrow Q_{in} = \dot{W} / \eta = 1000 \text{ MW} / 0.5 = 2000 \text{ MW} = 2,000,000,000 \text{ W!}$$

The previous example was a good demonstration of Fourier's law, but it cannot be used for more complex situations:

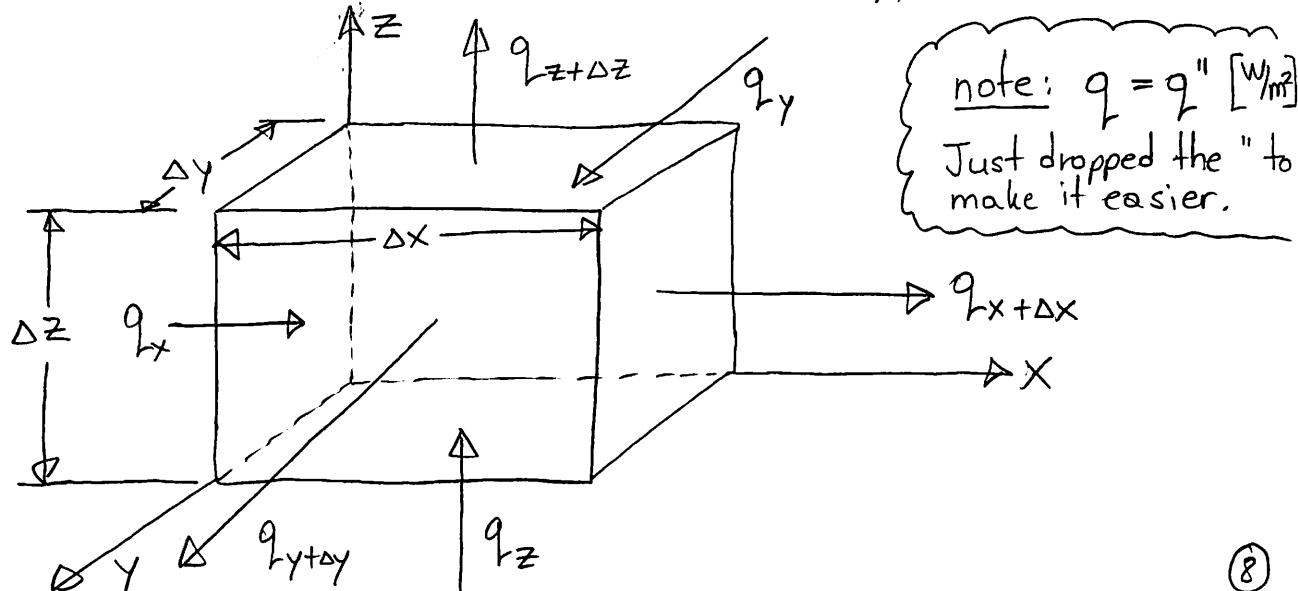
- Transient heat transfer
- Heat loss to the air along the length of the bar
- Radiation effects
- Heat generation.

We need a more general formulation to handle these problems.

Conduction Heat Transfer (Chapter 2 of Textbook)

The equation of conduction heat transfer (heat diffusion eqn.) is the first law of thermodynamics subject to the following assumptions:

- 1) No mass flow in or out of the control volume (CV) or system.
- 2) No change in Kinetic or Potential energy
- 3) There may exist a heat generation term, eg. I^2R from electrical losses, or radioactive decay, etc...



Writing out the first law of thermodynamics:

$$Q_{in} - Q_{out} + Q_{gen} = \dot{Q}_{ST}$$

(Energy conducted into the CV) + (Energy Generated Within the CV) = (The increase in energy in the CV per unit time)

Entering & leaving the surfaces we have:

$$Q_{in} = q_x A_x + q_y A_y + q_z A_z$$

$$Q_{out} = q_{x+\Delta x} A_{x+\Delta x} + q_{y+\Delta y} A_{y+\Delta y} + q_{z+\Delta z} A_{z+\Delta z}$$

For our cartesian coordinate system:

$$A_x = A_{x+\Delta x} = \Delta y \Delta z$$

$$A_y = A_{y+\Delta y} = \Delta x \Delta z$$

$$A_z = A_{z+\Delta z} = \Delta x \Delta y$$

Within our CV, we have energy generation, \dot{Q}'''

$$Q_{gen} = \dot{Q}''' \neq \dot{Q}''' \Delta x \Delta y \Delta z$$

Within the control volume, the change in stored energy is:

$$\dot{Q}_{ST} = \frac{dU}{dt} = \frac{d(Mu)}{dt} = \frac{d(p+u)}{dt}$$

U = internal energy
 u = specific internal energy

Assuming $p = \text{constant}$

$$\dot{Q}_{ST} = p \frac{du}{dt} \Delta x \Delta y \Delta z$$

M = total mass of CV
 p = density

But we know from thermodynamics that :

$$U = M C_p (T - T_{ref}) \Rightarrow \text{Equation of state for a simple incompressible substance with constant specific heat.}$$

or

$$u = C_p (T - T_{ref})$$

Back substituting:

$$Q_{ST} = \rho \frac{d(C_p(T - T_{ref}))}{dt} \Delta x \Delta y \Delta z$$

$T_{ref} = \text{constant}$ & $C_p = \text{constant}$, hence

$$\boxed{Q_{ST} = \rho C_p \frac{dT}{dt} \Delta x \Delta y \Delta z}$$

Putting everything together, we obtain :

$$Q_{in} - Q_{out} = \Delta y \Delta z (q_x - q_{x+\Delta x}) + \Delta x \Delta z (q_y - q_{y+\Delta y}) + \Delta x \Delta y (q_z - q_{z+\Delta z})$$

Using Tailor series expansion :

$$q_{x+\Delta x} = q_x + \frac{dq_x}{dx} \Delta x + \frac{1}{2} \frac{d^2 q_x}{dx^2} \Delta x^2 + \dots + \frac{1}{n!} \frac{d^n q_x}{dx^n} \Delta x^n$$

So

$$(q_x - q_{x+\Delta x}) = - \frac{dq_x}{dx} \Delta x - \frac{1}{2} \frac{d^2 q_x}{dx^2} \Delta x^2 - \dots - \frac{1}{n!} \frac{d^n q_x}{dx^n} \Delta x^n$$

$$(q_y - q_{y+\Delta y}) = - \frac{dq_y}{dy} \Delta y - \frac{1}{2} \frac{d^2 q_y}{dy^2} \Delta y^2 - \dots - \frac{1}{n!} \frac{d^n q_y}{dy^n} \Delta y^n$$

Same for $(q_z - q_{z+\Delta z})$.

Now let's put everything together

$$\begin{aligned}
 Q_{in} - Q_{out} &= -\Delta y \Delta z \left(\frac{dq_x}{dx} \Delta x + \frac{d^2 q_x}{2 dx^2} \Delta x^2 + \dots + \frac{1}{n!} \frac{d^n q_x}{dx^n} \Delta x^n \right) \\
 &\quad - \Delta x \Delta z \left(\frac{dq_y}{dy} \Delta y + \frac{d^2 q_y}{2 dy^2} \Delta y^2 + \dots + \frac{1}{n!} \frac{d^n q_y}{dy^n} \Delta y^n \right) \\
 &\quad - \Delta x \Delta y \left(\frac{dq_z}{dz} \Delta z + \frac{d^2 q_z}{2 dz^2} \Delta z^2 + \dots + \frac{1}{n!} \frac{d^n q_z}{dz^n} \Delta z^n \right)
 \end{aligned}$$

↳ Don't be scared, this will simplify a lot!

$$Q_{in} - Q_{out} + Q_{gen} = Q_{ST}$$

$$\begin{aligned}
 &- \Delta x \Delta y \Delta z \left[\left(\frac{dq_x}{dx} + \frac{1}{2} \frac{d^2 q_x}{dx^2} \Delta x + \dots \right) + \left(\frac{dq_y}{dy} + \frac{1}{2} \frac{d^2 q_y}{dy^2} \Delta y + \dots \right) \right. \\
 &\quad \left. + \left(\frac{dq_z}{dz} + \frac{1}{2} \frac{d^2 q_z}{dz^2} \Delta z + \dots \right) \right] + Q''' \Delta x \Delta y \Delta z = \rho C_p \frac{dT}{dE} \Delta x \Delta y \Delta z
 \end{aligned}$$

Note, we can divide both sides by $\Delta x \Delta y \Delta z$ to simplify

$$\begin{aligned}
 &\left(\frac{dq_x}{dx} + \frac{1}{2} \frac{d^2 q_x}{dx^2} \Delta x + \dots \right) + \left(\frac{dq_y}{dy} + \frac{1}{2} \frac{d^2 q_y}{dy^2} \Delta y + \dots \right) + \left(\frac{dq_z}{dz} + \frac{1}{2} \frac{d^2 q_z}{dz^2} \Delta z + \dots \right) \\
 &+ Q''' = \rho C_p \frac{dT}{dE}
 \end{aligned}$$

Taking the limit of $\Delta x, \Delta y, \Delta z \rightarrow 0$ (differential volume)
many of our terms drop out, & we are left with only:

$$-\left(\frac{dq_x}{dx} + \frac{dq_y}{dy} + \frac{dq_z}{dz}\right) + Q''' = \rho C_p \frac{dT}{dE}$$

But we have already shown that:

$$q_x = -k \frac{\partial T}{\partial x}; \quad q_y = -k \frac{\partial T}{\partial y}; \quad q_z = -k \frac{\partial T}{\partial z}$$

Back substituting, we obtain:

$$\frac{d}{dx} \left(k \frac{dT}{dx} \right) + \frac{d}{dy} \left(k \frac{dT}{dy} \right) + \frac{d}{dz} \left(k \frac{dT}{dz} \right) + Q''' = \rho C_p \frac{dT}{dt}$$

Note though, more rigorously written, we should have $d=2$ since $T=f(x, y, z, t)$

$$\frac{dT}{dx} \Rightarrow \frac{\partial T}{\partial x} \text{ (partial derivative)}$$

↳ Would be valid if $T=f(x)$ only.

Rewriting, we obtain:

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + Q''' = \rho C_p \frac{\partial T}{\partial t}$$

↳ Second order partial differential equation (PDE). Represents the conservation of thermal energy for an isotropic, incompressible substance, with density & specific heat independent of time.

A well posed problem requires two boundary conditions in each coordinate & an initial condition.

If thermal conductivity is isotropic (doesn't depend on location or direction):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

where $\alpha = \frac{k}{\rho C_p}$ \Rightarrow Thermal diffusivity material property.

For constant properties with 1D conduction:

$$\frac{\partial^2 T}{\partial x^2} + \frac{Q'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

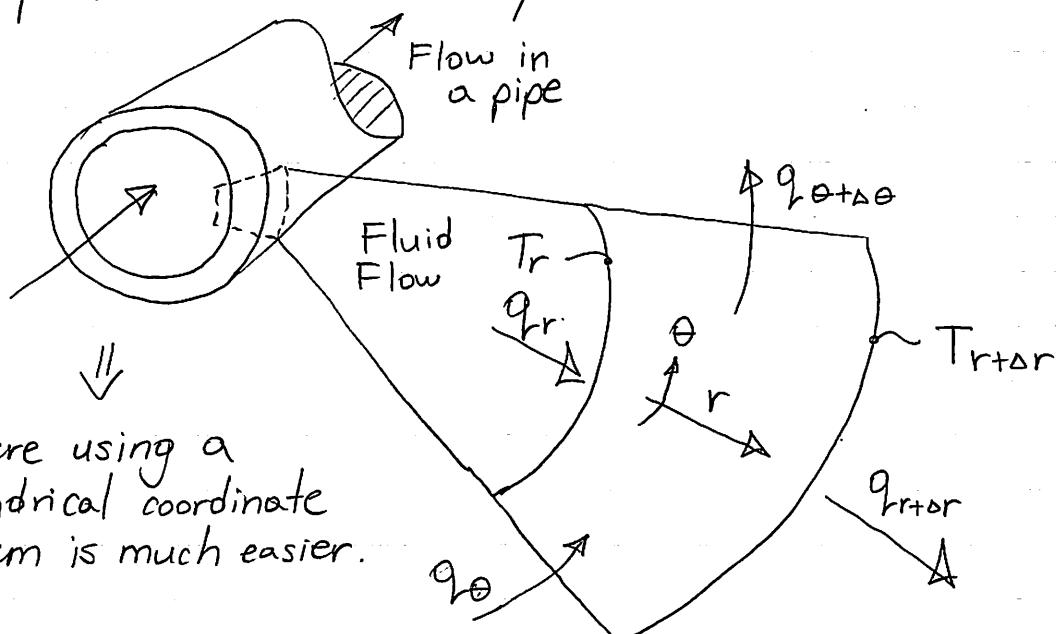
For no heat generation, 1-D conduction:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

For steady state, 1D, & no heat generation:

$$\frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \text{This is probably the most useful form to solve problems.}$$

So far, we've only used cartesian coordinates, but many heat transfer problems are easier to solve with cylindrical or spherical coordinate systems:



Here using a cylindrical coordinate system is much easier.

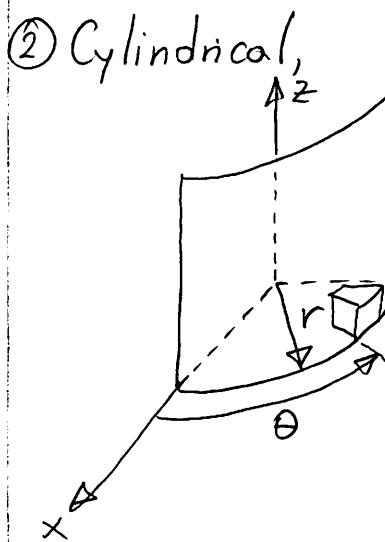
If we look back at our derivation of the heat equation, we can make things more general by:

$dH = ds_1 ds_2 ds_3$ where s_1, s_2, s_3 are the coordinates in consideration

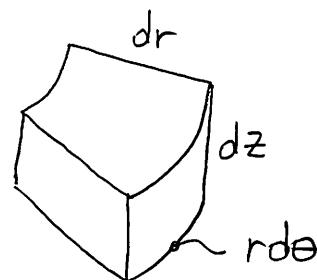
$$\frac{1}{ds_1 ds_2 ds_3} \left\{ \frac{\partial}{\partial s_1} \left(k \frac{\partial s_2 \partial s_3}{\partial s_1} \frac{\partial T}{\partial s_1} \right) ds_1 + \frac{\partial}{\partial s_2} \left(k \frac{\partial s_1 \partial s_3}{\partial s_2} \frac{\partial T}{\partial s_2} \right) ds_2 + \frac{\partial}{\partial s_3} \left(k \frac{\partial s_1 \partial s_2}{\partial s_3} \frac{\partial T}{\partial s_3} \right) ds_3 \right\} + Q''' = \frac{\partial}{\partial t} (\rho c_p T)$$

Specific Cases:

① Cartesian, $\begin{cases} ds_1 = dx \\ ds_2 = dy \\ ds_3 = dz \end{cases}$ } We'll get what we already solved



$$\begin{aligned} ds_1 &= dr \\ ds_2 &= dz \\ ds_3 &= r d\theta \end{aligned}$$



$$\frac{1}{r} \frac{\partial}{\partial r} \left(k r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + Q''' = \rho c_p \frac{\partial T}{\partial t}$$

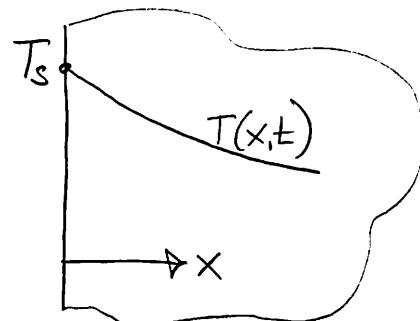
→ Heat equation in cylindrical coordinates.

Boundary Conditions

① Dirichlet (1'st kind)

T = specified on the boundary

$$T(0, t) \stackrel{\text{or}}{=} T_s$$

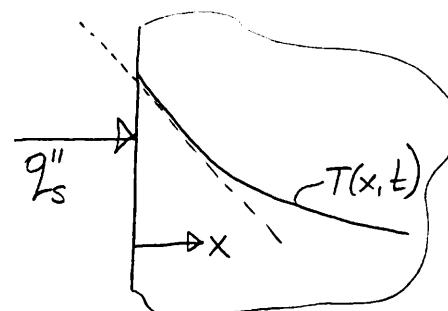


② Neumann (2'nd kind) or Constant surface heat flux

(a) Finite Heat Flux

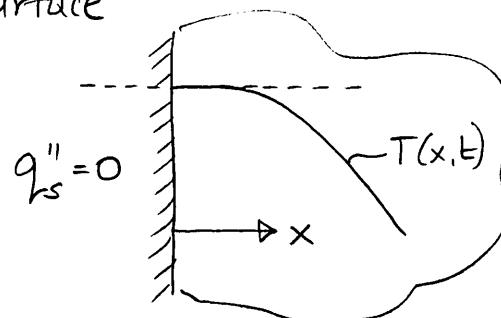
$$-k \frac{\partial T}{\partial x} \Big|_{x=0} = q''_s$$

or



(b) Adiabatic or Insulated Surface

$$\frac{\partial T}{\partial x} \Big|_{x=0} = 0$$



③ Robin (3'rd kind) or Convection surface condition

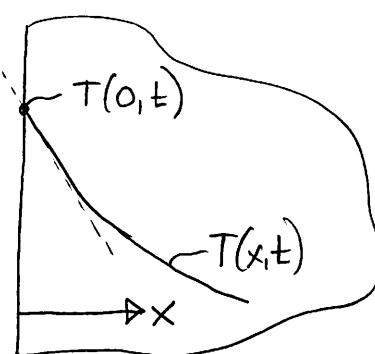
$$-k \frac{\partial T}{\partial x} \Big|_{x=0} = h(T_\infty - T(0, t))$$

Stems from Newton's Law of cooling:

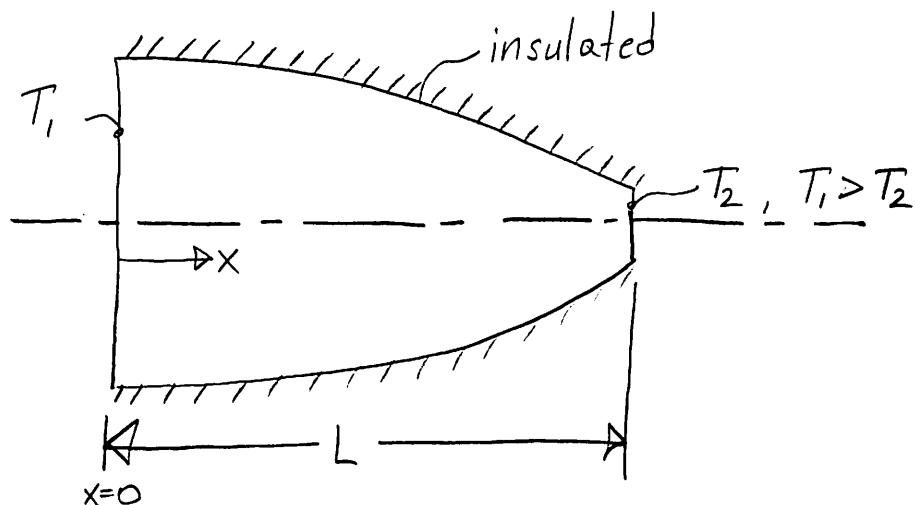
$$q'' = h(T_s - T_\infty)$$

Says heat flux is proportional to some constant h , and temperature difference ΔT .

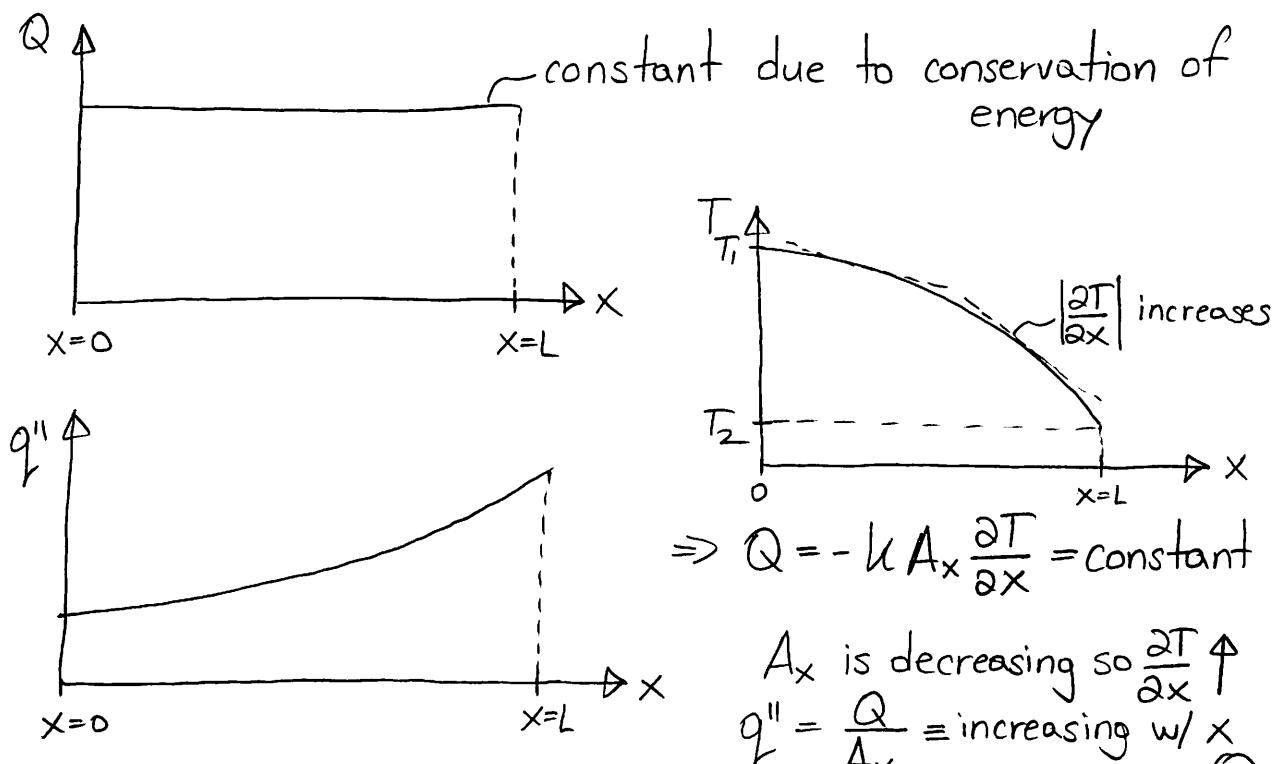
$$T_\infty, h$$



Example] Assume 1-D, steady-state, heat conduction through the axisymmetric shape below. Assume no heat generation & that it is well insulated on the outsides. Assume constant properties.



Sketch the temperature distribution, heat flux distribution, and heat transfer distribution as a function of x . Sketch only, no need for detailed calculations.



Steady-State 1D Conduction (Chapter 3 of textbook)

So far we've developed the governing equation of heat conduction in a stationary medium, and the associated boundary conditions. Let's apply them to some problems.

Assumptions:

- 1) $\vec{V} = 0 \Rightarrow$ Medium is stationary
- 2) Steady state \Rightarrow not a function of time
- 3) $Q''' = 0 \Rightarrow$ no heat generation
- 4) $k = \text{constant} \Rightarrow$ isotropic medium

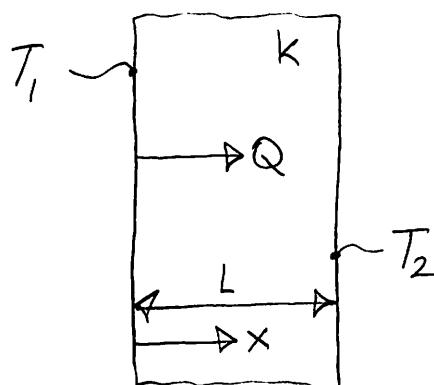
Let's write down our heat equation:

$$\frac{\partial^2 T}{\partial x^2} + \underbrace{\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}}_{1-D \text{ conduction}} + \underbrace{\frac{Q'''}{k}}_{Q''' = 0 \text{ Transient term}} = \frac{1}{k} \cancel{\frac{\partial T}{\partial t}}$$

So we are left with:

$$\frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow \text{We can solve this easily}$$

① Slab



Note; $Q = \text{heat transfer rate [W]}$

We already know our heat equation governs the heat transfer in the slab, so let's solve it.

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad (\text{integrate once})$$

$$\int \frac{\partial^2 T}{\partial x^2} dx = \int Q dx$$

$$\frac{\partial T}{\partial x} = C_1 \quad (\text{Integrate once more})$$

$$\int \frac{\partial T}{\partial x} dx = \int C_1 dx$$

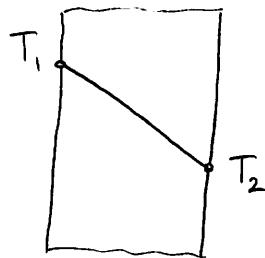
$T = C_1 x + C_2 \Rightarrow$ Now we apply our boundary conditions:

$$T|_{x=0} = T_1 = C_1(0) + C_2 \Rightarrow C_2 = T_1$$

$$T|_{x=L} = T_2 = C_1 L + C_2 = C_1 L + T_1 \Rightarrow C_1 = \frac{T_2 - T_1}{L}$$

$$T = \frac{T_2 - T_1}{L} x + T_1$$

↳ Temperature profile in the solid: linear and decreasing from hot to cold.

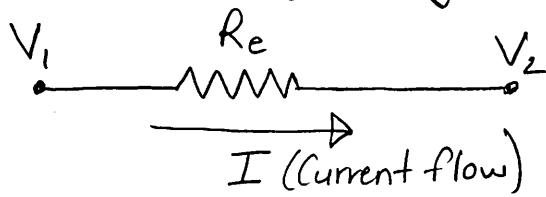


Heat transfer: Fourier's Law

$$Q = -kA \frac{\partial T}{\partial x} = -kA \left(\frac{T_2 - T_1}{L} \right)$$

$$\text{Heat flux: } q'' = -k \left(\frac{T_2 - T_1}{L} \right)$$

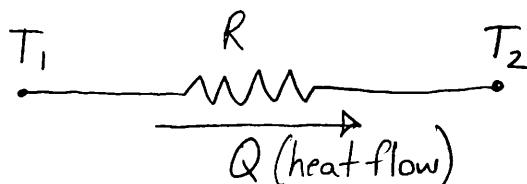
Note here we can see an interesting analogy.
In electrical engineering: Voltage difference



$$I = \frac{V_1 - V_2}{R_e} = \frac{\Delta V}{R_e}$$

Electrical resistance

Can we do the same for heat transfer:



$$Q = \frac{T_1 - T_2}{R} = \frac{\Delta T}{R}$$

Let's see what our thermal resistance would be:

$$Q = -kA \left(\frac{T_2 - T_1}{L} \right) = kA \left(\frac{T_1 - T_2}{L} \right) \Rightarrow \Delta T = T_1 - T_2$$

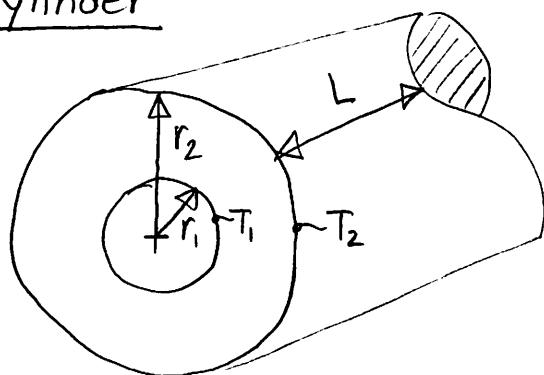
$$Q = kA \frac{\Delta T}{L}$$

But we know we want the form: $Q = \frac{\Delta T}{R}$

$$Q = kA \frac{\Delta T}{L} = \frac{\Delta T}{R}$$

$$\boxed{R = \frac{L}{kA}} \Rightarrow \text{Thermal resistance of a 1D slab}$$

② Cylinder



Writing out our heat equation in radial coordinates: (page. 14)

$$\frac{1}{r} \frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + Q''' = \rho C_p \frac{\partial T}{\partial t}$$

$T = f(r)$ only $T = f(r)$ only $Q''' = 0$ steady-state

$$\cancel{\frac{1}{r} \frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) = 0}$$

$$\frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) = 0 \quad (\text{integrate once})$$

$$\int \frac{\partial}{\partial r} \left(kr \frac{\partial T}{\partial r} \right) dr = \int 0 dr$$

$$r \frac{\partial T}{\partial r} = C_1 \quad (\text{Integrate once more})$$

$$\int \frac{\partial T}{\partial r} dr = \int \frac{C_1}{r} dr \quad * \text{ Remember } \int \frac{dr}{r} = \ln(r)$$

$T = C_1 \ln r + C_2 \Rightarrow$ Now we apply our boundary conditions

$$T \Big|_{r=r_1} = T_1 = C_1 \ln r_1 + C_2 \quad ①$$

$$T \Big|_{r=r_2} = T_2 = C_1 \ln r_2 + C_2 \quad ②$$

Subtract ② from ①

$$T_1 - T_2 = C_1 \ln r_1 - C_1 \ln r_2 + C_2 - C_2$$

$$T_1 - T_2 = C_1 (\ln r_1 - \ln r_2) \Rightarrow \text{Remember } \ln \text{ rules: } \begin{aligned} & (1) \ln(a) - \ln(b) \\ & = \ln(\frac{a}{b}) \end{aligned}$$

$$C_1 = \frac{T_1 - T_2}{\ln(r_1/r_2)} \quad ③$$

Back substitute ③ into ① & solve for C_2

$$C_2 = T_1 - \frac{(T_1 - T_2)}{\ln(r_1/r_2)} \ln(r_1) \quad ④$$

Back substitute ③ & ④ into our solution for T above

$$T = \frac{T_1 - T_2}{\ln(r_1/r_2)} \ln r + T_1 - \frac{(T_1 - T_2) \ln r_1}{\ln(r_1/r_2)}$$

$$T - T_1 = (T_1 - T_2) \left[\frac{\ln r}{\ln(r_1/r_2)} - \frac{\ln(r_1)}{\ln(r_1/r_2)} \right] \Rightarrow \text{Remember the } \ln \text{ rules}$$

$$\frac{T - T_1}{T_1 - T_2} = \frac{\ln(r/r_1)}{\ln(r_1/r_2)} \quad \text{or} \quad \boxed{\frac{T - T_1}{T_2 - T_1} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}} \Rightarrow \text{Radial temperature profile.}$$

Can we do the thermal resistance concept here too?

$$Q = \frac{\Delta T}{R} = -kA \frac{\partial T}{\partial r} \Big|_r = -kA \frac{\partial}{\partial r} \left[\frac{T_1 - T_2}{\ln(r_1/r_2)} \ln r + T_1 - \frac{(T_1 - T_2) \ln r_1}{\ln(r_1/r_2)} \right]$$

We know $\frac{\partial}{\partial r} [\ln r] = \frac{1}{r}$

$$\frac{\partial}{\partial r}(T_1) = 0$$

$$Q = -kA \frac{\partial}{\partial r} \left[\frac{T_1 - T_2}{\ln(r_1/r_2)} \ln r \right] = -kA \frac{(T_1 - T_2)}{\ln(r_1/r_2)} \cdot \frac{1}{r} \quad \frac{\partial}{\partial r}(\dots) = 0$$

But $\ln(r_1/r_2) = -\ln(r_2/r_1)$ \Rightarrow Back substitute into above
Length of tube (axial)

$$Q = kA \frac{T_1 - T_2}{\ln(r_2/r_1)} \cdot \frac{1}{r} \Rightarrow A = 2\pi r L$$

$$Q = 2\pi r L k \frac{T_1 - T_2}{\ln(r_2/r_1)} \cdot \frac{1}{r} = \boxed{2\pi L k \frac{T_1 - T_2}{\ln(r_2/r_1)} = Q}$$

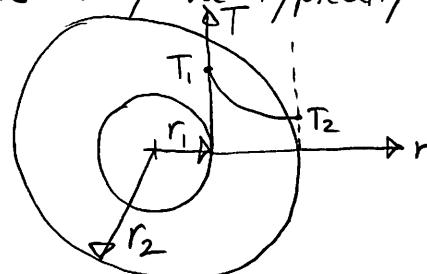
So for resistance, we want $R = \frac{\Delta T}{Q}$

$$R_{cyl} = \frac{T_1 - T_2}{2\pi L k \frac{T_1 - T_2}{\ln(r_2/r_1)}} = \frac{\ln(r_2/r_1)}{2\pi L k}$$

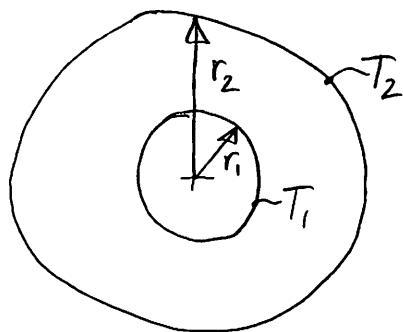
$$\boxed{R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi L k}} \Rightarrow \text{Cylindrical thermal resistance}$$

Note, if $r_2 < r_1$, expression is the same since then $\Delta T = T_2 - T_1$
so negatives would cancel. This is why we typically write:

$$\boxed{R_{cyl} = \frac{|\ln(r_2/r_1)|}{2\pi L k}}$$



③ Sphere



Writing out our spherical coordinate system heat equation:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = 0$$

Integrate once

$$\int \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = \int 0 \, dr$$

$$r^2 \frac{\partial T}{\partial r} = C_1$$

$$\frac{\partial T}{\partial r} = \frac{C_1}{r^2} \quad (\text{Integrate again})$$

$$\int 2T = \int \frac{C_1}{r^2} \, dr$$

$$T = -\frac{C_1}{r} + C_2 \Rightarrow \text{Apply boundary cond.}$$

$$T|_{r_1} = T_1 = -\frac{C_1}{r_1} + C_2 \quad ①$$

$$T|_{r_2} = T_2 = -\frac{C_1}{r_2} + C_2 \quad ②$$

$$\text{Subtract } ② \text{ from } ① \Rightarrow C_1 = \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \Rightarrow \text{Back substitute into } ②$$

$$C_2 = T_2 + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \cdot \frac{1}{r_2}$$

$$T = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \cdot \frac{1}{r} + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \cdot \frac{1}{r_2} + T_2$$

$$\boxed{\frac{T - T_2}{T_1 - T_2} = \frac{\left(\frac{1}{r} - \frac{1}{r_2}\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}}$$

\Rightarrow Spherical temperature profile

Let's try calculating the thermal resistance for the sphere:

$$Q = -kA \frac{\partial T}{\partial r} \Big|_r \Rightarrow \frac{\partial T}{\partial r} = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \cdot \left(-\frac{1}{r^2}\right)$$

$$A = 4\pi r^2$$

$$Q = +k(4\pi r^2) \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \left(-\frac{1}{r^2}\right) = \boxed{\frac{4\pi k(T_1 - T_2)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} = Q}$$

But we need $R = \frac{\Delta T}{Q}$

$$R_{\text{sph}} = \boxed{\frac{\left|\frac{1}{r_1} - \frac{1}{r_2}\right|}{4\pi k}}$$

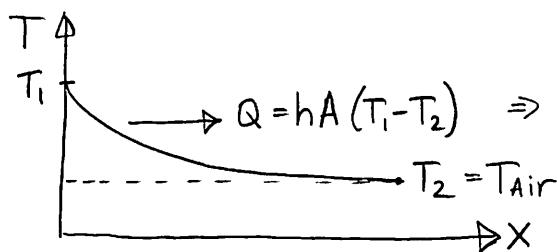
\Rightarrow Spherical thermal resistance

Note, same thing as before with the absolute value rule

④ Convection thermal resistance (while we're on the topic)

It is often very useful to handle convection the same way.
We learned before that: (pg. 15)

$$Q = hA\Delta T ; \quad h = \text{heat transfer coefficient [W/m}^2 \cdot \text{K]}$$



$Q = hA(T_1 - T_2) \Rightarrow$ hot surface exchanging heat w/ fluid

$$Q = hA\Delta T \Rightarrow R = \frac{\Delta T}{Q} = \frac{\Delta T}{hA\Delta T} \Rightarrow$$

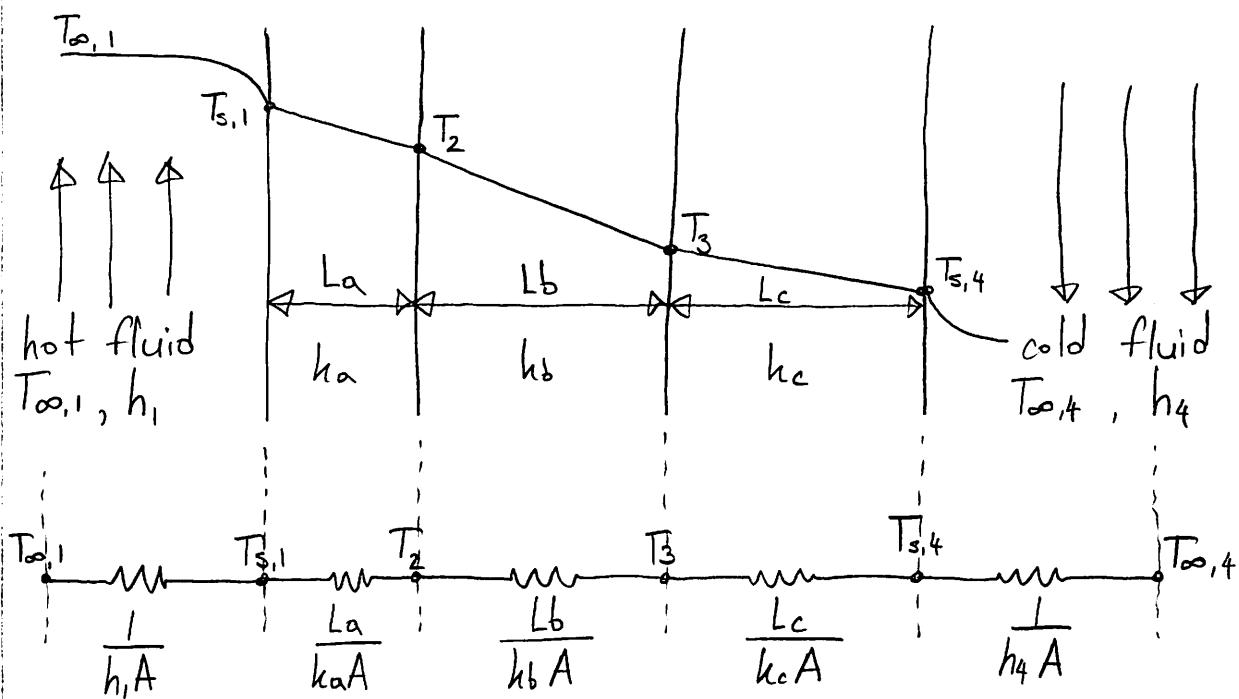
$$\boxed{R_{\text{conv}} = \frac{1}{hA}}$$

\hookrightarrow Thermal resistance associated with convection

Composite Problems

Just like in circuits, we simply add up our thermal resistances to solve for heat transfer and temperature at any node.

For a composite wall:



So if we want the overall heat transfer, we know $T_{\infty,1}$ & $T_{\infty,4}$ and all material properties, we can sum the resistances in series:

$$Q_{TOT} = \frac{\Delta T_{TOT}}{R_{TOT}} = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{h_1 A} + \frac{L_a}{k_a A} + \frac{L_b}{h_b A} + \frac{L_c}{k_c A} + \frac{1}{h_4 A}}$$

Also, we can write:

$$Q_{TOT} = \frac{\Delta T}{R} = \frac{T_{\infty,1} - T_{s,1}}{\frac{1}{h_1 A}} = \frac{T_{s,1} - T_2}{\frac{L_a}{k_a A}} = \frac{T_2 - T_3}{\frac{L_b}{h_b A}} = \dots$$

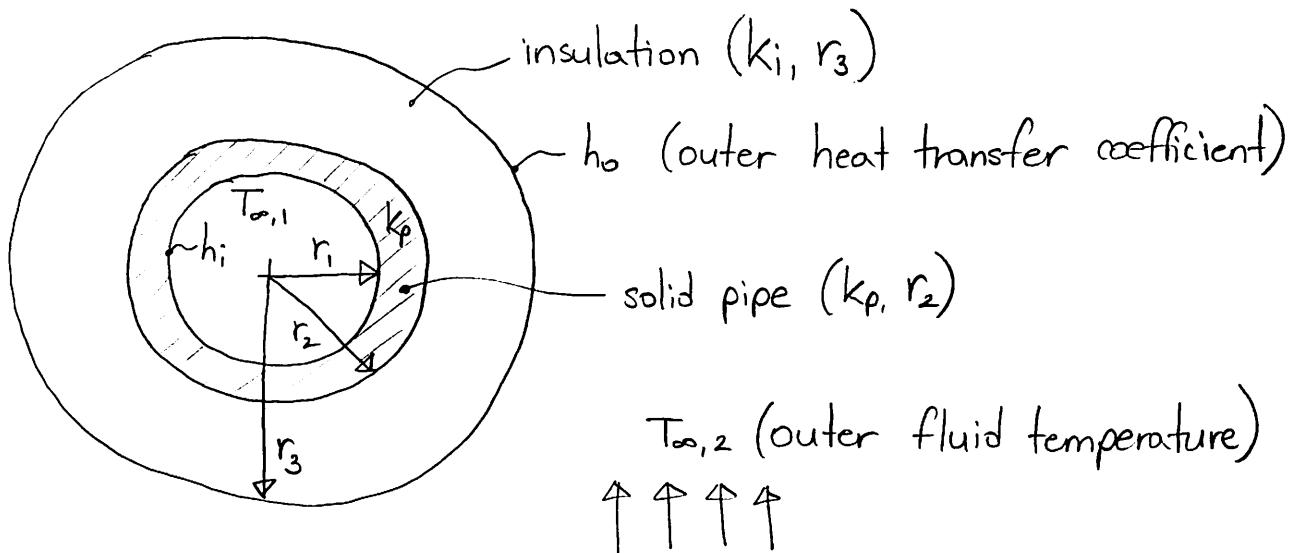
By doing the individual segment approach, we can then solve for intermediate temperatures $T_{s,1}, T_2, T_3, T_{s,4}$ which are typically unknowns in a design problem.

So in general, we can write: (for a wall with n layers)

$$Q = \frac{A (T_{\infty,1} - T_{\infty,n})}{\frac{1}{h_1} + \frac{1}{h_n} + \sum_{j=1}^n \frac{l_j}{k_j}}$$

$$R_{TOT} = \sum_{j=1}^n R_j$$

Critical Thickness of Insulation



What should r_3 be in order to minimize heat transfer from the fluid flowing within the tube to the outside fluid. Let's write out our total thermal resistance:

$$R_{TOT} = \underbrace{\frac{1}{2\pi r_1 h_i L}}_{\text{inner convection}} + \underbrace{\frac{\ln(r_2/r_1)}{2\pi k_p L}}_{\text{pipe conduction}} + \underbrace{\frac{\ln(r_3/r_2)}{2\pi k_i L}}_{\text{insulation conduction}} + \underbrace{\frac{1}{2\pi r_3 h_o L}}_{\text{outer convection}}$$

We can see from our equation that as we increase r_3 , the insulation conduction resistance increases, but the outer convection resistance decreases. So there must be an optimum!

We want to maximize R_{TOT} with respect to r_3 .

Remember, partial derivative

$$\frac{\partial R_{TOT}}{\partial r_3} = 0$$

$$\frac{\partial R_{TOT}}{\partial r_3} = \underbrace{\frac{2}{\partial r_3} \left(\frac{1}{2\pi r_i h_i L} \right)}_{=0 \text{ since } f(r_3)} + \underbrace{\frac{2}{\partial r_3} \left(\frac{\ln(r_2/r_1)}{2\pi h_p L} \right)}_{=0 \text{ since } f(r_3)} + \frac{2}{\partial r_3} \left(\frac{\ln(r_3/r_2)}{2\pi h_o L} \right) + \frac{2}{\partial r_3} \left(\frac{1}{2\pi k_i L} \right)$$

$$\frac{\partial R_{TOT}}{\partial r_3} = 0 = \frac{1}{2\pi k_i L} \frac{\partial}{\partial r_3} \left[\ln(r_3/r_2) \right] + \frac{1}{2\pi h_o L} \frac{\partial}{\partial r_3} \left[\frac{1}{r_3} \right]$$

$$\frac{1}{k_i} \frac{\partial}{\partial r_3} \ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_o} \frac{\partial}{\partial r_3} \left(\frac{1}{r_3}\right) = 0$$

$$\frac{1}{k_i} \cdot \frac{r_2}{r_3 \cdot r_2} + \frac{1}{h_o} \left(-\frac{1}{r_3^2}\right) = 0 \Rightarrow \text{Remember: } \frac{\partial}{\partial x} \ln(x) = \frac{1}{x}$$

$$\frac{1}{k_i} - \frac{1}{h_o r_3} = 0$$

$$\frac{\partial}{\partial x} \ln(ax) = \frac{1}{x} \cdot a$$

$$\boxed{r_{3,crit} = \frac{k_i}{h_o}} \Rightarrow \text{Critical thickness of insulation for a pipe.}$$

So is this a maximum or a minimum?
We can check with a second derivative.

$$\frac{\partial^2 R_{TOT}}{\partial r_3^2} = \left[\frac{2}{\partial r_3} \left(\frac{1}{k_i r_3} \right) - \frac{2}{\partial r_3} \left(\frac{1}{h_o r_3^2} \right) \right] \frac{1}{2\pi L}$$

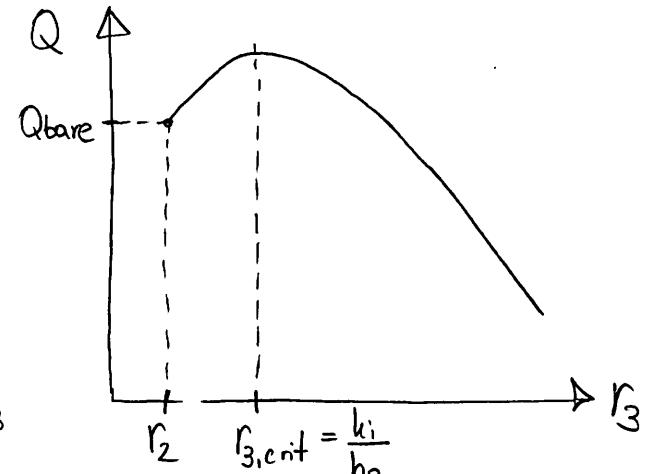
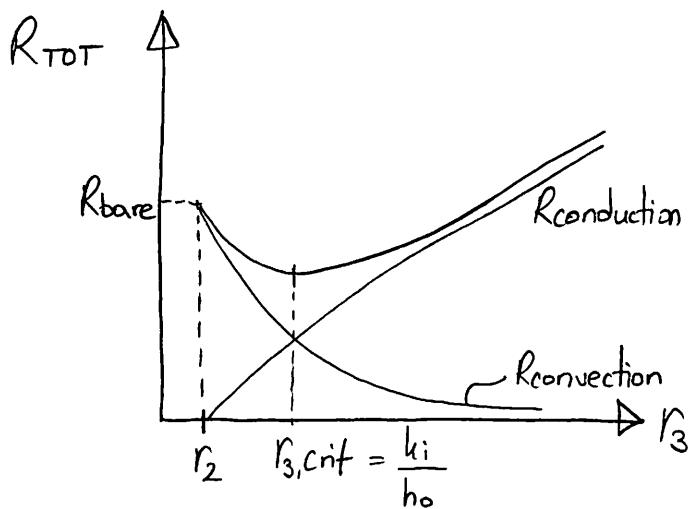
$$\frac{\partial^2 R_{TOT}}{\partial r_3^2} = \left(-\frac{1}{k_i r_3^2} + \frac{2}{h_o r_3^3} \right) \cdot \frac{1}{2\pi L} \quad ①$$

Back substituting $r_{3,crit}$ into equation ①

$$\frac{\partial^2 R_{TOT}}{\partial r_3^2} \Big|_{r_{3,crit}} = \left(-\frac{1}{k_i \left(\frac{k_i^2}{h_o^2} \right)} + \frac{2}{h_o \left(\frac{k_i^3}{h_o^3} \right)} \right) \frac{1}{2\pi L} = \left(-\frac{h_o^2}{k_i^3} + \frac{2h_o^2}{k_i^3} \right) \cdot \frac{1}{2\pi L}$$

$$\frac{\partial^2 R_{TOT}}{\partial r_3^2} \Big|_{r_{3,crit}} = +\frac{h_o^2}{2\pi k_i^3} > 0 \quad (\text{Always})$$

So our calculated $r_{3,crit}$ is a global minimum.
No optimum thickness exists, only a critical insulation thickness.



This result tells us when it is ok to add insulation.
For example, for a hot water pipe, $r_2 = 10\text{ cm}$

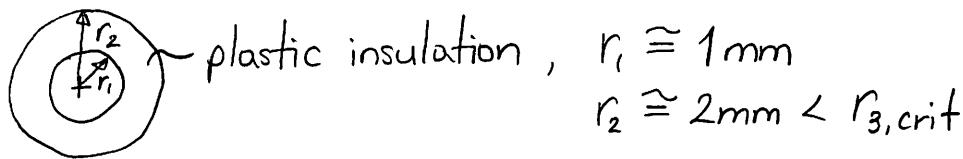
$$\left. \begin{array}{l} k_i = 0.1 \text{ W/m}\cdot\text{K} \\ h_o = 5 \text{ W/m}^2\cdot\text{K} \end{array} \right\} r_{3,crit} = \frac{0.1 \text{ W/m}\cdot\text{K}}{5 \text{ W/m}^2\cdot\text{K}} = 0.02\text{ m} = 2\text{ cm}$$

Here $r_{3,crit} < r_2 \Rightarrow$ means it is OK to insulate!

In general: $k_i \approx 0.1 \text{ W/m}\cdot\text{K}$ (common insulating material)
 $h_o \approx 5 \text{ W/m}^2\cdot\text{K}$ (natural convection)
 $r_{3,\text{crit}} \approx 1 \text{ cm} \Rightarrow$ We typically design our systems (HVAC&R) to be larger than this, hence OK to insulate.

∴ We can insulate hot water and steam pipes without worrying about increasing external heat transfer losses

So how about electrical wires?

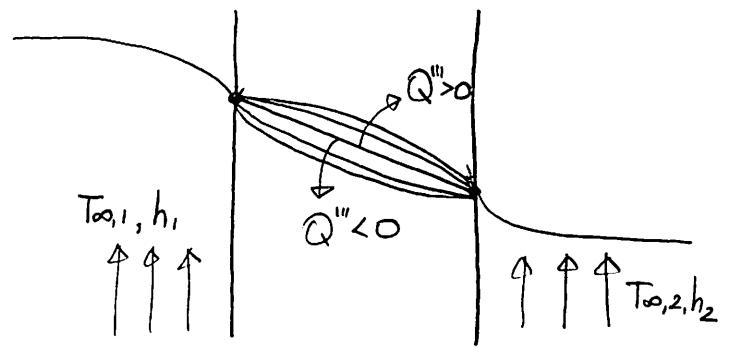


Although $r_2 < r_{3,\text{crit}}$, we typically want to cool our electrical wires & NOT thermally insulate them, so using a bigger or thicker insulation with a lower R_{TOT} is good!

Heat Generation

Assumptions:

- 1) 1D
- 2) Steady State
- 3) Constant properties
- 4) Uniform Q''' ($Q''' = \text{constant}$)



Note, the heat transfer rate and heat flux now need not be constant with respect to x .

Writing out our heat equation: (in cartesian)

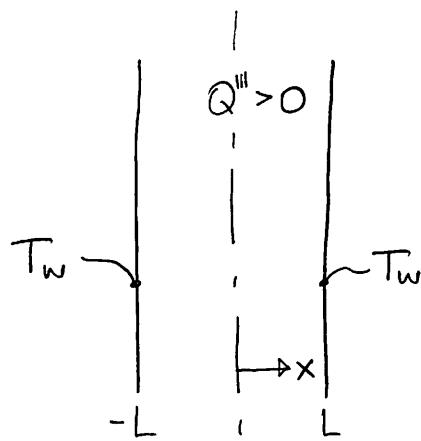
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q'''}{k} = \frac{\rho C_p}{k} \frac{\partial T}{\partial t}$$

$\underbrace{\qquad}_{O(10)}$ $\underbrace{\qquad}_{O(ss)}$

So our equation becomes:

$$\frac{\partial^2 T}{\partial x^2} + \frac{Q'''}{k} = 0 \quad ①$$

Let's do the simplest case BC's (1'st kind)



$$T(x=L) = T_w$$

$$T(x=-L) = T_w$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \quad (\text{Symmetry or insulated})$$

According to symmetry, the temperatures to the left and right of the plane of symmetry are the same, hence $\frac{\partial T}{\partial x} = 0$.

Integrating ①:

$$\frac{\partial^2 T}{\partial x^2} = -\frac{Q'''}{k}$$

$$\int \frac{\partial^2 T}{\partial x^2} dx = \int -\frac{Q'''}{k} dx$$

$$\frac{\partial T}{\partial x} = -\frac{Q'''}{k} x + C_1$$

$$\text{But we know } \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \Rightarrow C_1 = 0$$

$$\text{Integrating once more: } T(x) = -\frac{Q''' x^2}{2k} + C_2$$

We know $T(x = \pm L) = T_w$

$$T(x=L) = T_w = -\frac{Q'''L^2}{2k} + C_2 \Rightarrow C_2 = T_w + \frac{Q'''L^2}{2k}$$

$$T(x) = T_w + \frac{Q'''L^2}{2k} \left(1 - \left(\frac{x}{L}\right)^2\right) \Rightarrow \text{Temperature profile inside the slab.}$$

So where is T_{max} ?

$$\frac{dT}{dx} = -\frac{Q'''x}{k} = 0 \Rightarrow x=0 \text{ is the solution}$$

Let's check if it is a maximum at $x=0$:

$$\frac{\partial^2 T}{\partial x^2} \Big|_{x=0} = -\frac{Q'''}{k} < 0 \quad (\text{So } T \text{ is a max at } x=0)$$

$$T_{max} = T_w + \frac{Q'''L^2}{2k}$$

How about heat flux q'' :

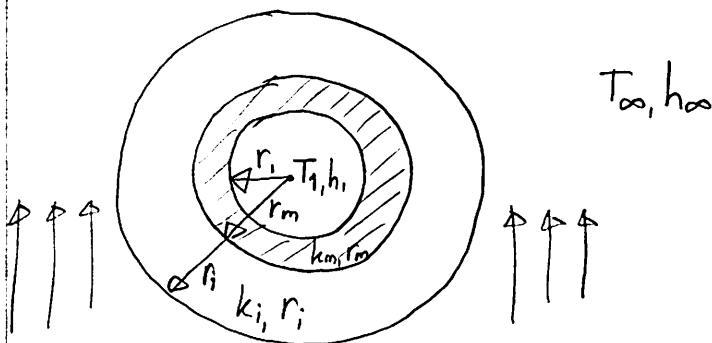
$$q'' = -k \frac{\partial T}{\partial x} = Q'''x = q'' = Q'''x$$

$$\text{At the boundaries: } (x=\pm L) \quad q'' \Big|_{x=L} = Q'''L \Rightarrow \text{makes sense.}$$

Other Cases:

If we had different wall temperatures: $T(x=-L) = T_{s1}$
 We would solve the same problem but $T(x=L) = T_{s2}$
 with the new B.C.'s.

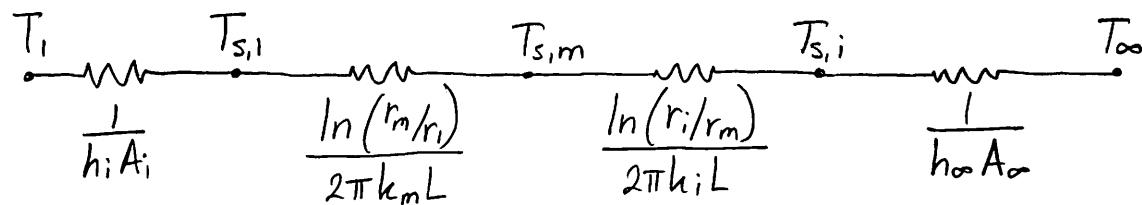
$$T(x) = \frac{Q'''L^2}{2k} \left(1 - \left(\frac{x}{L}\right)^2\right) + \frac{T_{s2} - T_{s1}}{2} \cdot \frac{x}{L} + \frac{T_{s1} + T_{s2}}{2}$$

Example | Pipe flow & freezing water

$$\begin{aligned}T_i &= 10^\circ\text{C}, h_i = 100 \text{ W/m}^2\cdot\text{K} \\r_i &= 2 \text{ cm} \\r_m &= 2.5 \text{ cm}, k_m = 400 \text{ W/m}\cdot\text{K} \\r_i &= 5 \text{ cm}, h_i = 2 \text{ W/m}\cdot\text{K} \\T_{\infty} &= ?, h_{\infty} = 50 \text{ W/m}^2\cdot\text{K}\end{aligned}$$

At what outside temperature T_{∞} , will the water in the pipe begin to freeze?

We can solve this with our thermal resistance approach



We know when freezing starts that $T_{s,1} = 0^\circ\text{C} \Rightarrow$ freezing point of water.

Let's solve the first leg of the resistance diagram:

$$Q = \frac{\Delta T}{R} = \frac{T_i - T_{s,1}}{\frac{1}{h_i A_i}} = \frac{10^\circ\text{C} - 0^\circ\text{C}}{\underbrace{(100 \text{ W/m}^2\cdot\text{K})(2\pi(0.02\text{m})(1\text{m}))}_{h_i A_i}}$$

$$Q = 125.6 \text{ W}$$

(assuming 1m long tube, doesn't matter in the end)

Since we don't have any heat generation, we know that Q is constant in our thermal circuit. So now we can solve for T_{∞} .

$$\begin{aligned}
 R_{TOT} &= \frac{1}{h_i A_i} + \frac{\ln(r_m/r_i)}{2\pi k_m L} + \frac{\ln(r_i/r_m)}{2\pi h_i L} + \frac{1}{h_\infty A_\infty} \\
 &= \frac{1}{(100)(2\pi(0.02)(1))} + \frac{\ln(2.5/2)}{2\pi(400)(1)} + \frac{\ln(5/2.5)}{2\pi(2)(1)} + \frac{1}{(50)(2\pi(0.05)(1))} \\
 &= 0.0796 \text{ K/W} + \underbrace{8.88 \times 10^{-5} \text{ K/W}}_{\text{Thermal resistance of metal}} + \underbrace{0.055 \text{ K/W}}_{\text{Thermal resistance of insulation}} + 0.064 \text{ K/W}
 \end{aligned}$$

$$R_{TOT} = 0.20 \text{ K/W}$$

Now we can solve for T_∞ :

$$Q = \frac{\Delta T_{TOT}}{R_{TOT}} = 125.6 \text{ W} = \frac{10^\circ\text{C} - T_\infty}{0.2 \text{ K/W}}$$

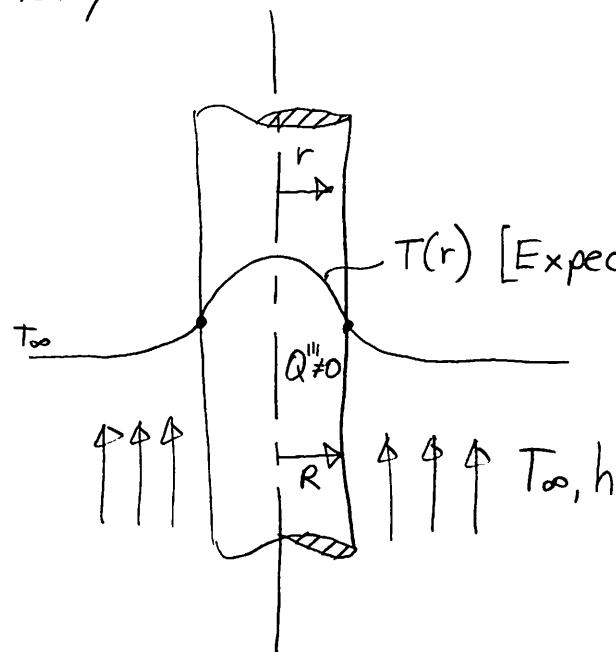
$$10^\circ\text{C} - T_\infty = 24.98^\circ\text{C}$$

$$\boxed{T_\infty = -15^\circ\text{C}}$$

Therefore, we don't need to worry about our pipe freezing until the outside air reaches -15°C . In reality, we probably should add some more insulation just in case since on occasion, temperatures in Illinois can reach below -15°C .

Circular Cylinder with Heat Generation

Many times the constant boundary temperature condition is not valid. Here, we will explore this case in a radial geometry:



- Assumptions:
- 1) SS and 1D
 - 2) Constant properties
 - 3) Uniform heat gen: $Q'' > 0$

Writing out our heat equation in radial form:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(k_r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + Q''' = \rho C_p \frac{\partial T}{\partial t}$$

10 10 SS

$$\frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + Q''' = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = - \frac{Q'''}{k}$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = - \frac{Q'''}{k} r \quad (\text{Integrate wrt. } r)$$

$$\int \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = - \frac{Q'''}{k} \int r dr$$

$$r \frac{\partial T}{\partial r} = - \frac{Q'''}{2k} r^2 + C \quad (\text{Integrate again})$$

$$\int \frac{\partial T}{\partial r} = -\frac{Q'''}{2k} \int r \partial r + \int \frac{C_1}{r} \partial r$$

$$T(r) = -\frac{Q'''r^2}{4k} + C_1 \ln r + C_2$$

Our boundary conditions are a little different now:

$$1) -k \left. \frac{\partial T}{\partial r} \right|_{r=R} = h \left(T \Big|_{r=R} - T_\infty \right) \quad [\text{Energy balance at the surface}]$$

$$2) \left. \frac{\partial T}{\partial r} \right|_{r=0} = 0 \quad [\text{Symmetry of the problem}]$$

Right away we can solve for one of our constants. Since $\lim_{r \rightarrow 0} \ln r = \infty$, then $C_1 = 0$. since we know we have a finite temperature.

The other way to see this is to apply B.C. #2 to our solution

$$\left. \frac{\partial T}{\partial r} \right|_{r=0} = -\frac{Q'''(0)}{2k} + \underbrace{\frac{C_1}{0}}_{\infty}$$

hence the only way this works is if $C_1 = 0$

Now we apply BC #1 to our equation:

$$T(r) = -\frac{Q'''r^2}{4k} + C_2$$

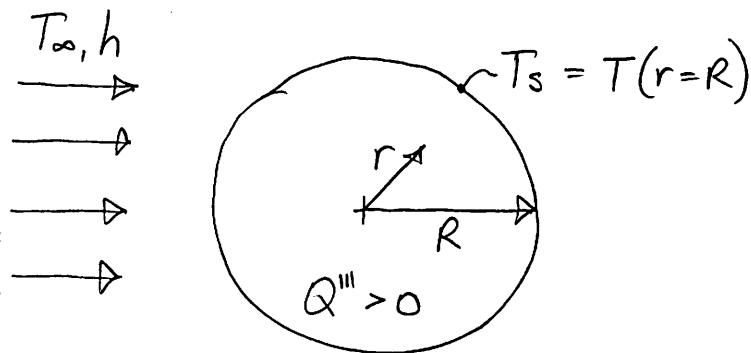
$$-k \left. \frac{\partial T}{\partial r} \right|_{r=R} = +k \frac{Q'''R}{2k} = h \left(-\frac{Q'''R^2}{4k} + C_2 - T_\infty \right) \quad [\text{BC #1}]$$

$$\frac{1}{h} \left[\frac{kQ'''R}{2k} + \frac{hQ'''R^2}{4k} + hT_\infty \right] = C_2$$

$$\frac{1}{h} \left[\frac{Q'''R}{2} \left(1 + \frac{hR}{2k} \right) + hT_\infty \right] = C_2 \Rightarrow \text{Done.}$$

Sphere with Heat Generation

Here we'll do the same approach as with the cylinder but with a different flavor:



- Assumptions:
- 1) SS & 1D
 - 2) Constant properties
 - 3) Uniform heat generation

Writing out our spherical coordinate heat equation with terms already canceled:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{Q'''}{h} = 0$$

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = - \frac{Q'''}{h} r^2 \quad (\text{Integrate once})$$

$$\int \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = - \frac{Q'''}{h} \int r^2 dr$$

$$r^2 \frac{\partial T}{\partial r} = - \frac{Q'''}{h} \frac{r^3}{3} + C_1$$

$$\frac{\partial T}{\partial r} = - \frac{Q'''}{3h} r + \frac{C_1}{r^2} \quad (\text{Integrate again})$$

$$\int \frac{\partial T}{\partial r} = - \int \frac{Q'''}{3h} r dr + \int \frac{C_1}{r^2} dr$$

$$T(r) = - \frac{Q'''}{6h} r^2 - \frac{C_1}{r} + C_2$$

Now we can apply our boundary conditions:

$$1) T(r=R) = T_S \quad [\text{Surface temperature}]$$

$$2) \frac{\partial T}{\partial r} \Big|_{r=0} = 0 \quad [\text{Symmetry of the problem}]$$

B.C. # 2 is handled just like in the radial solution:

$$\frac{\partial T}{\partial r} \Big|_{r=0} = - \frac{Q'''(0)}{3k} + \underbrace{\frac{C_1}{0^2}}_{\infty} = 0$$

\Rightarrow Only way this goes to 0 is if $C_1 = 0$

$$\therefore C_1 = 0$$

Applying B.C. #1:

$$T(r=R) = - \frac{Q'''R^2}{6k} + C_2 = T_S$$

$$\boxed{C_2 = T_S + \frac{Q'''R^2}{6k}}$$

So our solution becomes:

$$\boxed{T(r) = \frac{Q'''R^2}{6k} \left(1 - \frac{r^2}{R^2}\right) + T_S}$$

\Rightarrow But note, we didn't specify T_S , we specified T_∞ & h . So how do we reconcile this.

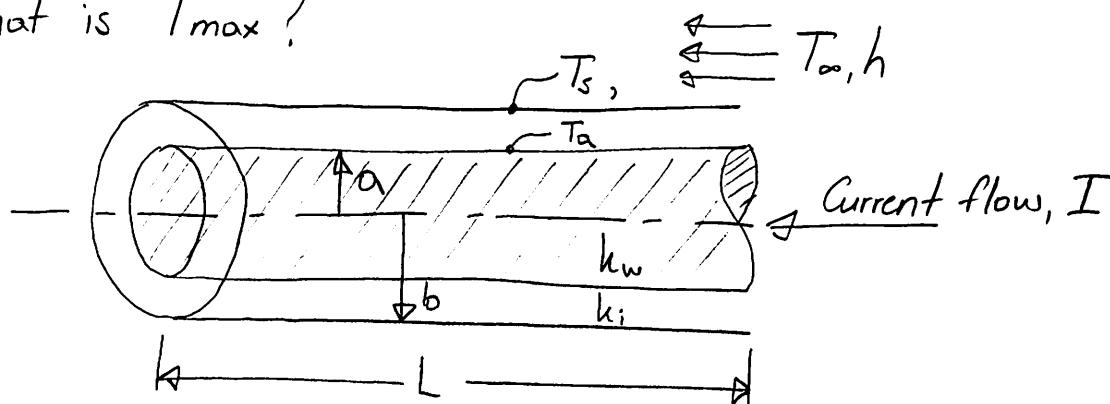
We can solve for T_S with an energy balance at the surface:

$$\text{Total energy generated in the sphere} = \frac{4}{3}\pi R^3 Q'''$$

$$\text{Total heat transfer rate at the surface} = 4\pi R^2 h (T_S - T_\infty)$$

$$\frac{4}{3}\pi R^3 Q''' = 4\pi R^2 h (T_S - T_\infty) \Rightarrow \boxed{T_S = T_\infty + \frac{R Q'''}{3h}}$$

Example / Wire, with electrical resistance R_e and current I , what is T_{max} ?



Assumptions:

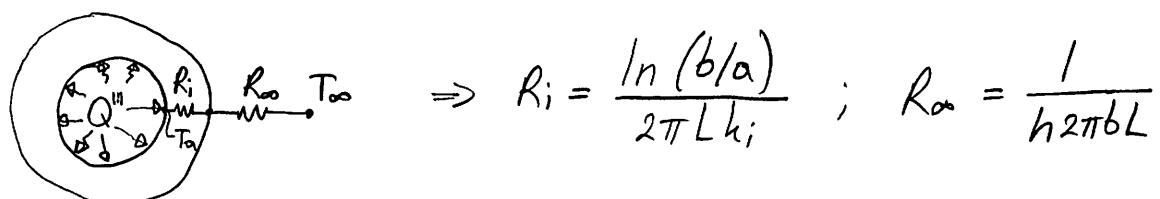
1) 1D & SS

2) Constant properties : k_i = insulation thermal conductivity
 k_w = wire thermal conductivity
 h = heat transfer coefficient on outside of wire.

This problem seems complicated as it involves a heat generation problem with a thermal resistance problem. Let's put the two together:

$$\text{Energy generation in the wire} = \frac{I^2 R_e}{\pi a^2 L} = Q''' \quad [\text{heat generation term}]$$

Now that we have the total heat generation, we can use our thermal resistance approach:



$$Q_{TOT} = \pi a^2 L Q''' = \frac{\Delta T}{R_{TOT}} = \frac{T_a - T_\infty}{\frac{1}{2\pi L k_i} + \frac{1}{2\pi b L h}} \quad (\text{Solve for } T_a)$$

$$T_a = T_\infty + \frac{a^2 Q'''}{2} \left[\frac{1}{k_i} + \frac{1}{bh} \right] \quad (1)$$

So we've solved for T_a but is that the maximum temperature?

Looking back at our notes to the cylindrical solution with heat generation (pg. 34)

$$T(r) = -\frac{Q'''r^2}{4k_w} + C_2$$

Applying our boundary condition here: $T(r=a) = T_a$

$$T_a = -\frac{Q'''a^2}{4k_w} + C_2 \Rightarrow \boxed{C_2 = T_a + \frac{Q'''a^2}{4k_w}}$$

$$T(r) = \frac{Q'''a^2}{4k_w} \left(1 - \frac{r^2}{a^2}\right) + T_a$$

To obtain the maximum temperature in the wire, we can differentiate (note: we already know it is at $r=0$, however its good to be rigorous)

$$\frac{\partial T}{\partial r} = -\frac{Q''2ar}{4k_w} = \frac{2r}{4k_w} = 0 \Rightarrow r=0 \text{ is the maximum}$$

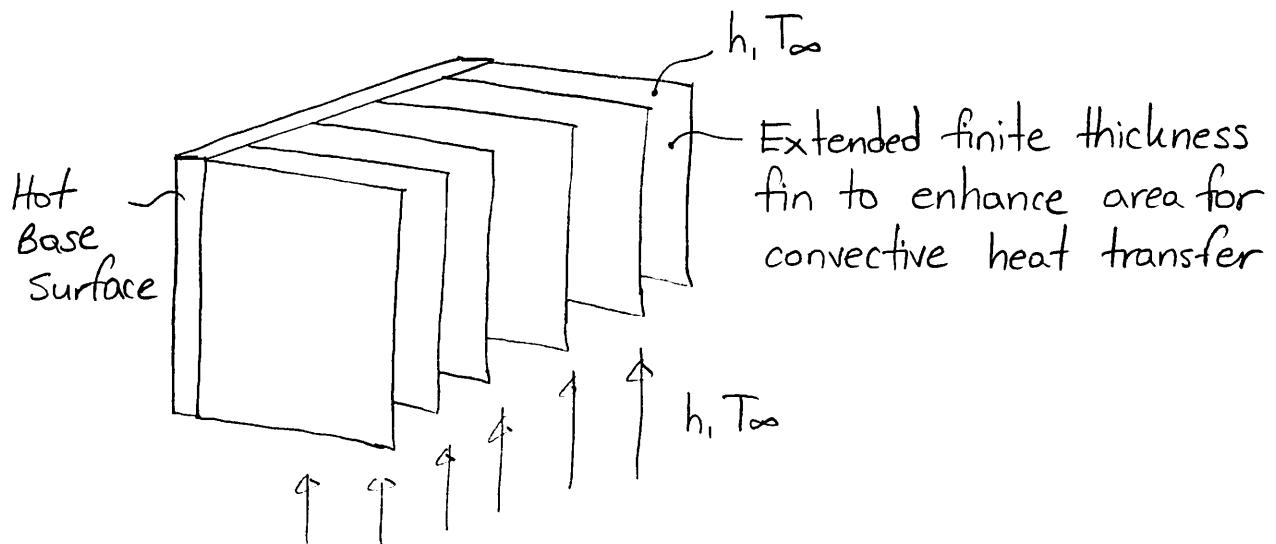
$$\frac{\partial^2 T}{\partial r^2} \Big|_{r=0} = -\frac{2Q'''}{4k_w} < 0 \text{ hence always a maximum for } Q''' > 0$$

Hence: $\boxed{T_{max} = T(r=0) = T_a + \frac{Q'''a^2}{4k_w}} \quad \textcircled{2} \text{ Back sub. into } \textcircled{1}$

$$\boxed{T_{max} = T_\infty + \frac{a^2 Q'''}{2} \left[\frac{1}{2h_w} + \frac{\ln(b/a)}{h_i} + \frac{1}{bh} \right]}$$

Quasi - 1D Conduction : Fins

Fins are generally used when the heat transfer coefficient on the surface is not large and we want to augment heat transfer.

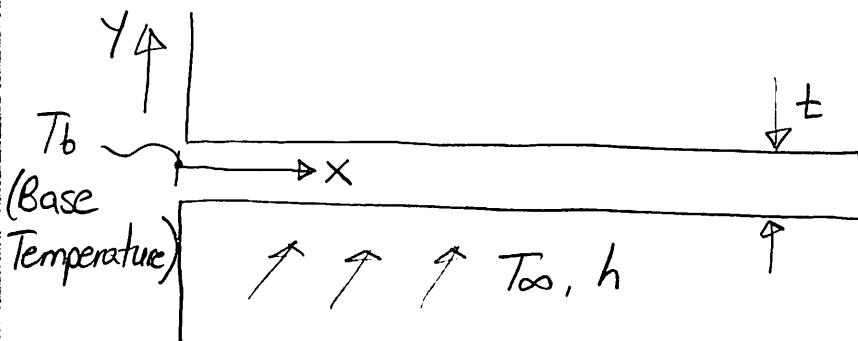


The primary function of the fin is to add surface area for convection heat transfer.

Assumptions:

- 1) SS
- 2) No heat generation
- 3) Fin cross-sectional area & perimeter are constant
- 4) 1D heat transfer

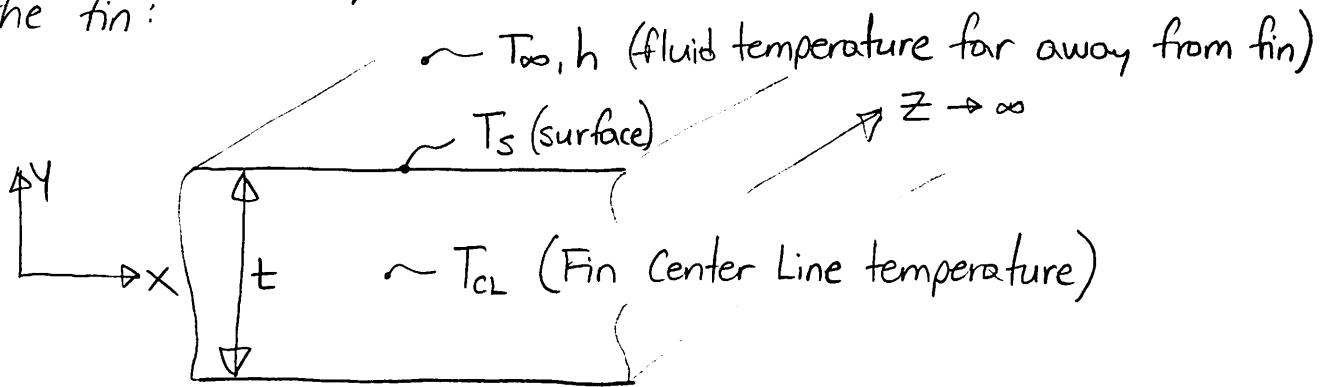
What does assumption ④ mean & how do we ensure it is valid



For 1D heat transfer, our fin temperature should be a function of x-only.

$$T = f(x) \neq f(y)$$

So how do we quantify this? Let's look at a segment of the fin:



Energy balance across the fin (y -direction), we obtain

$$\underbrace{-k \frac{\partial T}{\partial y} (Ldx)}_{\text{Conduction}} = \underbrace{h(Ldx)(T_s - T_\infty)}_{\text{Convection}}$$

$$-k \frac{\partial T}{\partial y} = h(T_s - T_\infty)$$

We can approximate $\frac{\partial T}{\partial y}$ as: $\frac{T_s - T_{CL}}{t/2} = -\frac{(T_{CL} - T_s)}{t/2}$

$$+ h \frac{T_{CL} - T_s}{t/2} = h(T_s - T_\infty)$$

$$\frac{T_{CL} - T_s}{T_s - T_\infty} = + \frac{1}{2} \frac{ht}{k} \approx 0.05 \quad (\text{Set it equal to this})$$

$$\frac{1}{2} \left(\frac{ht}{k} \right) \leq \frac{1}{20} \Rightarrow \boxed{\frac{ht}{k} \leq \frac{1}{10} = Bi_L} \Rightarrow \text{Biot number}$$

So how do we interpret this approximation or condition?

- 1) Temperature changes across the fin thickness are small compared to those external to the fin

2) Internal thermal resistance to conduction heat transfer across the fin (y -direction) is small compared to the external convective heat transfer resistance. $R_{\text{conv}} \gg R_{\text{cond}}$

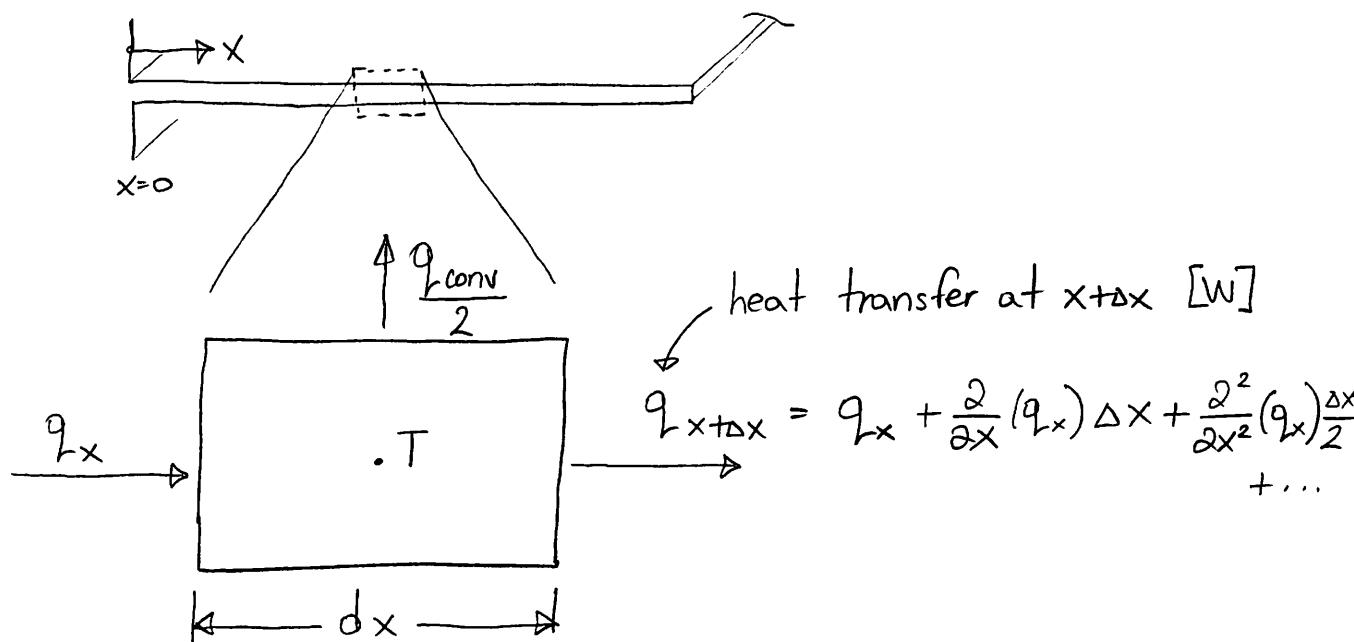
We can double check claim #2 from our previous results:

$$\frac{R_{\text{cond}}}{R_{\text{conv}}} = \frac{\frac{t}{kA}}{\frac{1}{hA}} = \frac{ht}{k} \ll 1$$

$\underbrace{ht}_{B_{\text{it}}} \ll 1$

So now we can model the heat transfer in the x -direction as $10 \Rightarrow$ Makes our lives a lot easier!

Let's take a differential fin element:



$$q_{\text{conv}} = hPdx(T - T_{\infty}) \Rightarrow P = \text{perimeter of the fin}$$

$$q_x = -kA \frac{\partial T}{\partial x}$$

$P_{\Delta x}$ = surface area/unit length
 $P_{\Delta x}$ = area for convective heat transfer

Let's combine the two equations and do an energy balance on our differential element:

$$E_{in} - E_{out} + E_{gen} = E_{STORED}$$

$\downarrow \quad \downarrow$
 $Q'''=0 \quad SS$

$$q_x - \left[q_x + \frac{\partial q_x}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 + \dots \right] - hP \Delta x (T - T_\infty) = 0$$

$$- \frac{\partial q_x}{\partial x} \Delta x - \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 - \dots = hP \Delta x (T - T_\infty)$$

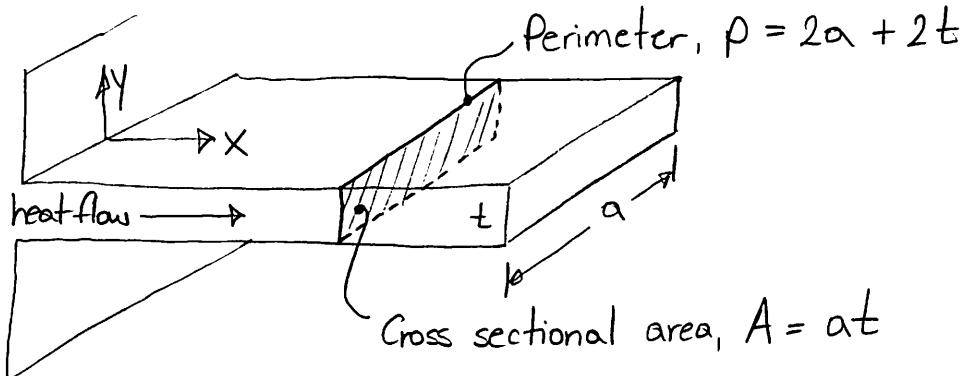
Since we are working with a differential volume: $\Delta x \rightarrow 0$

$$- \frac{\partial q_x}{\partial x} = hP(T - T_\infty) \quad ①$$

$$q_x = -kA \frac{\partial T}{\partial x} = \text{Fourier's Law} \Rightarrow \text{Back sub into } ①$$

$$kA \frac{\partial^2 T}{\partial x^2} = hP(T - T_\infty) \quad ②$$

Note, A is the fin cross sectional area:



To help us solve equation ②, we can assume the following:

$$\underbrace{\theta = T - T_{\infty}}_a ; \quad \frac{d\theta}{dT} = 1 \Rightarrow \underbrace{d\theta = dT}_b$$

Convert equation ② into θ coordinates by back subbing add

$$kA \frac{d^2\theta}{dx^2} - hP\theta = 0$$

$$\frac{d^2\theta}{dx^2} - \frac{hP}{kA}\theta = 0 \Rightarrow \text{Let } m^2 = \frac{hP}{kA}$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \Rightarrow \text{Linear, second order ODE}$$

Remember from calculus class, if you have an equation of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

\nwarrow $n-1$ derivative \searrow a_0 constant

Then you can write a characteristic equation:

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda^1 + a_0 = 0 \quad (\text{polynomial})$$

and solve for the roots of the equation: $\lambda = \dots$

For our case:

$$\begin{aligned} \lambda^2 - m^2 &= 0 \\ \lambda^2 &= m^2 \\ \lambda &= \pm m \end{aligned}$$

So our general solution is: $\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$

We can show that this works by the following:

Assume a solution of the form:

$$\begin{aligned}\theta &= C_1 e^{mx} + C_2 e^{-mx} \\ \theta' &= mC_1 e^{mx} - mC_2 e^{-mx} \quad (\text{First derivative, } \frac{\partial \theta}{\partial x}) \\ \theta'' &= m^2 C_1 e^{mx} + m^2 C_2 e^{-mx}\end{aligned}$$

Back substitute into our ODE: $\theta'' - m^2 \theta = 0$

$$\underbrace{m^2(C_1 e^{mx} + C_2 e^{-mx})}_{\theta''} - m^2(C_1 e^{mx} + C_2 e^{-mx}) = 0$$

θ
 $0=0 \checkmark \text{ (Works)}$

OK, so now back to our problem. How do we solve for C_1 & C_2 (constants). That depends on our B.C.'s

We have a linear second order ODE, so we need 2 B.C.'s

① Infinite Fin ($x \rightarrow \infty$)

Our B.C.'s become: $\theta(x=0) = T_B - T_\infty = \Theta_b$
 $\theta(x \rightarrow \infty) = T_\infty - T_\infty = 0$

As we move out to ∞ , our fin will approach the temperature of the surrounding fluid.

$$\begin{aligned}\theta(x) &= C_1 e^{mx} + C_2 e^{-mx} \\ \theta(\infty) &= 0 = \underbrace{C_1 e^{m(\infty)}}_{C_1=0} + C_2 e^{-m(\infty)}\end{aligned}$$

$C_1 = 0$ for the solution to be valid since $e^\infty \rightarrow \infty$

Applying our first B.C.:

$$\Theta(x=0) = \Theta_b = C_2 e^{-m(0)}$$

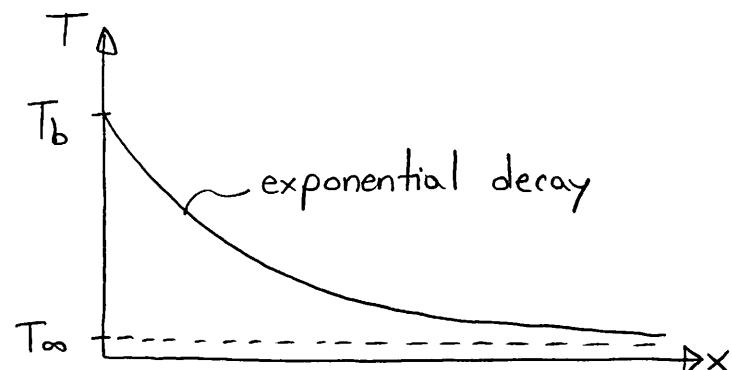
$$e^0 = 1 \Rightarrow C_2 = \Theta_b$$

$$\boxed{\Theta(x) = \Theta_b e^{-mx}}$$

Or in dimensional form:

$$\boxed{\frac{T - T_\infty}{T_b - T_\infty} = e^{-\sqrt{\frac{hP'}{kA}} \cdot x}}$$

\Rightarrow Fin temperature profile:



So how do we calculate heat transfer?

Let's calculate it at the base because all of our energy comes from the base:

$$Q_{net} = Q(x=0) = -kA \frac{\partial \Theta}{\partial x} \Big|_{x=0} = +kA \Theta_b e^{-m(0)} \Big|_1$$

$$\boxed{Q_{net} = q_{net} = +kA \Theta_b \sqrt{\frac{hP'}{kA}} = \sqrt{kAhP'} \Theta_b}$$

Note, if we compare our heat transfer to that of the case with no fin, we get:

$$\frac{Q_{net}}{Q_0} = \frac{\sqrt{kAhP'} \Theta_b}{hA \Theta_b} = \sqrt{\frac{kP'}{hA}} = \sqrt{\frac{k}{ht}} = Bi_t^{-\frac{1}{2}} \gg 1 \text{ since } Bi_t \ll 1$$

Hence adding the fin enhances the heat transfer very much!

Note here I assumed: $\frac{\rho}{A} = \frac{1}{E}$

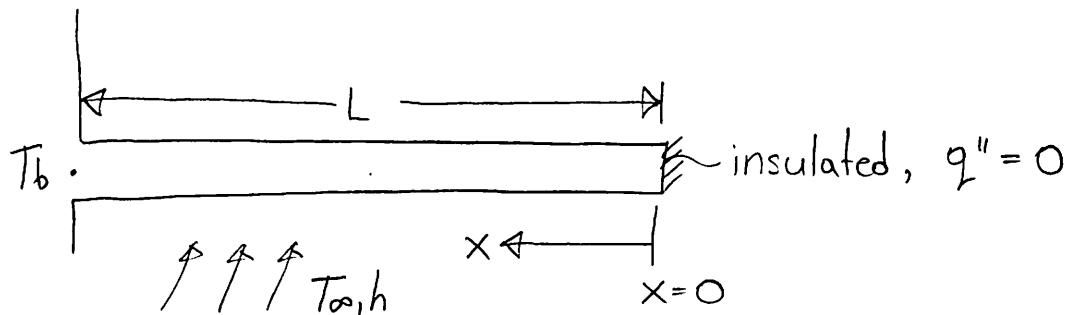
We can do another sanity check:

$$\begin{aligned} Q_{\text{net}} &= \int_0^{\infty} h P \theta dx = \int_0^{\infty} h P \theta_b e^{-mx} dx \\ &= h P \theta_b \left. \frac{e^{-mx}}{m} \right|_0^{\infty} = \frac{h P \theta_b}{m} = \sqrt{k A h P} \theta_b \end{aligned}$$

"Works!"

Same answer as before!

② Insulated Tip ($-k \frac{\partial T}{\partial x} \Big|_{x=0} = 0$)



We know our fundamental differential fin equation remains the same, so we will have the same general solution:

$$\Theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

We can re-write this as:

$$\Theta(x) = C_1 \cosh(mx) + C_2 \sinh(mx)$$

Our new B.C.'s are:

$$1) \quad \Theta(x=L) = \Theta_b ; \quad 2) \quad -k \frac{\partial \Theta}{\partial x} \Big|_{x=0} = 0$$

Remember:

$$\sinh(mx) = \frac{e^{mx} - e^{-mx}}{2}$$

$$\cosh(mx) = \frac{e^{mx} + e^{-mx}}{2}$$

Back substituting our B.C.'s

$$\frac{\partial \Theta}{\partial x} = C_1 m \sinh(mx) + C_2 m \cosh(mx)$$

$$\left. \frac{\partial \Theta}{\partial x} \right|_{x=0} = C_1 m \underbrace{\sinh(0)}_0 + C_2 m \underbrace{\cosh(0)}_1 = 0$$

$$C_2 = 0$$

Applying our first B.C.

$$\Theta(x=L) = \Theta_b = C_1 \cosh(mL)$$

$$C_1 = \frac{\Theta_b}{\cosh(mL)}$$

Back substituting C_1 & C_2 into our solution:

$$\Theta(x) = \frac{\Theta_b \cosh(mx)}{\cosh(mL)}$$

$$\boxed{\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh(mx)}{\cosh(mL)}} \Rightarrow \text{Temperature profile along the fin for insulated tip.}$$

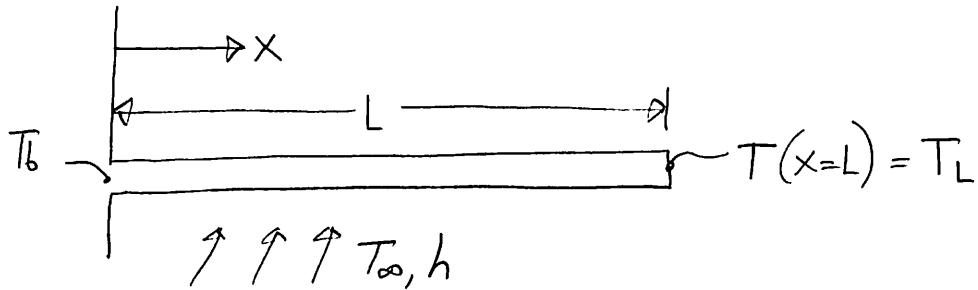
How about heat transfer?

$$Q_{net} = -kA \left. \frac{\partial \Theta}{\partial x} \right|_{x=L} = +kA \Theta_b m \frac{\sinh(mL)}{\cosh(mL)} = kA \Theta_b \underbrace{\frac{hP \sinh(mL)}{kA \cosh(mL)}}_{\tanh(mL)}$$

$$\boxed{Q_{net} = \sqrt{kAhP}(T_b - T_\infty) \tanh(mL)}$$

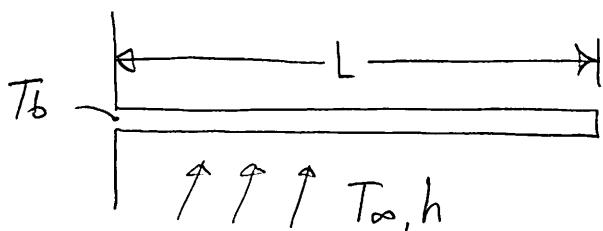
\hookrightarrow Heat transfer from a fin with an insulated tip.

③ Perscribed Tip Temperature ($T(x=L) = T_L$)



$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{T_L - T_\infty}{T_b - T_\infty} \cdot \frac{\sinh(mx) + \sinh[m(L-m)]}{\sinh(mL)}$$

④ Convection at the Tip ($h(T(x=L) - T_\infty) = -k \frac{\partial T}{\partial x} \Big|_{x=L}$)



$$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh(m(L-x)) + \frac{h}{mk} \sinh(m(L-x))}{\cosh(mL) + \frac{h}{mk} \sinh(mL)}$$

Fin Efficiency

We can define a fin efficiency as:

$$\begin{aligned}\eta_{\text{fin}} &= \frac{\text{Actual Heat Transfer with Fin}}{\text{Heat Transfer if } \theta = \theta_b \text{ everywhere on Fin}} \\ &= \frac{q_{\text{actual}}}{q_{\text{ideal}}}\end{aligned}$$

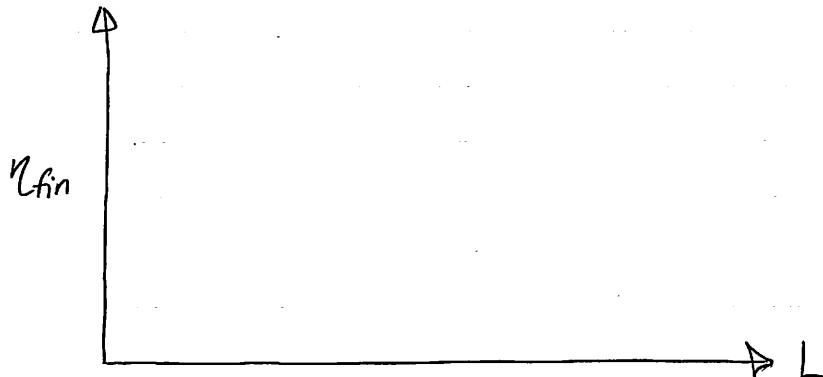
$q_{\text{ideal}} = h(PL)(T_b - T_\infty) \Rightarrow$ Note this is for the insulated tip case.

$$q_{\text{actual}} = \int_0^L hP(T - T_\infty) dx = \sqrt{kA}hP \theta_b \tanh(mL)$$

$$\eta_{\text{fin}} = \frac{\sqrt{kA}hP \theta_b \tanh(mL)}{hPL \theta_b} = \underbrace{\frac{\sqrt{kA}}{hP}}_{1/m} \cdot \frac{\tanh(mL)}{L}$$

$$\boxed{\eta_{\text{fin}} = \frac{\tanh(mL)}{mL}}$$

\Rightarrow Insulated Tip Fin Efficiency
For others, see Table 3.5 of Textbook
(pg. 168)



Fin Resistance ($R = \Delta T/Q$)

$$R_{\text{fin}} = \frac{\theta_b}{q_{\text{actual}}} = \frac{T_b - T_\infty}{q_{\text{actual}}} = \frac{\theta_b}{\sqrt{kA}hP \tanh(mL)\theta_b} = \frac{1}{hA_{\text{fin}}\eta_{\text{fin}}}$$

$$\boxed{R_{\text{fin}} = \frac{1}{hA_{\text{fin}}\eta_{\text{fin}}}} \Rightarrow A_{\text{fin}} = \text{fin outside area (PL)} \\ \eta_{\text{fin}} = \text{fin efficiency}$$

Fin Effectiveness (ϵ_f)

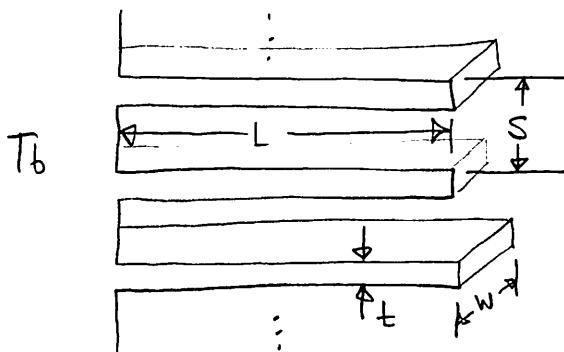
$$\epsilon_f = \frac{\text{actual heat transfer}}{\text{heat transfer if no fin}} = \frac{q_{\text{ideal}} \cdot \eta_{\text{fin}}}{q_{\text{ideal}} \left(\frac{A}{PL}\right)} \Rightarrow \boxed{\epsilon_f = \eta_{\text{fin}} \cdot \frac{PL}{A}}$$

Overall Surface Efficiency (η_o)

The overall surface efficiency characterizes the performance of an array of fins.

$$\eta_o = \frac{q_t}{q_{\max}} = \frac{q_t}{h A_t \theta_b}$$

q_t = total heat transfer of fins & base of area (A_t)



Assuming:

- 1) N = number of fins in array
- 2) A_f = exposed area per fin
- 3) A_b = base area between fins
- 4) η_f = individual fin efficiency

$$A_t = N A_f + A_b$$

$$q_t = N \eta_f h A_f \theta_b + h A_b \theta_b$$

$$q_t = h [N \eta_f A_f + (A_t - N A_f)] \theta_b = h A_t \left[1 - \frac{N A_f}{A_t} (1 - \eta_f) \right] \theta_b$$

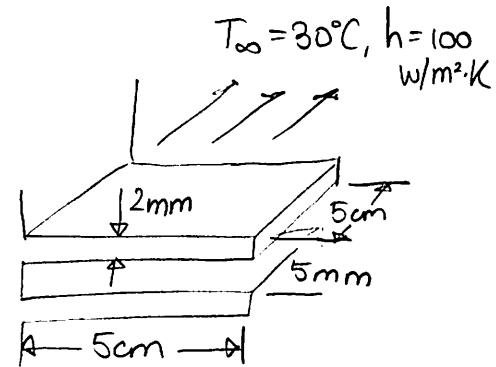
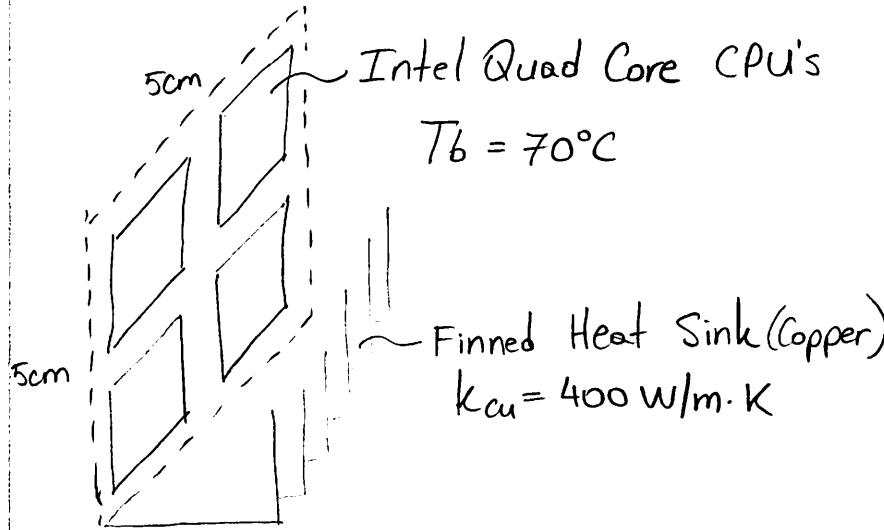
$$\eta_o = 1 - \frac{N A_f}{A_t} (1 - \eta_f)$$

⇒ Overall surface efficiency

We can now also calculate the thermal resistance of a fin array,

$$R_o = \frac{\theta_b}{q_t} = \frac{1}{h A_t \eta_o}$$

⇒ Thermal resistance of a fin array.

Example / CPU cooling

What is the increase in heat transfer due to the use of fins?

Due to the small fin thickness, we can ignore the fin edges, and use 1D fin analysis.

$$A_f = 2w \left(L + \frac{t}{2} \right) = 2(0.05m) \left(0.05m + \frac{0.002m}{2} \right) = 0.0051m^2$$

$$A_t = \underbrace{N(A_f)}_{\substack{\text{Total fin} \\ \text{Area}}} + \underbrace{A_b}_{\substack{\text{Total} \\ \text{Base Area}}} = \left[0.0051m^2 + (0.003m)(0.05m) \right] \underbrace{\left[\frac{0.05m}{0.005m} \right]}_N$$

$$A_t = 0.0525m^2$$

$$\text{But wait, we didn't check the Biot #! } Biot = \frac{ht}{k} = \frac{(100)(0.002)}{400}$$

We know for our individual fin:

$$Biot = 5 \times 10^{-4} \ll 0.1$$

$$\eta_{\text{fin}} = \frac{\tanh(mL)}{mL} = \frac{\tanh\left(\sqrt{\frac{hP}{kA}}L\right)}{\sqrt{\frac{hP}{kA}} \cdot L} = \frac{\tanh\left(\sqrt{\frac{(2)(0.05)(100)}{(400)(0.05)(0.002)}}(0.05)\right)}{\sqrt{\frac{(2)(0.05)(100)}{(400)(0.05)(0.002)}}(0.05)} \xrightarrow{\text{OK!}}$$

$$\eta_0 = 1 - \frac{NA_f}{A_t} (1 - \eta_f) = 1 - \frac{10(0.0051)}{0.0525} (1 - 0.833) = 0.84$$

$$q_t = hA_t \theta_b \eta_0 = (100)(0.0525)(70-30)(0.84) = 176.4 \text{ W}, \quad q_0 = hA_0 \theta_b = \frac{100(10)(0.0051)}{100(0.002)(40)} = 10 \text{ W}$$

Steady Multi-Dimensional Heat Transfer (Shape Factor)

If a problem is steady (ss) and has isothermal surfaces, we can define what's called a "shape factor"

$$Q = Sk\Delta T$$

↳ Obtained analytically or numerically

$$R_{th} = \frac{1}{kS}$$

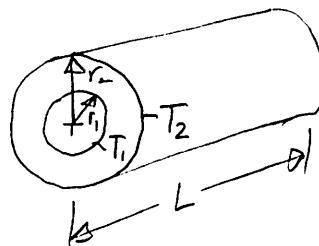
For example, let's see how it works out for a plane wall:

$$R_{wall} = \frac{L}{kA} = \frac{1}{kS} \Rightarrow S_{wall} = \frac{A}{L}$$

$$Q = kA \frac{\Delta T}{L} = kA \frac{\partial T}{\partial x} \Rightarrow \text{Makes sense}$$

How about a cylinder:

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi k L}$$



$$S_{cyl} = \frac{1}{k R_{cyl}} = \frac{2\pi L}{\ln(r_2/r_1)}$$

⇒ Very simple and in general not needed for simple geometries as R_{th} is available & our previous analysis holds.

However for more complex shapes, it is a very useful concept.

Table 5.4 Conduction shape factors: $Q = S k \Delta T$, $R_t = 1/(kS)$.

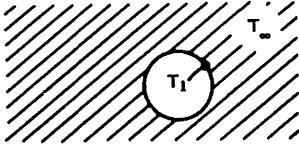
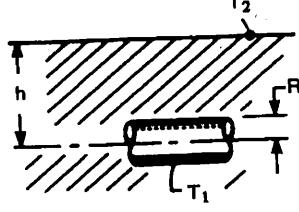
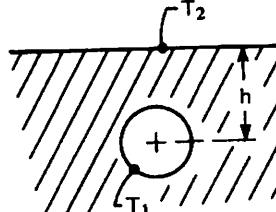
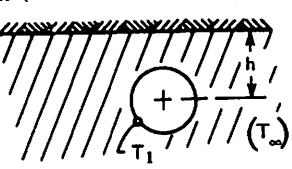
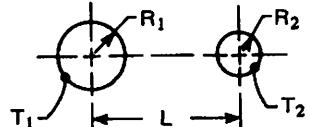
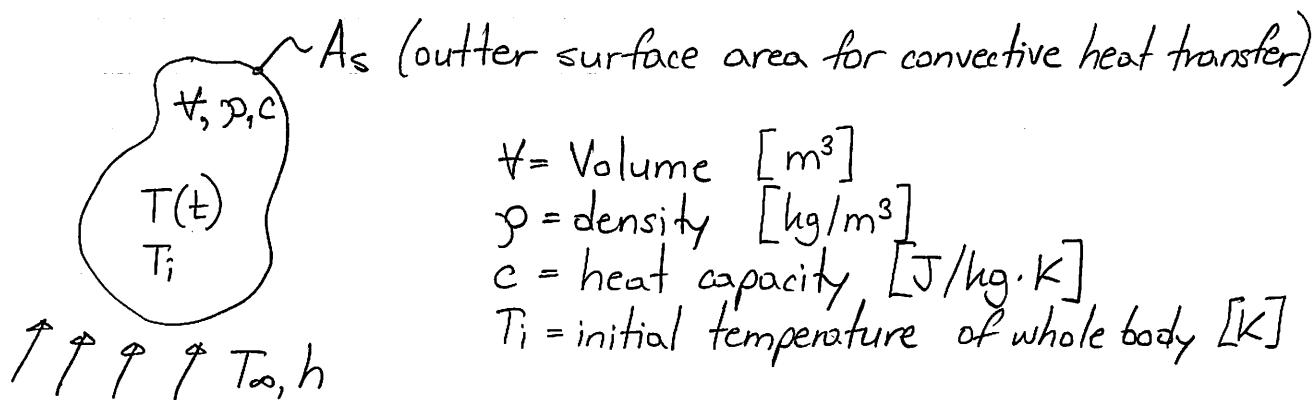
Situation	Shape factor, S	Dimensions	Source
1. Conduction through a slab	A/L	meter	Example 2.2
2. Conduction through wall of a long thick cylinder	$\frac{2\pi}{\ln(r_o/r_i)}$	none	Example 5.9
3. Conduction through a thick-walled hollow sphere	$\frac{4\pi(r_o r_i)}{r_o - r_i}$	meter	Example 5.10
4. The boundary of a spherical hole of radius R conducting into an infinite medium	$4\pi R$	meter	Problems 5.19 and 2.15
			
5. Cylinder of radius R and length L , transferring heat to a parallel isothermal plane; $h \ll L$	$\frac{2\pi L}{\cosh^{-1}(h/R)}$	meter	[5.16]
			
6. Same as item 5, but with $L \rightarrow \infty$ (two-dimensional conduction)	$\frac{2\pi}{\cosh^{-1}(h/R)}$	none	[5.16]
7. An isothermal sphere of radius R transfers heat to an isothermal plane; $R/h < 0.8$ (see item 4)	$\frac{4\pi R}{1 - R/2h}$	meter	[5.16, 5.17]
			

Table 5.4 Conduction shape factors: $Q = S k \Delta T$, $R_t = 1/(kS)$ (*con't.*)

Situation	Shape factor, S	Dimensions	Source
8. An isothermal sphere of radius R , near an insulated plane, transfers heat to a semi-infinite medium at T_∞ (see items 4 and 7)	$\frac{4\pi R}{1 + R/2h}$	meter	[5.18]
			
9. Parallel cylinders exchange heat in an infinite conducting medium	$\frac{2\pi}{\cosh^{-1} \left(\frac{L^2 - R_1^2 - R_2^2}{2R_1 R_2} \right)}$	none	[5.6]
			
10. Same as 9, but with cylinders widely spaced; $L \gg R_1$ and R_2	$\frac{2\pi}{\cosh^{-1} \left(\frac{L}{2R_1} \right) + \cosh^{-1} \left(\frac{L}{2R_2} \right)}$	none	[5.16]
11. Cylinder of radius R_i surrounded by eccentric cylinder of radius $R_o > R_i$; centerlines a distance L apart (see item 2)	$\frac{2\pi}{\cosh^{-1} \left(\frac{R_o^2 + R_i^2 - L^2}{2R_o R_i} \right)}$	none	[5.6]
12. Isothermal disc of radius R on an otherwise insulated plane conducts heat into a semi-infinite medium at T_∞ below it	$4R$	meter	[5.6]
13. Isothermal ellipsoid of semimajor axis b and semiminor axes a conducts heat into an infinite medium at T_∞ ; $b > a$ (see 4)	$\frac{4\pi b \sqrt{1 - a^2/b^2}}{\tanh^{-1} \left(\sqrt{1 - a^2/b^2} \right)}$	meter	[5.16]

Lumped Capacitance Analysis (Transient Problems)

This analysis method is very valuable for a whole host of problems



Doing an energy balance on our body of interest:

$$\cancel{E_{in}} - \cancel{E_{out}} + \cancel{E_{gen}} = E_{stored}$$

$$-hA_s(T - T_{\infty}) = \frac{\partial}{\partial t} (\rho V c T) \Rightarrow \text{Assume constant properties}$$

$$\cancel{\rho V c} \frac{\partial T}{\partial t} + hA_s(T - T_{\infty}) = 0$$

$$\frac{\partial T}{\partial t} + \frac{hA_s}{\cancel{\rho c}} (T - T_{\infty}) = 0 \Rightarrow \text{Let } \Theta = T - T_{\infty}$$

$$\frac{\partial \Theta}{\partial t} + \frac{hA_s}{\cancel{\rho c}} \Theta = 0 \Rightarrow \text{Let } \lambda = \frac{hA_s}{\cancel{\rho c}}$$

$$\frac{\partial \Theta}{\Theta} = -\int \lambda dt$$

$$\ln \Theta = -\lambda t + C_2$$

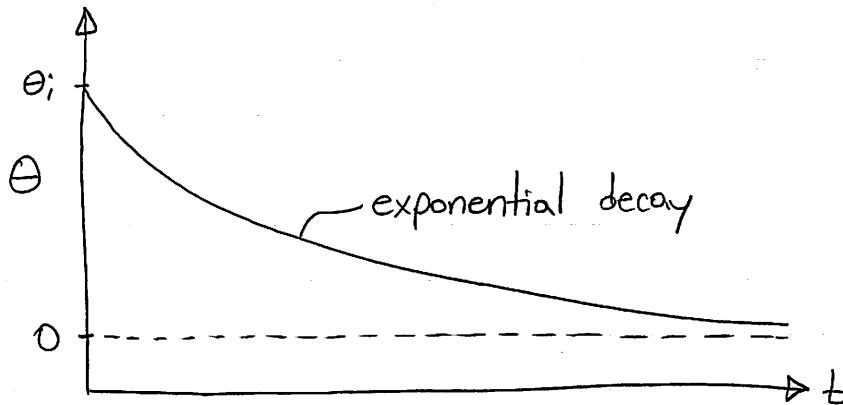
$$e^{\ln \Theta} = e^{-\lambda t + C_2} \Rightarrow \Theta(t) = C_3 e^{-\lambda t}$$

Our B.C. or I.C. is: $\Theta(t=0) = \Theta_i = T_i - T_\infty$

$$\Theta(t=0) = C_3 \underbrace{e^{-\frac{(0)}{1}}}_{1} = \Theta_i \Rightarrow C_3 = \Theta_i$$

$$\Theta(t) = \Theta_i e^{-\frac{hA_s t}{\kappa c}} \Rightarrow \text{Lumped capacitance model.}$$

Exponential decay in body temp.



Note, the lumped capacitance model is only valid for bodies that change temperature uniformly as they cool or heat up.

$$Bi \leq 0.1 \quad t \equiv \text{slab thickness not time!}$$

Remember, for a slab: $Bi_t = \frac{ht}{k} \leq 0.1$

What about a cylinder or sphere? What thickness do we use in our Bi ? slab thickness

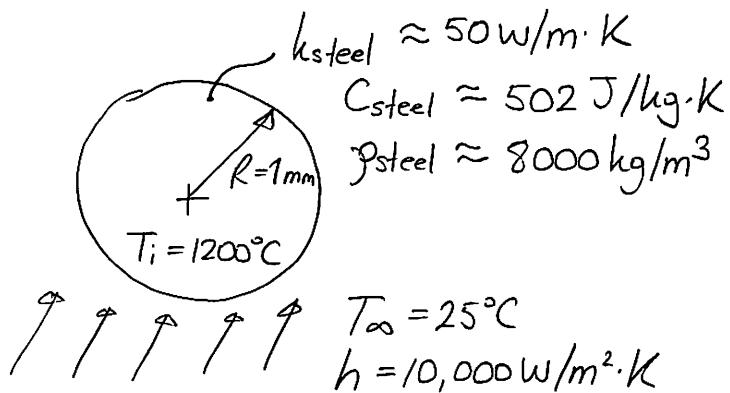
$$t = \frac{\text{Volume}}{\text{Surface Area}} \Rightarrow \text{Slab} \Rightarrow \frac{t \cdot L \cdot W}{2(L \cdot W)} = t ; Bi_{\text{slab}} = \frac{ht}{k}$$

Sometimes labeled
Lc for length scale.

$$\text{Cylinder} \Rightarrow \frac{\pi R^2 K}{2\pi R K} = \frac{R}{2} ; Bi_{\text{cyl}} = \frac{hR}{2k}$$

$$\text{Sphere} \Rightarrow \frac{\frac{4}{3}\pi R^3}{4\pi R^2} = \frac{R}{3} ; Bi_{\text{sph}} = \frac{hR}{3k}$$

Example | A steel quenching process for ball bearings requires the ball bearing to reach 100°C in order to work. The initial ball temperature is 1200°C , and it is dipped in cold water at 25°C . The ball radius is $R = 1\text{ mm}$. The heat transfer coefficient is $h = 10,000 \text{ W/m}^2 \cdot \text{K}$. Find how long the quenching time should be.



First step : Check the B_{sphere} ≤ 0.1

$$B_{\text{sph}} = \frac{hR_{\text{sph}}}{k} = \frac{(10,000 \text{ W/m}^2 \cdot \text{K})(0.001\text{m})}{3(50 \text{ W/m} \cdot \text{K})} = 0.066 \leq 0.1$$

$\therefore B_{\text{sph}} \leq 0.1 \Rightarrow$ Lumped Capacitance OK to use.

We just solved that :

$$\Theta(t) = T - T_\infty = (T_i - T_\infty) e^{-\frac{hAt}{\rho C}}$$

$$\frac{100^{\circ}\text{C} - 25^{\circ}\text{C}}{1200^{\circ}\text{C} - 25^{\circ}\text{C}} = e^{-\frac{(10,000)(3)t}{(0.001)(8000)(502)}}$$

$$\begin{aligned} 0.064 &= e^{-7.47t} \\ \ln(0.064) &= -7.47t \\ t &= 0.375 \end{aligned}$$

If we look back at our solution for lumped capacitance

$$\frac{\Theta}{\Theta_i} = \exp \left[-\frac{h A_s t}{\rho C} \right]$$

Dimensionless Dimensionless \Rightarrow Implies that:

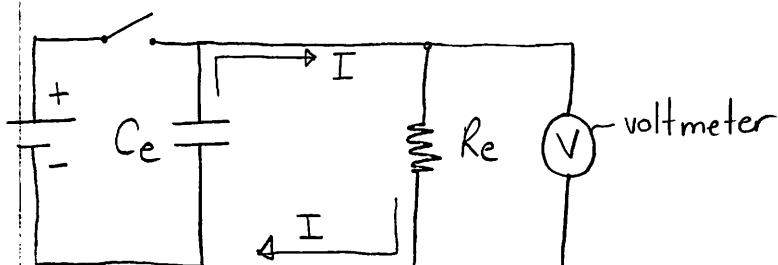
$$\tau = \left(\frac{1}{h A_s} \right) (\rho + c) = R_t \cdot C_t \quad (\text{Thermal time constant})$$

Convective Thermal
 Resistance Capacitance
 $= R_t$ $= C_t$

The larger τ is, the slower the response of the body.

Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment.

Analogous to the voltage decay when discharging a capacitor through a resistor in an RC circuit.



$$V(t) = V_0 e^{-\frac{t}{R_e C_e}} \quad \text{solution}$$

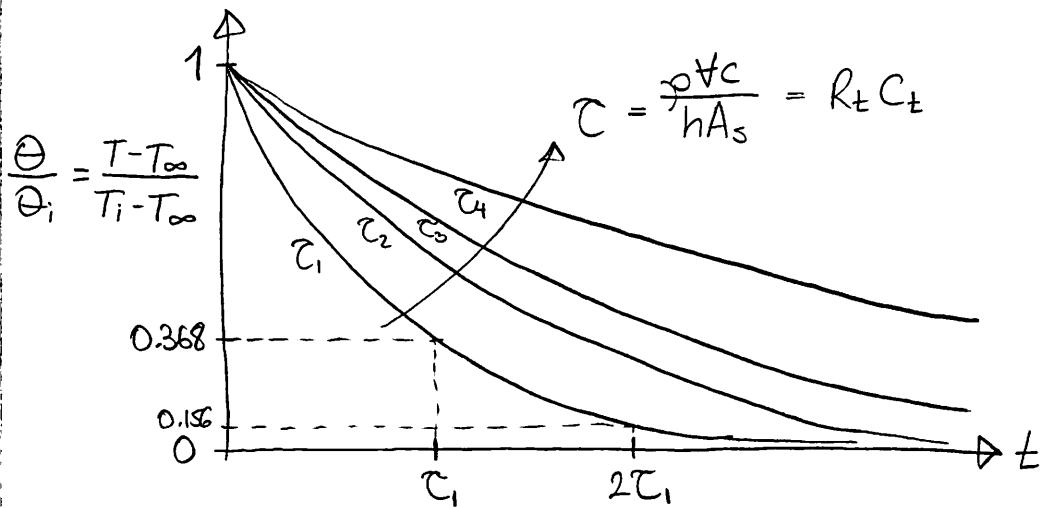
$$V = I R_e \quad (\text{Voltage})$$

$$C_e \frac{dV}{dt} + \frac{V}{R_e} = 0$$

Current at Current through
capacitor resistor

- ② - The larger the capacitance C_e , the more energy stored and the slower the voltage decay
- The larger the thermal capacitance $(\rho + c)$, the more energy stored and the slower the temperature decay
- The larger the resistor (electrical or thermal), the smaller the current (electrical or thermal) and the slower the temp decay

We can now draw our solutions in dimensionless form:



We can take the analysis one step further and say:

$$\frac{hA_s t}{\rho \kappa c} = \frac{ht}{\rho c L_c} \Rightarrow L_c = \sqrt{\frac{t}{A_s}} \text{ (length scale)}$$

$$= \frac{h L_c}{k} \cdot \frac{k}{\rho c} \cdot \frac{t}{L_c^2} = \underbrace{\frac{h L_c}{k}}_{Bi} \cdot \underbrace{\frac{\alpha t}{L_c^2}}_{F_o} \Rightarrow \frac{\Theta}{\Theta_i} = e^{-Bi F_o}$$

↳ Lumped capacitance analysis in dimensionless form.

$$F_o = \text{Fourier \#} = \frac{\alpha t}{L_c^2} \Rightarrow \text{Dimensionless time that characterises transient heat transfer prob.}$$

$$\alpha = \text{Thermal diffusivity} = \frac{k}{\rho c} \Rightarrow \text{Describes how quickly a material responds to a change in temperature. (Thermal inertia)}$$

Note:

$$F_o = \frac{\text{Diffusive transport rate}}{\text{Storage rate}}$$

$$\left. \begin{aligned} F_o &= \frac{\alpha t}{L_c^2} = \frac{kt}{\rho c L_c^2} \cdot \left(\frac{\Delta T}{\Delta T} \right) \\ &= \left(\frac{k \Delta T}{L} \right) / \left(\frac{\rho c \Delta T L}{t} \right) \\ &= q''_{\text{cond}} / q''_{\text{stored}} \end{aligned} \right\}$$

Transient Conduction

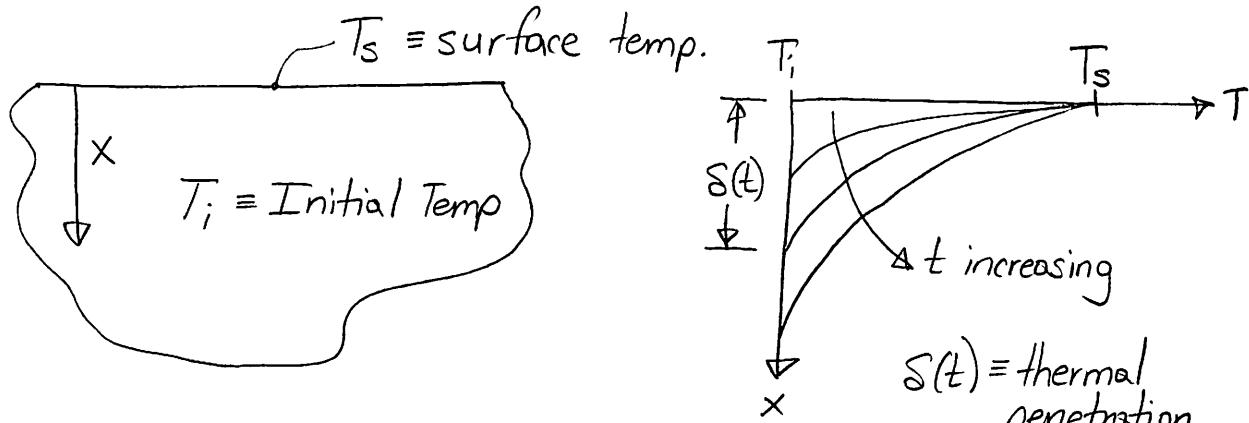
So far, we have been mainly dealing with steady-state conduction problems. What if transient effects are important and $Bi > 0.1$?

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$\downarrow 0 \neq 0 \quad \downarrow 0 \neq 0 \quad \downarrow 0 (Q'''=0)$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \Rightarrow \text{Second order PDE. Need } 2 \text{ BC's \& 1 IC}$$

Semi-Infinite Body



The surface of the semi-infinite body suddenly experiences a finite temperature (T_s), how does the body temperature $T(x)$ change with time (t)?

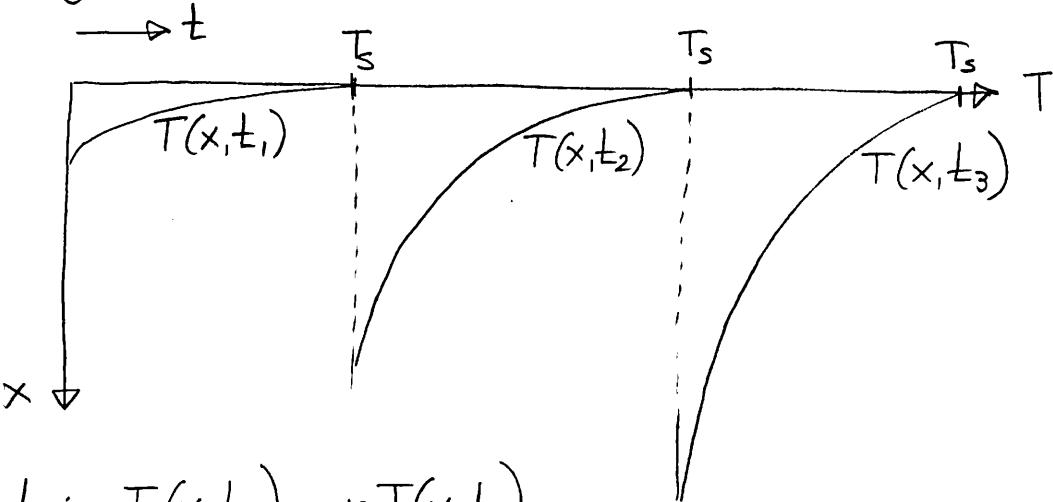
Let's non-dimensionalise our temperature & solve our PDE:

$$\Theta = \frac{T - T_s}{T_i - T_s} \quad \left. \begin{array}{l} 1) x=0, \Theta = 0 \quad (T(x=0)=T_s) \\ 2) x \rightarrow \infty, \Theta = 1 \quad (T(x \rightarrow \infty)=T_i) \\ 3) t=0, \Theta = 1 \quad (T(x,t=0)=T_i) \end{array} \right\}$$

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \Theta}{\partial t}$$

↪ Note, separation of variables won't work due to infinite med. (60)

The best way to solve this equation is to look for a scaling parameter or a similarity variable.



Note: $T(x, t_2) = n T(x, t_1)$
 $T(x, t_3) = n T(x, t_2) = m T(x, t_1)$

All of the temperature profiles look self similar in nature, just scaled up or down. This tells us there is a fundamental underlying similarity variable:

Assume: $\eta = \frac{x}{f(t)}$, so $\Theta(\eta)$, $f(t)$ = function of time

Transforming our equation:

$$\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{1}{f} \frac{\partial \Theta}{\partial \eta}$$

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \Theta}{\partial x} \right) = \frac{1}{f} \frac{\partial}{\partial x} \left(\frac{\partial \Theta}{\partial \eta} \right) = \frac{1}{f} \frac{2}{\partial \eta} \left(\frac{\partial \Theta}{\partial \eta} \right) \frac{\partial \eta}{\partial x} = \frac{1}{f^2} \frac{\partial^2 \Theta}{\partial \eta^2}$$

$$\frac{\partial \Theta}{\partial t} = \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = - \frac{\partial \Theta}{\partial \eta} \cdot x \cdot \frac{f'}{f^2} = - \eta \frac{\partial \Theta}{\partial \eta} \frac{f'}{f}$$

Back substituting into our PDE

$$\frac{1}{f^2} \frac{\partial^2 \theta}{\partial n^2} + \frac{f'}{f\alpha} n \frac{\partial \theta}{\partial n} = 0$$

$$\frac{\partial^2 \theta}{\partial n^2} + \underbrace{\left(\frac{f \cdot f'}{\alpha} \right)}_{\text{Let this equal to a constant } = 2} n \frac{\partial \theta}{\partial n} = 0$$

Let this equal to a constant = 2 (could choose any number)

$$\frac{ff'}{\alpha} = 2$$

$$\frac{f \partial f}{\partial t} = 2\alpha \Rightarrow \int f \partial f = \int 2\alpha \partial t$$

$$\frac{f^2}{2} = 2\alpha t$$

$$f = 2\sqrt{\alpha t} \quad \Rightarrow \quad n = \frac{x}{2\sqrt{\alpha t}}$$

Note, this may seem arbitrary but it ends up working out & providing a solution.

So our equation becomes:

$$\frac{\partial^2 \theta}{\partial n^2} + 2n \frac{\partial \theta}{\partial n} = 0 \Rightarrow \text{ODE only! Nice!}$$

B.C.'s : $n=0, \theta=0$ $n \rightarrow \infty, \theta=1$ $\left. \begin{array}{l} \\ \end{array} \right\}$ Turned our 2BC's & IC into 2BC's only.

Rewriting our ODE:

$$\frac{\theta''}{\theta'} = 2n$$

$$\frac{2}{\partial n} (\ln \theta') = \frac{\theta''}{\theta}$$

$$\frac{2}{\partial n} (\ln \theta') = -2n \quad (\text{Integrate})$$

$$\ln(\theta') = -\frac{2n^2}{2} + C = -n^2 + C, \quad (\text{take exponent on both sides})$$

$$\theta' = C_2 e^{-n^2}$$

Integrating one more time:

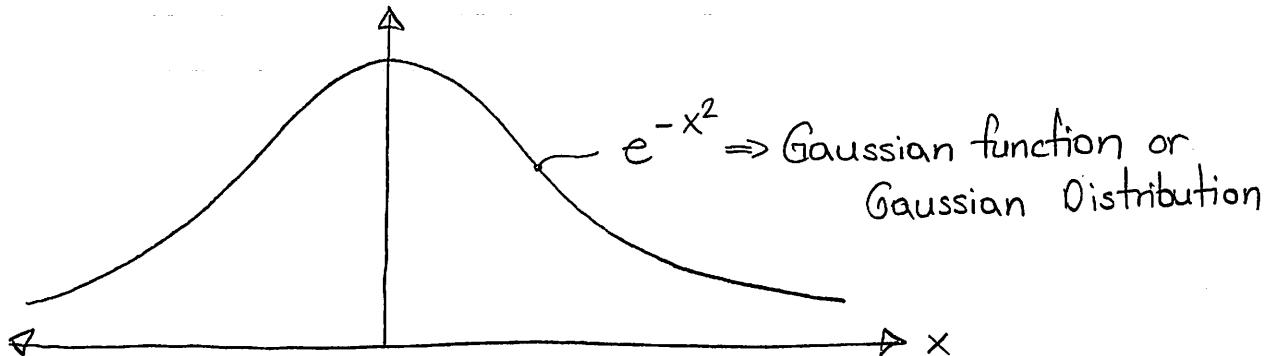
$$\Theta = C_2 \int_0^{\eta} e^{-\eta^2} d\eta + C_3$$

Applying our B.C.'s

$$\Theta(\eta=0) = 0 \Rightarrow C_3 = 0$$

$$\Theta(\eta \rightarrow \infty) = 1 \Rightarrow 1 = C_2 \int_0^{\infty} e^{-\eta^2} d\eta$$

Let's chat about this a bit!



The function describes "error" or difference between a measurement and its unbiased estimator (or mean).

$\int_0^x e^{-x^2} dx$ cannot be calculated analytically, but it can be shown analytically that:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

⇒ Symmetric function about $x=0$

So now we can define the following:

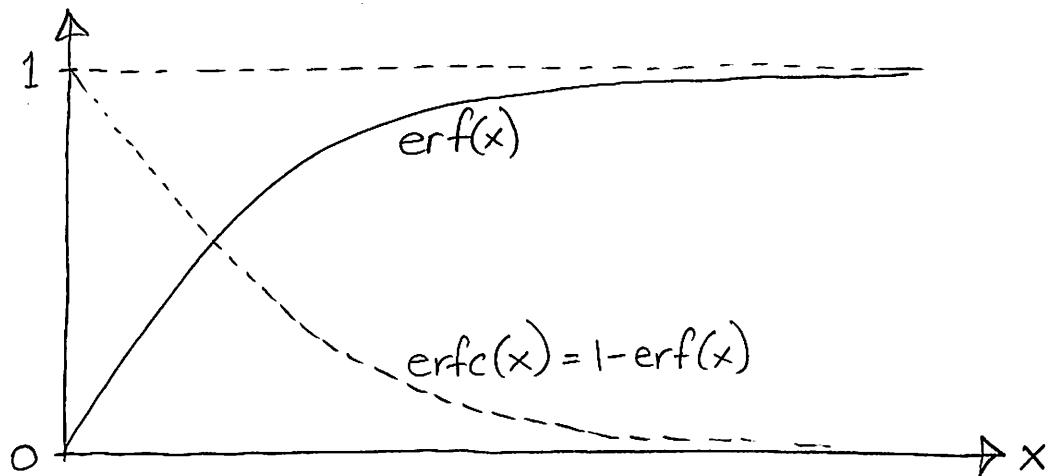
We define the "error" function as:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Error Function

$$\text{erfc}(x) = 1 - \text{erf}(x)$$

Complimentary Error Function



So now we can go back and solve. We had the following:

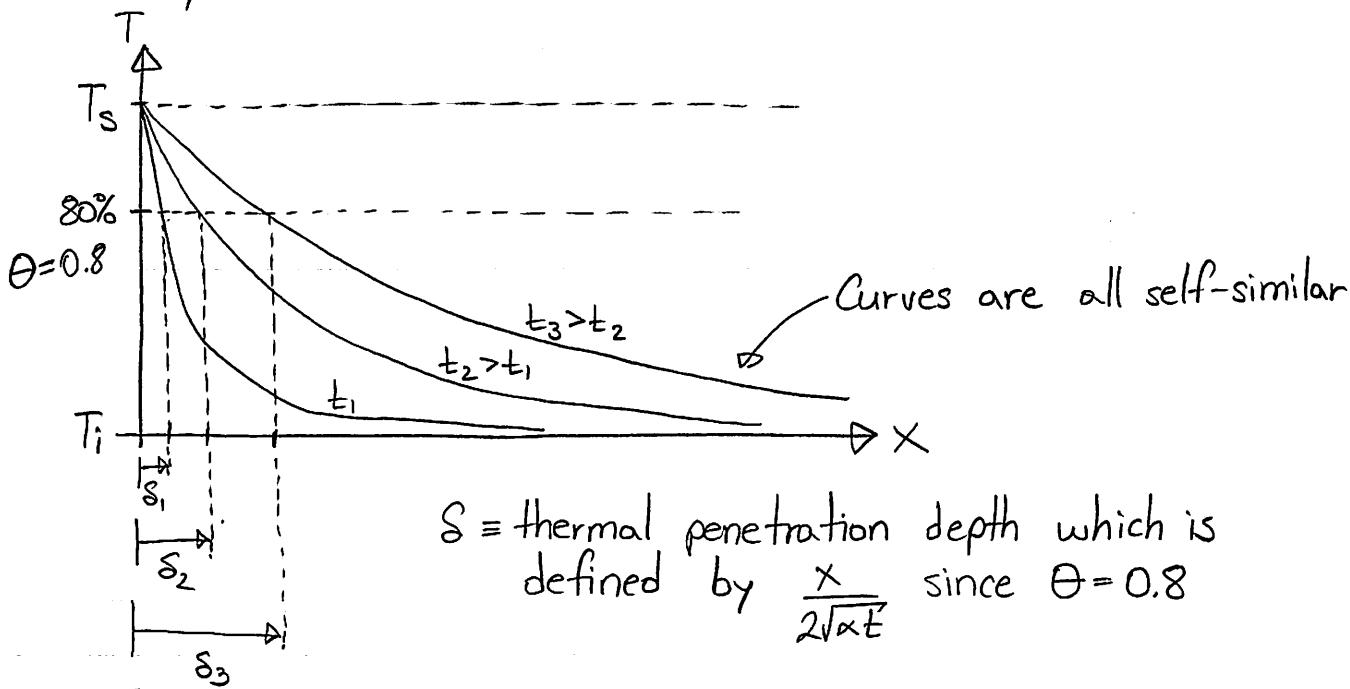
$$1 = C_2 \int_0^\infty e^{-n^2} dn$$

$\underbrace{\qquad}_{\frac{\sqrt{\pi}}{2}} \Rightarrow C_2 = \frac{2}{\sqrt{\pi}}$

$$\therefore \Theta = \frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_0^2 e^{-n^2} dn = \text{erf}(2) = \text{erf}\left(\frac{x}{2\sqrt{\alpha E}}\right)$$

\hookrightarrow Tabulated results for $\text{erf}(n)$ on pg. 1015 of Textbook
Appendix B, Table B.2

If we plot our result:



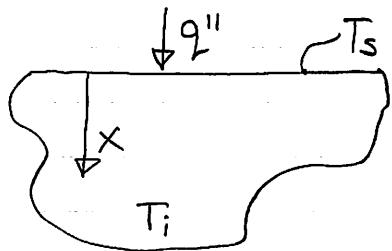
In our equation, we had:

$$\Theta = \frac{T - T_s}{T_i - T_s} = 0.8 = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) = \operatorname{erf}\left(\frac{s}{2\sqrt{\alpha t}}\right)$$

$$\frac{s}{2\sqrt{\alpha t}} = \text{constant} \Rightarrow s \sim 2\sqrt{\alpha t}$$

So we can say that the heat penetration depth proceeds with $\sqrt{\alpha t}$. Usually we say s is found at $\Theta = 0.95$ or 0.99 .

How do we determine heat flux?



$$q''|_{x=0} = -k \frac{\partial T}{\partial x}|_{x=0}$$

$$\Theta = \frac{T - T_s}{T_i - T_s}; \quad d\Theta = \frac{1}{T_i - T_s} dT$$

$$\left. q'' \right|_{x=0} = -k(T_i - T_s) \frac{\partial \theta}{\partial x} \Big|_{x=0} = k \underbrace{(T_s - T_i)}_{\Delta T} \underbrace{\frac{\partial \theta}{\partial x}}_1 \underbrace{\frac{\partial \theta}{\partial n}}_2 \Big|_{x=0}$$

Remember: $\frac{\partial \theta}{\partial n} = \theta' = C_2 e^{-n^2} = \frac{2}{\sqrt{\pi}} e^{-n^2}$

$$\frac{1}{2\sqrt{\alpha t}}$$

$$\frac{2}{\sqrt{\pi}}$$

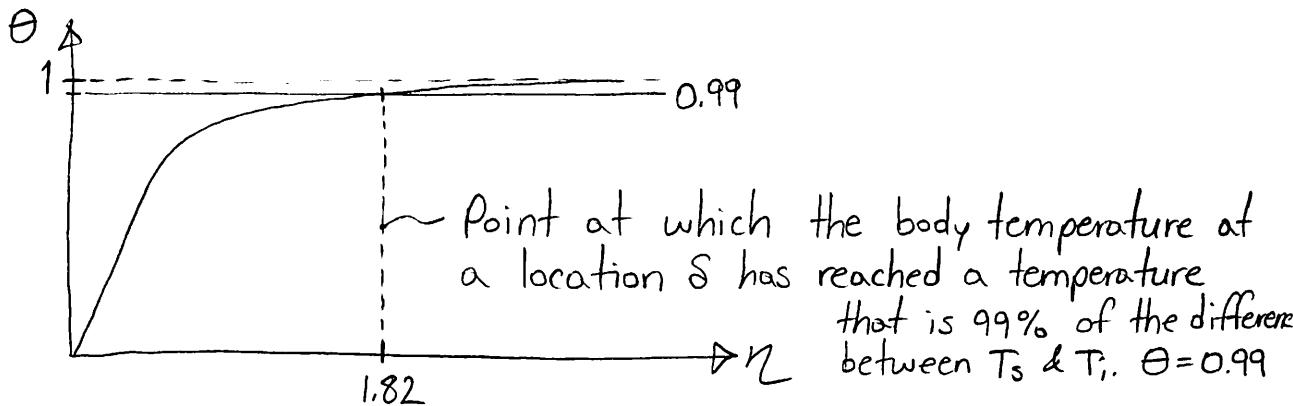
$$q'' = \frac{k \Delta T}{\sqrt{\pi \alpha t}}$$

$$\alpha = \frac{k}{\rho c}$$

\Rightarrow Thermal Diffusivity [Material prop.]

Note, to get total energy transferred [J], you need to integrate q'' with respect to t .

One last thing:

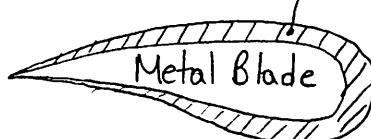


$$\frac{S}{2\sqrt{\alpha t}} = 1.82 \Rightarrow S(t) = 3.64\sqrt{\alpha t} \Rightarrow \text{Thermal penetration depth.}$$

Application Ancient samurai swords from Japan

Clay sheath (thermal insulator)

Thicker on back side, very thin on edge



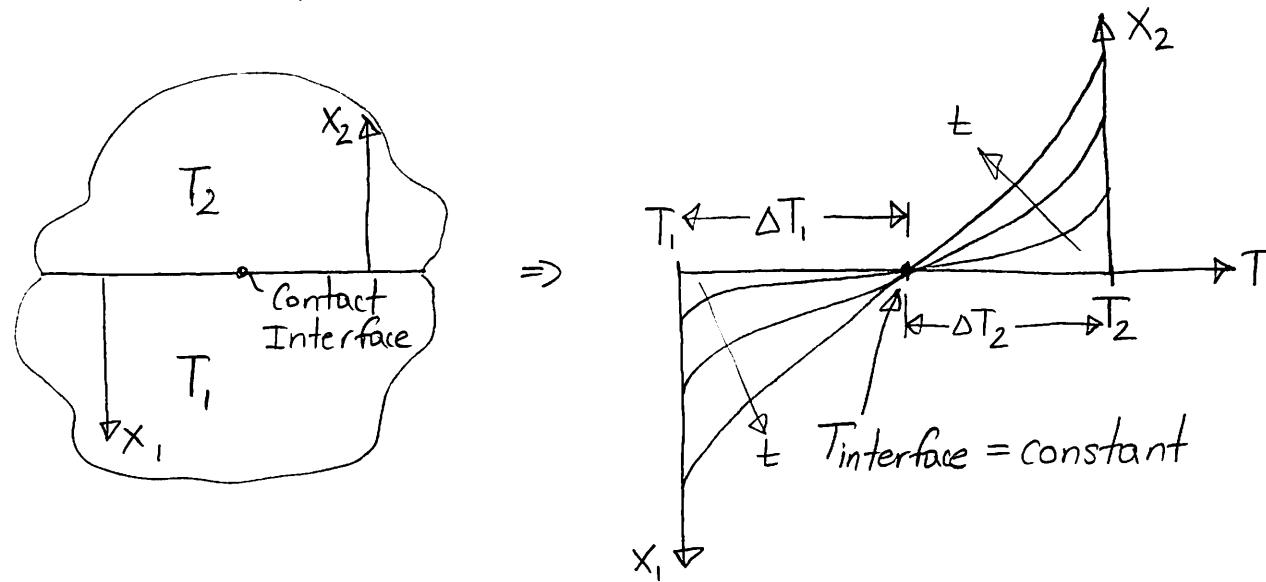
\Rightarrow Quench in Cold water \Rightarrow

Cools faster, Case hardened

Cools slower Ductile (won't crack)

Contact Between Two Semi-Infinite Solids

When two bodies at different initial temperatures are brought into contact, they initially achieve a temperature that is constant at their interface:



How do we make sure this is indeed true?
Let's apply an energy balance on the interface:

$$\dot{q}'' = \frac{k \Delta T}{\sqrt{\pi \alpha t}} = \frac{(k \rho c)^{1/2}}{\sqrt{\pi}} \cdot \frac{\Delta T}{\sqrt{t}}$$

$$\dot{q}_1'' = \dot{q}_2'' = \frac{(k_1 \rho_1 c_1)^{1/2}}{\sqrt{\pi}} \cdot \frac{\Delta T_1}{\sqrt{t}} = \frac{(k_2 \rho_2 c_2)^{1/2}}{\sqrt{\pi}} \cdot \frac{\Delta T_2}{\sqrt{t}}$$

$$(k_1 \rho_1 c_1)^{1/2} \Delta T_1 = (k_2 \rho_2 c_2)^{1/2} \Delta T_2 \quad (1)$$

But we also know that: $\Delta T_1 + \Delta T_2 = \Delta T \quad (2)$
Back substituting (2) into (1):

$$\Delta T_2 = \frac{(k \rho c)_1^{1/2}}{(k \rho c)_2^{1/2}} \Delta T_1 \Rightarrow \Delta T = \left(1 + \sqrt{\frac{(k \rho c)_1}{(k \rho c)_2}} \right) \Delta T_1$$

$$\Delta T_1 = \frac{\Delta T}{1 + \sqrt{\frac{(kpc)_1}{(kpc)_2}}} = \frac{\sqrt{(kpc)_2} \cdot \Delta T}{\sqrt{(kpc)_1} + \sqrt{(kpc)_2}} \neq f(t) \Rightarrow \text{not a function of time!}$$

$$q|_{x=0} = \frac{\sqrt{(kpc)_1} \sqrt{(kpc)_2}}{\sqrt{(kpc)_1} + \sqrt{(kpc)_2}} \cdot \frac{\Delta T}{\sqrt{\pi E}} \Rightarrow \text{Heat flux between two contacting semi-infinite bodies.}$$

This is why when you touch certain objects at room temperature, they feel "colder" than others, even though they are at the same temperature.

Example | Brass & Wooden Doorknobs

Brass: $k_1 = 109 \text{ W/m}\cdot\text{K}$
 $\rho_1 = 8730 \text{ kg/m}^3$
 $C_1 = 380 \text{ J/kg}\cdot\text{K}$

$$(k_1 \rho_1 C_1)^{1/2} = 19016 \text{ J/m}^2\text{K}^{1/2}$$

Wood: $k_2 = 0.17 \text{ W/m}\cdot\text{K}$
 $\rho_2 = 750 \text{ kg/m}^3$
 $C_2 = 1700 \text{ J/kg}\cdot\text{K}$

$$(k_2 \rho_2 C_2)^{1/2} = 466 \text{ J/m}^2\text{K}^{1/2}$$

Your body temperature is $\approx 37^\circ\text{C}$
Room temperature of a cold room $\approx 17^\circ\text{C}$

$$\Delta T = T_{\text{body}} - T_{\text{Room}} = 20^\circ\text{C}$$

Now we need human flesh properties:

Human hand: $k_3 = 0.6 \text{ W/m}\cdot\text{K}$ (Water)
 $\rho_3 \approx 1000 \text{ kg/m}^3$
 $C_3 = 4190 \text{ J/kg}\cdot\text{K}$

$$(k_3 \rho_3 C_3)^{1/2} = 1586 \text{ J/m}^2\text{K}^{1/2}$$

Case 1: You reach for the brass doorknob:

$$q''_{3,1} = \frac{(19016)(1586)}{(19016)+(1586)} \frac{\Delta T}{\sqrt{\pi E}} = 1463 \frac{\Delta T}{\sqrt{\pi E}} \Rightarrow \text{Heat transfer from your hand to the brass doorknob.}$$

Case 2: You reach for the wooden doorknob:

$$q''_{3,2} = \frac{(466)(1586)}{(466)+(1586)} \cdot \frac{\Delta T}{\sqrt{\pi E}} = 360 \frac{\Delta T}{\sqrt{\pi E}} \Rightarrow \text{Heat transfer from your hand to the wooden doorknob.}$$

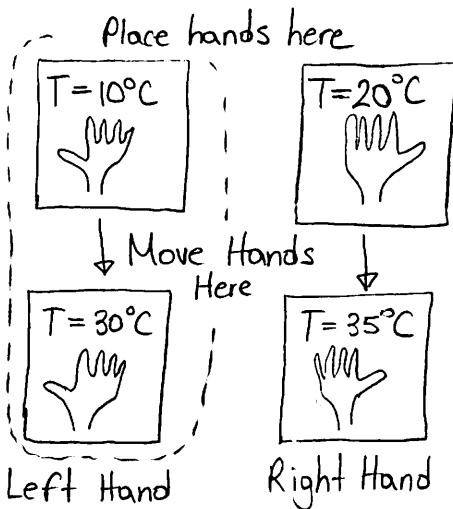
Comparing, we see that:

$$\left[\frac{q''_{3,1}}{q''_{3,2}} = 4.1 \right]$$

\Rightarrow This is why the brass doorknob feels so much colder. It pulls out heat from your hand at a rate 4 times faster than the wood.

On another note, your body translates temperature or feeling hot or cold by sensing heat flow, not absolute temperature. The faster you lose heat, the colder something feels.

Try the following:



Your left hand (dotted circle) will feel much hotter than your right, even though the second right plate is at a higher temperature!

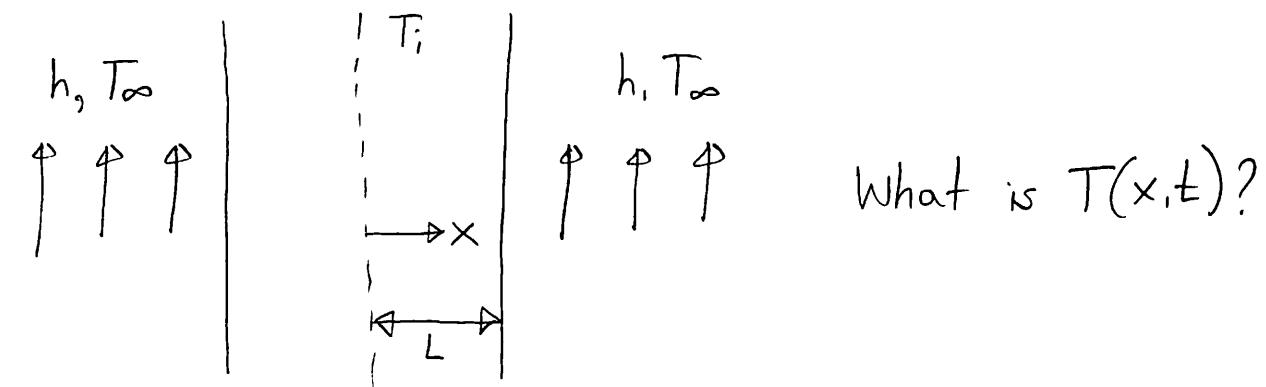
Transient Heat Conduction in Finite Bodies

So far, we've covered the cases of:

- 1) Lumped capacitance, $T \neq f(x)$
- 2) Infinite media, $0 \leq x \leq \infty$

How do we handle bodies that are finite in size & have $B_i z > 0.1$?

Let's solve the plane wall problem. Slab initially placed in a medium with T_∞ & h . The slab has initial temp. T_i .



Looking at our heat equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{Q'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Assuming:

1) $T(0, y, z, t) = T_i$,	$\left. \frac{\partial T}{\partial x} \right _{x=0} = 0$	}
2) $Q''' = 0$,	$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$	

We need 2 B.C.'s and an IC to solve:

1) $T(x, t=0) = T_i$ (Initial condition)

2) $\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0$ (1st B.C.)

3) $-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h(T(x=L) - T_\infty)$ (2nd B.C.)

To make our solution more tractable, we will non-dimensionalize

Let: $\bar{x} = \frac{x}{L}$ (Dimensionless distance from center, $0 \leq \bar{x} \leq 1$)

$$\frac{\partial \bar{x}}{\partial x} = \frac{1}{L} \Rightarrow \partial \bar{x} = \frac{\partial x}{L}$$

$$\Theta = \frac{T - T_{\infty}}{T_i - T_{\infty}} \quad (\text{Dimensionless temperature, } 0 \leq \Theta \leq 1) \\ \text{O } (T_i, T_{\infty} = \text{constant})$$

$$\frac{\partial \Theta}{\partial T} = \frac{\partial}{\partial T} \left(\frac{T}{T_i - T_{\infty}} \right) - \frac{\cancel{\partial}}{\cancel{\partial T}} \left(\frac{T_{\infty}}{T_i - T_{\infty}} \right) = \frac{1}{T_i - T_{\infty}} \Rightarrow \partial \Theta = \frac{\partial T}{T_i - T_{\infty}}$$

How about dimensionless time?

$$\frac{\partial T}{\partial x} = \frac{(T_i - T_{\infty})}{L} \frac{\partial \Theta}{\partial \bar{x}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_i - T_{\infty}}{L^2} \frac{\partial^2 \Theta}{\partial \bar{x}^2}$$

$$\frac{\partial T}{\partial t} = (T_i - T_{\infty}) \frac{\partial \Theta}{\partial t} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Back substituting into our PDE}$$

$$\frac{T_i - T_{\infty}}{L^2} \frac{\partial^2 \Theta}{\partial \bar{x}^2} = (T_i - T_{\infty}) \frac{\partial \Theta}{\partial t} \cdot \frac{1}{\alpha}$$

$$\frac{\partial^2 \Theta}{\partial \bar{x}^2} = \frac{L^2}{\alpha} \frac{\partial \Theta}{\partial t} \quad \textcircled{1} \rightarrow \text{We still have a dimensional time here.}$$

Let's use F_0 # or dimensionless time (look on pg. 59 of notes)

$$F_0 = C = \frac{\alpha t}{L^2} = \frac{\text{diffusive transport rate}}{\text{storage rate}}$$

$$t = \frac{L^2 C}{\alpha} \Rightarrow \frac{\partial t}{\partial C} = \frac{L^2}{\alpha} \Rightarrow \partial t = \frac{L^2}{\alpha} \partial C \Rightarrow \text{back substitute into } \textcircled{1}$$

Now our equation becomes:

$$\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{k^2}{\alpha} \cdot \frac{\partial \theta}{\partial C} \cdot \frac{\alpha}{k^2} \Rightarrow \boxed{\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{\partial \theta}{\partial C} \quad \text{or} \quad \frac{\partial \theta}{\partial F_0}}$$

↳ Dimensionless 1D, transient heat eq.

Now we can non-dimensionalize our IC & B.C.'s

1) Our IC becomes: $\boxed{\theta(\bar{x}, C=0) = 1}$

2) Our 1'st B.C. becomes: $\boxed{\left. \frac{\partial \theta}{\partial \bar{x}} \right|_{\bar{x}=0} = 0}$

3) Our 2'nd B.C. becomes:

$$-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h(T(x=L) - T_\infty)$$

$$-h \left. \frac{(T_i - T_\infty)}{L} \cdot \frac{\partial \theta}{\partial \bar{x}} \right|_{\bar{x}=1} = h \theta(\bar{x}=1) \cdot (T_i - T_\infty)$$

$$\boxed{\left. \frac{\partial \theta}{\partial \bar{x}} \right|_{\bar{x}=1} = - \underbrace{\frac{hL}{k}}_{Bi} \theta(\bar{x}=1)}$$

$$Bi \Rightarrow Bi \# = \frac{\text{Conduction resistance}}{\text{convection resistance}}$$

Now we've reduced the problem to:

$$T(x, t) = f(x, L, t, \alpha, h, T_\infty, T_i)$$

$$\downarrow \\ \theta = f(\bar{x}, Bi, F_0) \Rightarrow \text{Very powerful!}$$

Now we can solve using the separation of variables:

$$\Theta(\bar{x}, \bar{c}) = F(\bar{x}) \cdot G(\bar{c}) \Rightarrow \text{Back substitute into our PDE and divide through by } F \cdot G$$

$$\underbrace{\frac{1}{F} \frac{\partial^2 F}{\partial \bar{x}^2}}_{f(\bar{x}) \text{ only}} = \underbrace{\frac{1}{G} \cdot \frac{\partial G}{\partial \bar{c}}}_{f(\bar{c}) \text{ only}} = \text{constant} \quad (\text{Must be a constant only since two independent functions})$$

Assuming our constant $= -\lambda^2$

$$\left. \begin{array}{l} \frac{\partial^2 F}{\partial \bar{x}^2} + \lambda^2 F = 0 \quad (1) \\ \frac{\partial G}{\partial \bar{c}} + \lambda^2 G = 0 \quad (2) \end{array} \right\} \text{Homogeneous ODE's. Use characteristic equation.}$$

Solving ① first: $\lambda'^2 + \lambda^2 = 0$
 $\lambda' = \pm \sqrt{-1} \lambda = \lambda i$

So our solution becomes:

$$F = C_1 e^{i\lambda \bar{x}} + C_2 e^{-i\lambda \bar{x}} \Rightarrow \text{since}$$

We can write our solution as:

$$F = C_1 \cos(\lambda \bar{x}) + C_2 \sin(\lambda \bar{x}) \quad (3)$$

Now solving ②: $\lambda' + \lambda^2 = 0 \Rightarrow \lambda' = -\lambda^2$

$$G = C_3 e^{-\lambda^2 \bar{c}} \quad (4)$$

Combining our two solutions, we obtain:

Identities

$\sin \bar{x} = \frac{e^{i\bar{x}} - e^{-i\bar{x}}}{2i}$
$\cos \bar{x} = \frac{e^{i\bar{x}} + e^{-i\bar{x}}}{2}$

$$\Theta = F \cdot G = C_3 e^{-\lambda^2 C} (C_1 \cos(\lambda \bar{x}) + C_2 \sin(\lambda \bar{x}))$$

$$\Theta = e^{-\lambda^2 C} (A \cos(\lambda \bar{x}) + B \sin(\lambda \bar{x})) \quad ⑤$$

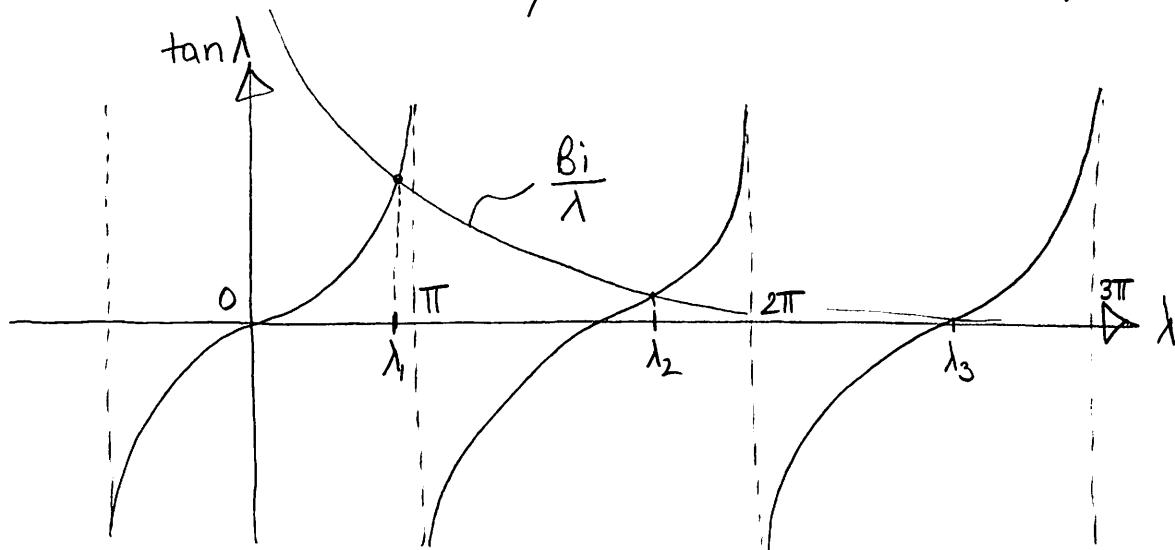
Now we apply our B.C.'s :

$$\left. \frac{\partial \Theta}{\partial \bar{x}} \right|_{\bar{x}=0} = 0 = -e^{-\lambda^2 C} (A \lambda \sin(0) + B \lambda \cos(0)) \Rightarrow B = 0$$

$$\left. \frac{\partial \Theta}{\partial \bar{x}} \right|_{\bar{x}=1} = -B_i \Theta(\bar{x}=1, C) \Rightarrow -A e^{-\lambda^2 C} \lambda \sin \lambda = -B_i A e^{-\lambda^2 C} \cos \lambda$$

$$\lambda \tan \lambda = B_i \quad ⑥$$

We know $\tan \lambda$ is a periodic function with period π , so solution can lie anywhere between $0 \leq \lambda \leq \pi$, $\pi \leq \lambda \leq 2\pi$, etc...



Multiple roots exist, so we have multiple solutions

$$\lambda_n \tan \lambda_n = B_i \quad (\text{Eigenfunction with eigenvalues } \lambda_n)$$

There exist an infinite number of solutions of the form $A e^{-\lambda^2 C} \cos(\lambda \bar{x})$. The final solution is a linear superposition of all of them:

$$\Theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 C} \cos(\lambda_n \bar{x}) \quad (7)$$

Our IC allows us to determine our constants A_n :

$$\Theta(\bar{x}, C=0) = 1$$

$$1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n \bar{x})$$

Using orthogonality \Rightarrow multiply both sides by $\cos(\lambda_m \bar{x})$ & integrate:

$$\int_0^1 \cos(\lambda_m \bar{x}) d\bar{x} = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos(\lambda_n \bar{x}) \cos(\lambda_m \bar{x}) d\bar{x}}_{=0 \text{ if } n \neq m}$$

So our solution becomes:

$$\begin{aligned} \int_0^1 \cos(\lambda_n \bar{x}) d\bar{x} &= A_n \int_0^1 \cos^2(\lambda_n \bar{x}) d\bar{x} \\ &\underbrace{\sin(\lambda_n \bar{x}) \Big|_0^1}_{= \sin(\lambda_n(1)) - \sin(\lambda_n(0))} \quad \underbrace{\cos^2 x + \sin^2 x = 1}_{\cos^2 x - \sin^2 x = \cos 2x} \\ &= \sin(\lambda_n(1)) - \sin(\lambda_n(0)) \quad \cos^2 x = (1 + \cos 2x)/2 \\ &= \sin(\lambda_n) \quad \Rightarrow \left[2\lambda_n \bar{x} + \sin(2\lambda_n \bar{x}) \right] \Big|_0^1 \\ &\quad = 2\lambda_n + \sin(2\lambda_n) \end{aligned}$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} \quad (8)$$

\Rightarrow Combine with eq. 7 & you're done!

There exist an infinite number of solutions of the form $A e^{-\lambda^2 C} \cos(\lambda \bar{x})$. The final solution is a linear superposition of all of them:

$$\Theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 C} \cos(\lambda_n \bar{x}) \quad (7)$$

Our IC allows us to determine our constants A_n :

$$\Theta(\bar{x}, C=0) = 1$$

$$1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n \bar{x})$$

Using orthogonality \Rightarrow multiply both sides by $\cos(\lambda_m \bar{x})$ & integrate:

$$\int_0^1 \cos(\lambda_m \bar{x}) d\bar{x} = \underbrace{\sum_{n=1}^{\infty} A_n \int_0^1 \cos(\lambda_n \bar{x}) \cos(\lambda_m \bar{x}) d\bar{x}}_{=0 \text{ if } n \neq m}$$

So our solution becomes:

$$\begin{aligned} \int_0^1 \cos(\lambda_n \bar{x}) d\bar{x} &= A_n \int_0^1 \cos^2(\lambda_n \bar{x}) d\bar{x} \\ &\underbrace{\frac{1}{\lambda_n} [\sin(\lambda_n \bar{x})]_0^1}_{=0} \quad \underbrace{\int_0^1 \cos^2(\lambda_n \bar{x}) d\bar{x}}_{\cos^2 x + \sin^2 x = 1} \\ &= \frac{1}{\lambda_n} (\sin(\lambda_n(1)) - \sin(\lambda_n(0))) \\ &= \sin(\lambda_n) \cdot \frac{1}{\lambda_n} \quad \cos^2 x - \sin^2 x = \cos 2x \\ &\quad \cos^2 x = (1 + \cos 2x)/2 \\ &\Rightarrow A_n \left[\frac{1}{2} + \frac{\sin(2\lambda_n)}{4\lambda_n} \right] \end{aligned}$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} \quad (8)$$

\Rightarrow Combine with eq. 7 & you're done!

Since the solution involves an infinite series, not very useful from an analytical standpoint.

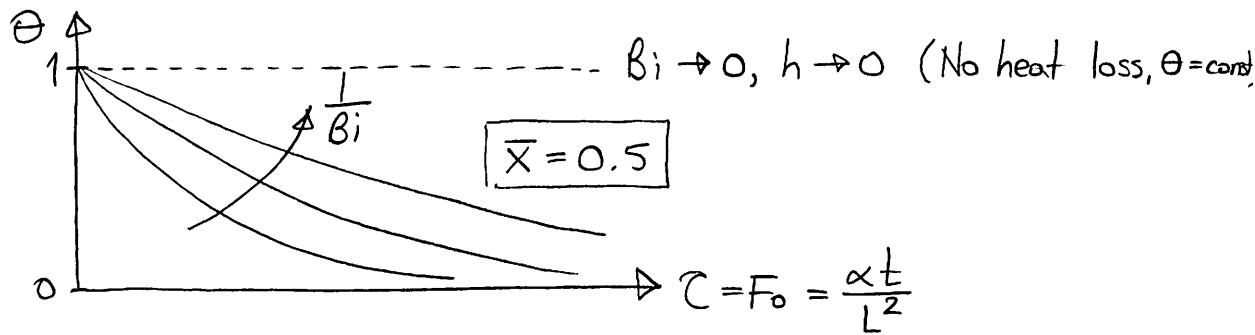
Turns out a good approximation is available by using only the first few terms, since the exponential decay term diminishes the higher order terms.

Tabular Solutions

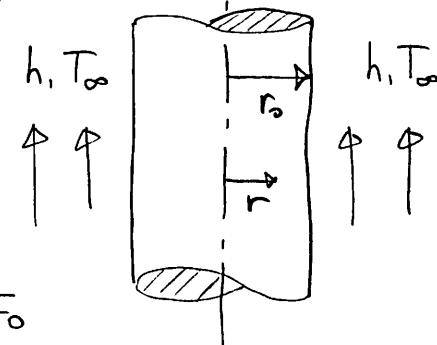
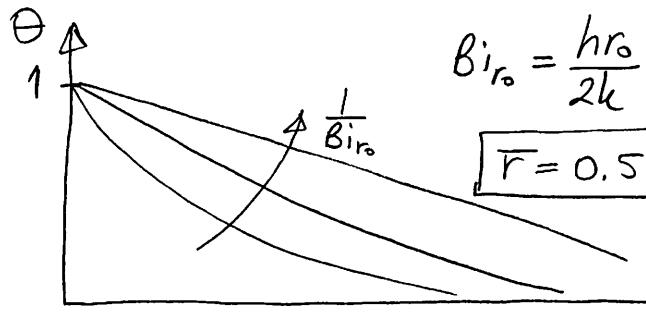
People have conveniently tabulated the solutions for specific geometries:

We know: $\Theta = f(\bar{x}, Bi, \tau \text{ or } F_o)$ for a slab

We can plot our results for a given \bar{x} as a function of Bi & F_o



Same for a cylinder: for a given $\bar{r} = \frac{r}{r_o}$, $\Theta = f(F, Bi_{r_o}, F_{o,r_o})$



In general, these tabular solutions are provided & simple to use.

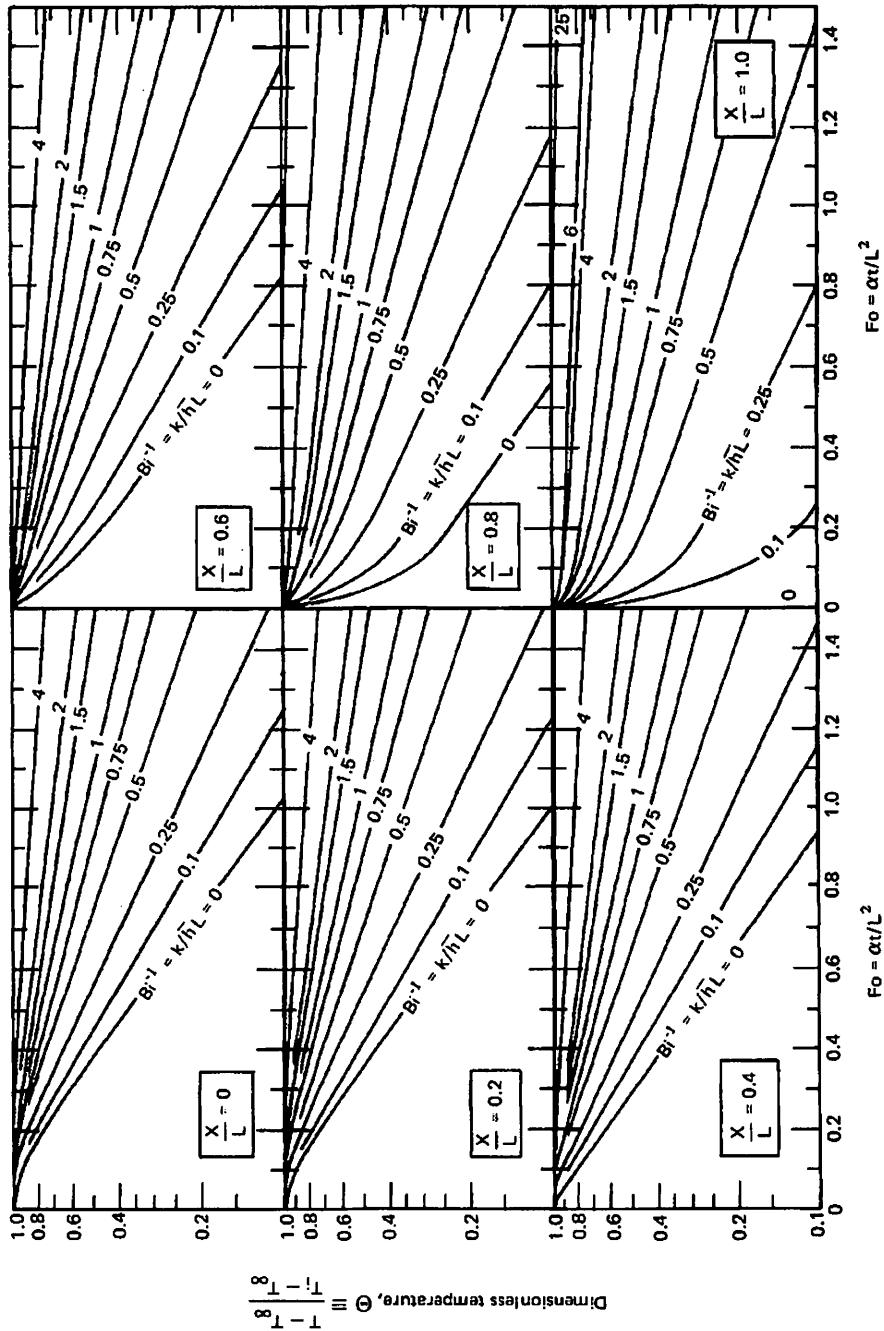


Figure 5.7 The transient temperature distribution in a slab at six positions: $x/L = 0$ is the center, $x/L = 1$ is one outside boundary.

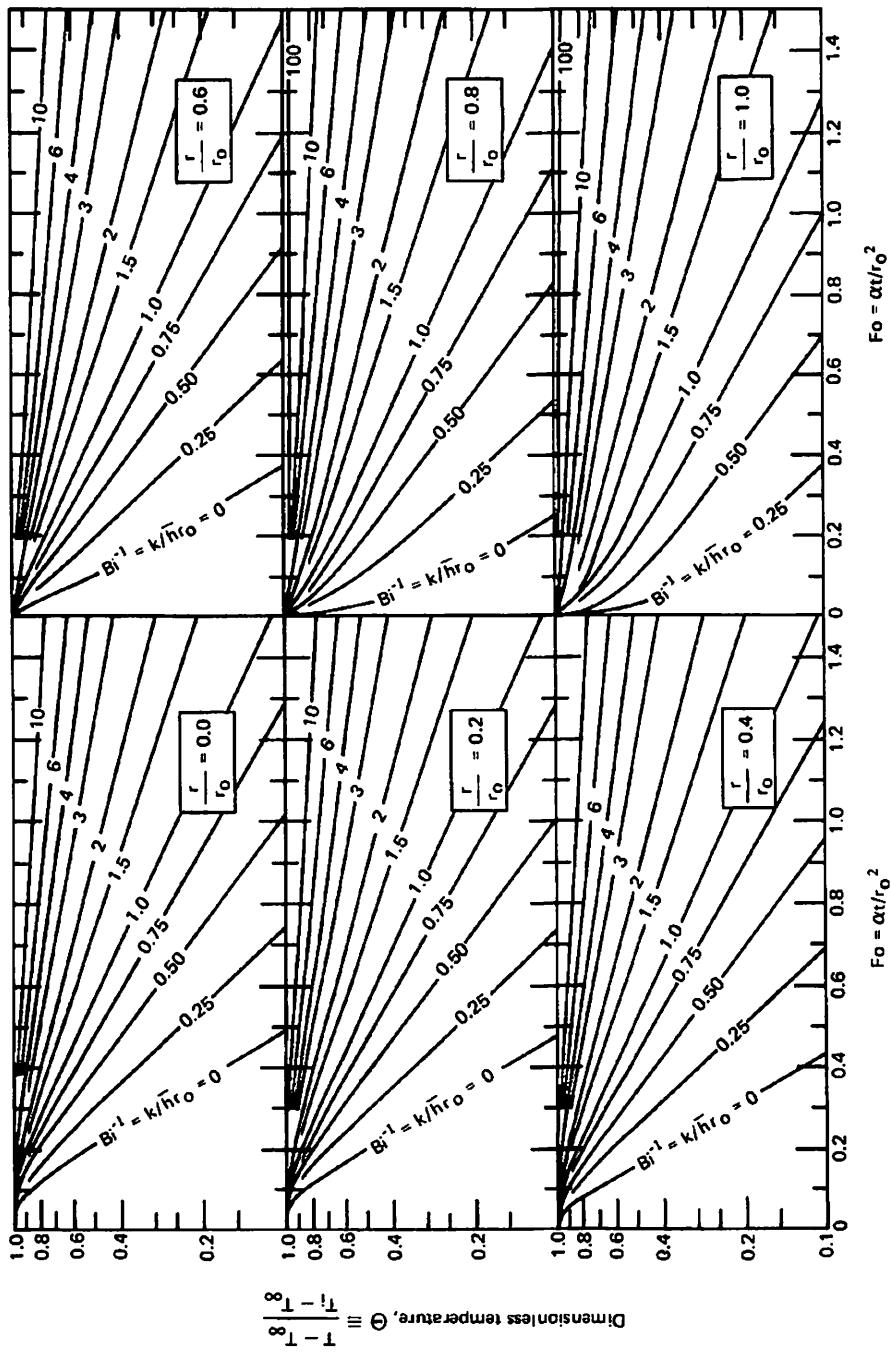


Figure 5.8 The transient temperature distribution in a long cylinder of radius r_o at six positions:
 $r/r_o = 0$ is the centerline; $r/r_o = 1$ is the outside boundary.

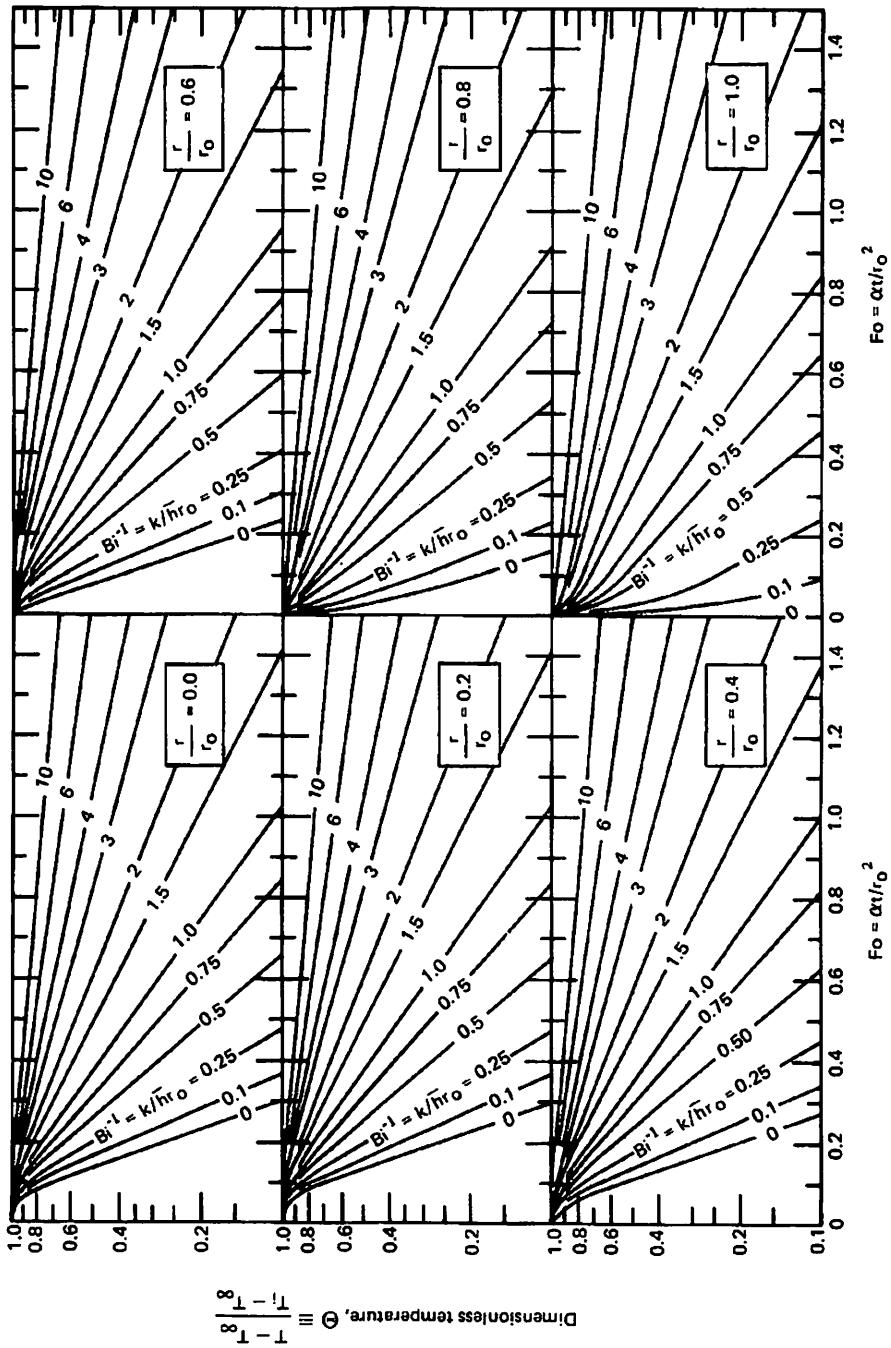
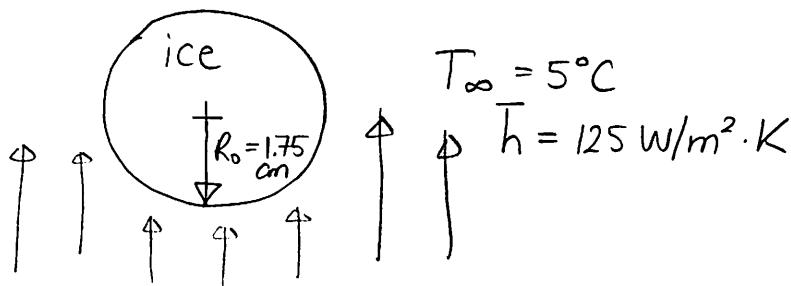


Figure 5.9 The transient temperature distribution in a sphere of radius r_o at six positions: $r/r_o = 0$ is the center; $r/r_o = 1$ is the outside boundary.

Example] A large hailstone (ice sphere), 3.5 cm in diameter & initially at $T_i = -20^\circ\text{C}$ falls in the surrounding air at $T_\infty = 5^\circ\text{C}$ at terminal velocity.



- (a) How long will it take before the outer surface begins to melt?
- (b) What is the temperature of the center when it begins to melt at the surface?

To solve we can simply use our chart on pg. 79 of notes.

$$Bi_{R_o} = \frac{h R_o}{k} = \frac{(125 \text{ W/m}^2 \cdot \text{K})(0.0175 \text{ m})}{(2.22 \text{ W/m} \cdot \text{K})} = 0.985 \approx 1$$

At the instant the surface begins to melt, the surface temperature is $T_s = 0^\circ\text{C}$

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \frac{0^\circ\text{C} - 5^\circ\text{C}}{-20^\circ\text{C} - 5^\circ\text{C}} = 0.2$$

From our chart at $\frac{r}{R_o} = \frac{R}{R_o} = 1$ (surface), & $\Theta = 0.2$, $Bi^{-1} = 1$

$$Fo = 0.6 = \frac{\alpha_{\text{ice}} t}{R_o^2} \Rightarrow \boxed{t = \frac{R_o^2 Fo}{\alpha_{\text{ice}}} = 146 \text{ seconds}}$$

For part (b) we can use the same Bi_{R_o} , Fo , and solve for Θ_c

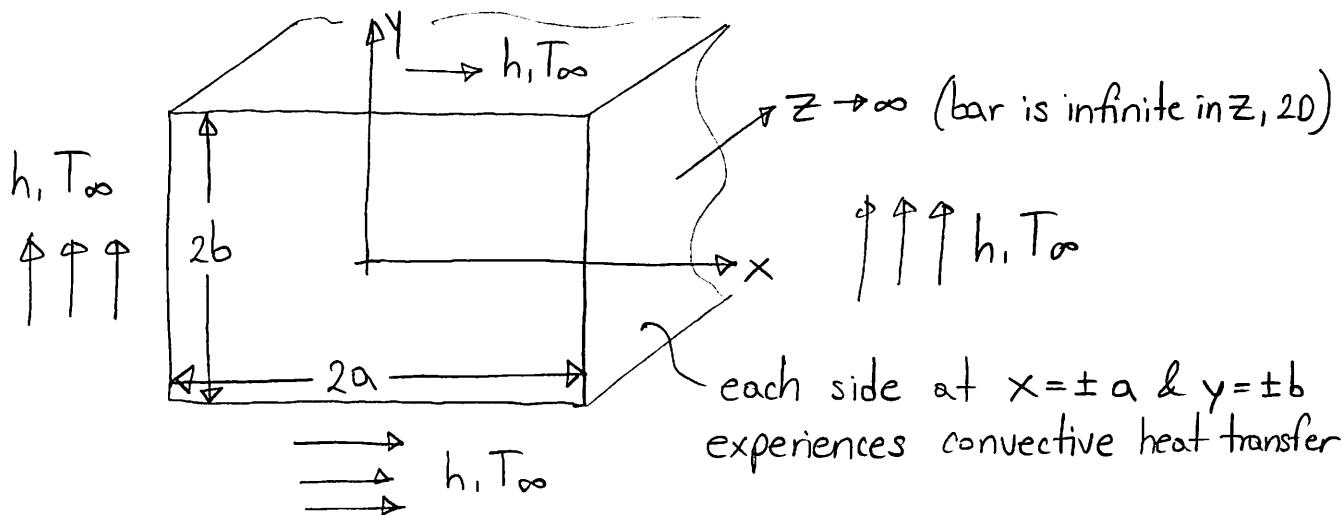
$$\Theta_c = \frac{T_c - T_\infty}{T_i - T_\infty} = 0.3 \Rightarrow \boxed{T_c = -2.5^\circ\text{C}}$$

Conduction in 2D & 3D bodies of Finite Extent

So far we've learned to handle 1D, transient, finite body problems. How do we deal with 2D or 3D problems?

It's easier than it seems.

Assume we have the following 2D bar being cooled:



Our governing equation is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \Rightarrow \text{Needs 4 B.C.'s and 1 I.C to solve}$$

B.C.'s 1) $\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0$ (Symmetry)

2) $-k \left. \frac{\partial T}{\partial x} \right|_{x=a} = h(T(x=a) - T_\infty)$ (Energy balance at the wall)

3) $\left. \frac{\partial T}{\partial y} \right|_{y=0} = 0$ (symmetry)

4) $-k \left. \frac{\partial T}{\partial y} \right|_{y=b} = h(T(y=b) - T_\infty)$ (Energy balance at the top wall)

Our I.C. is: i) $T(x,y,t=0) = T_i$

Note, for the analysis, the h on each side (Top & side) need not be the same.

If we apply the separation of variables again:

$$T(x,y,t) = \bar{X}(x,t) \cdot \bar{Y}(y,t) \Rightarrow \text{Back substitute into PDE}$$

$$\bar{Y} \frac{\partial^2 \bar{X}}{\partial x^2} + \bar{X} \frac{\partial^2 \bar{Y}}{\partial y^2} = \frac{\bar{Y}}{\alpha} \frac{\partial \bar{X}}{\partial t} + \frac{\bar{X}}{\alpha} \cdot \frac{\partial \bar{Y}}{\partial t}$$

Divide both sides by $\bar{X} \cdot \bar{Y}$ & rearrange

$$\underbrace{\frac{1}{\bar{X}} \cdot \left(\frac{\partial^2 \bar{X}}{\partial x^2} - \frac{1}{\alpha} \frac{\partial \bar{X}}{\partial t} \right)}_{f(x,t)} = - \underbrace{\frac{1}{\bar{Y}} \left(\frac{\partial^2 \bar{Y}}{\partial y^2} - \frac{1}{\alpha} \frac{\partial \bar{Y}}{\partial t} \right)}_{g(y,t)} = h(t)$$

Where $h(t) = 0$ since the x & y solutions are fundamentally similar in character:

$$\left. \begin{array}{l} \frac{\partial^2 \bar{X}}{\partial x^2} - \frac{1}{\alpha} \frac{\partial \bar{X}}{\partial t} = 0 \\ \frac{\partial^2 \bar{Y}}{\partial y^2} - \frac{1}{\alpha} \frac{\partial \bar{Y}}{\partial t} = 0 \end{array} \right\} \text{Note, these are the exact same equations as in the 1D case we just did.}$$

It turns out the separate directions can be treated independently. The only condition is that for the prescribed F_0 in each direction, that t is the same since time is same for each direction.

Hence we can use our graphical approach!

Hence for our 2D case:

$$\Theta = f(\bar{x}, \bar{y}, Bi_x, Bi_y, Fo_x, Fo_y) = \frac{T - T_\infty}{T_i - T_\infty}$$

$$\Theta = \Theta_{x,t} \cdot \Theta_{y,t} \quad (\text{non-dimensional form of } T(x,y,t) = \bar{X} \cdot \bar{Y})$$

$$\Theta_{x,t} = f(\bar{x}, Bi_x, Fo_x) \qquad \Theta_{y,t} = f(\bar{y}, Bi_y, Fo_y)$$

$$\bar{x} = \frac{x}{a}$$

$$\bar{y} = \frac{y}{b}$$

$$Bi_x = \frac{ha}{k}$$

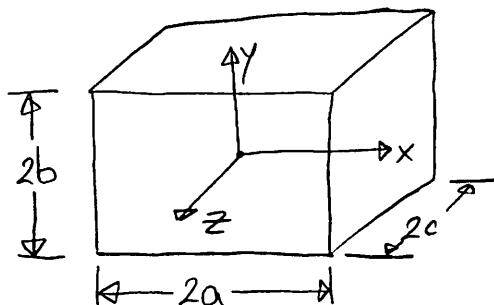
$$Bi_y = \frac{hb}{k}$$

$$Fo_x = \frac{\alpha t}{a^2} \quad \xleftarrow[\text{note*, t must be the same.}]{} \quad Fo_y = \frac{\alpha t}{b^2}$$

Use charts on pages 77 to 79 of notes to solve for each independent direction and then combine & multiply.

$$\boxed{\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \Theta_{x,t} \cdot \Theta_{y,t}}$$

The same would be true for a 3D slab or other geometries:



$$\boxed{\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \Theta_{x,t} \cdot \Theta_{y,t} \cdot \Theta_{z,t}}$$

Remember, use same t for all Fo calculations.



Example] To cook a roast, all portions of the roast must attain a temperature of 80°C . How long will it take to cook a 2.25 kg roast?

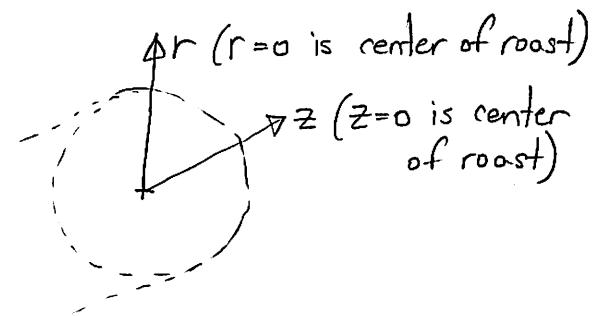
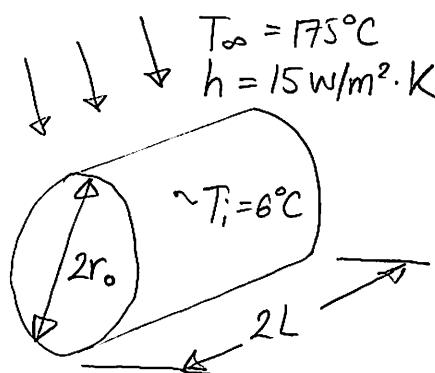
$$T_i = 6^\circ\text{C}$$

$$T_{\text{ooven}} = T_\infty = 175^\circ\text{C}$$

$$h = 15 \text{ W/m}^2 \cdot \text{K}$$

Assumptions:

- 1) Treat roast as cylinder with $D=2L$
- 2) Roast properties similar to water
- 3) Constant properties.



We can model this as 2D conduction in r & z coordinates
We need to solve for:

$$T(r, z, t) \Rightarrow T(0, 0, t) = 80^\circ\text{C}$$

We just learned that:

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \Theta_{r,t} \cdot \Theta_{z,t}$$

Cylinder solution Slab solution

$$\frac{T(0, 0, t) - T_\infty}{T_i - T_\infty} = \frac{80^\circ\text{C} - 175^\circ\text{C}}{6^\circ\text{C} - 175^\circ\text{C}} = 0.56$$

$$0.56 = \left. \frac{T(0, t) - T_\infty}{T_i - T_\infty} \right|_{\text{slab}} \cdot \left. \frac{T(0, t) - T_\infty}{T_i - T_\infty} \right|_{\text{cylinder}}$$

We just learned for each solution, we need Bi & Fo

$$Bi = \frac{hr_o}{k} = \frac{hL}{k} = \frac{(15 \text{ W/m}^2 \cdot \text{K})(L)}{0.6 \text{ W/m} \cdot \text{K}}$$

\rightarrow Water thermal conductivity

But we need L & r_o :

$$M = \rho V = \rho \cdot 2L \cdot \pi r_o^2$$

$$r_o = L = \left[\frac{M}{2\pi\rho} \right]^{1/3} = \left[\frac{2.25 \text{ kg}}{2\pi(1000 \text{ kg/m}^3)} \right]^{1/3}$$

$L = r_o = 0.0712 \text{ m}$

Plugging back into our Bi #:

$$Bi = 1.68 \quad \text{or} \quad Bi^{-1} = 0.6$$

Now for Fourier # (Fo): Note*: here $L=r_o$ but not always the case, hence Bi & Fo may differ.

$$Fo = \frac{\alpha t}{r_o^2} = \frac{\alpha t}{L^2} = (1.53 \times 10^{-7} \text{ m}^2/\text{s}) \cdot t / (0.0712 \text{ m})^2$$

$Fo = 3.02 \times 10^{-5} \cdot t$

Now we need to use trial and error to guess from our tabular solutions:

Trial	Fo	t (hours)	Θ_o/Θ_i slab	Θ_o/Θ_i cylinder	$\Theta_{\text{slab}} \cdot \Theta_{\text{cyl}}$
1	0.4	3.68	0.8 (pg. 77)	0.52 (pg. 78)	0.42
2	0.3	2.75	0.82	0.65	0.53

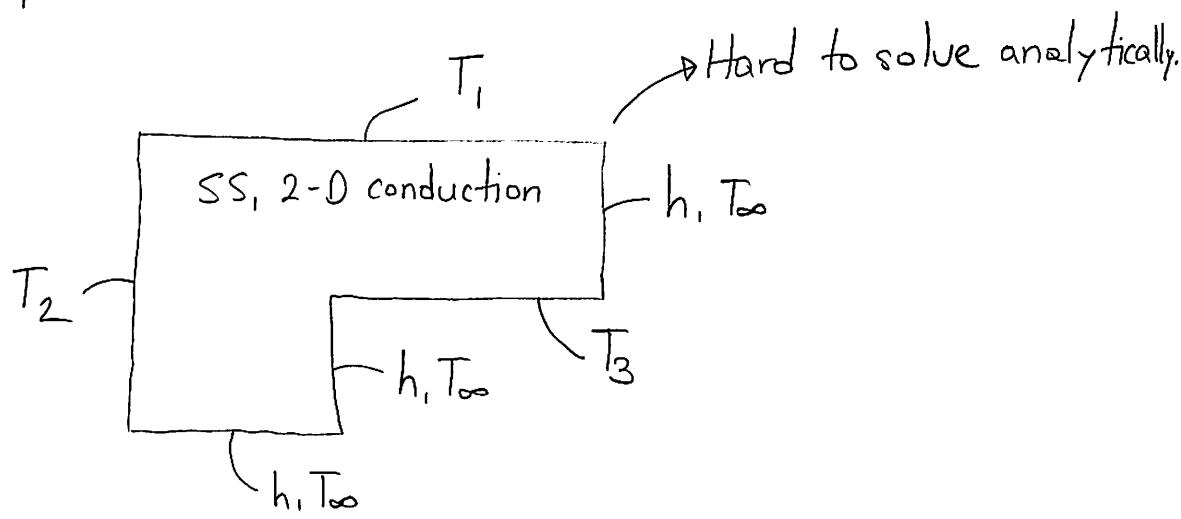
So on our second trial, we have good agreement with $\Theta=0.56$
 It takes $2\frac{3}{4}$ hours to roast the meal.

Numerical Analysis of Heat Conduction

So far, we have been using strictly analytical methods to solve heat transfer problems.

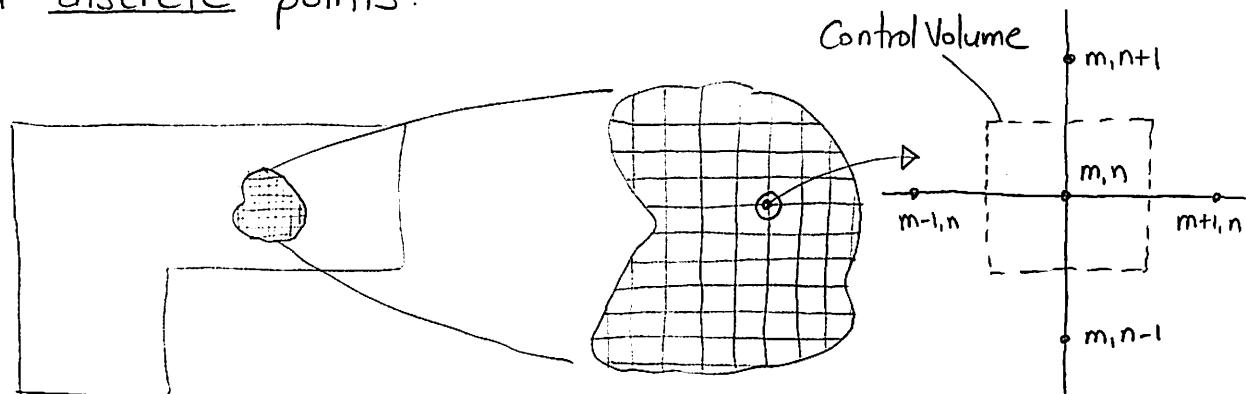
Sometimes, we are presented with non-trivial geometries which require numerical solution of the heat equation.

For example:



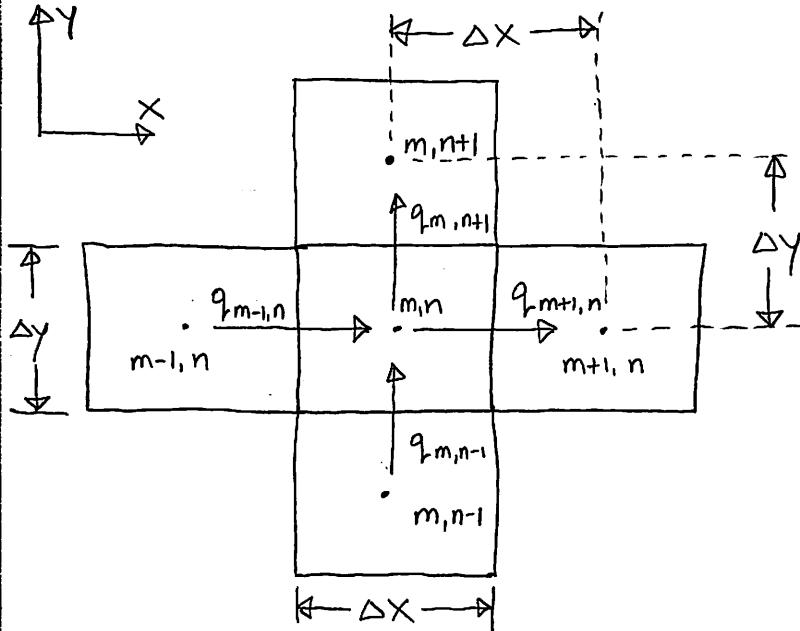
So how do we solve numerically:

First we need to break up our domain into a finite set of discrete points:



Assume $T_{n,m}$ is the average temperature of the control volume. (86)

Applying an energy balance on our control volume:



$$\dot{E}_{in} - \dot{E}_{out} = 0 \quad (\text{Assuming SS, unit depth (20), } Q''=0)$$

Applying Fourier's Law:

$$q = -kA \frac{\partial T}{\partial x}$$

$$q_{m-1,n \rightarrow m,n} = k \Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_{m,n \rightarrow m+1,n} = k \Delta y \frac{T_{m,n} - T_{m+1,n}}{\Delta x}$$

Note, negative sign dropped because I flipped ΔT .

We can do the same for the y-direction & substitute into our energy balance:

$$k \frac{\Delta y}{\Delta x} (T_{m-1,n} - T_{m,n}) - k \frac{\Delta y}{\Delta x} (T_{m,n} - T_{m+1,n}) + k \frac{\Delta x}{\Delta y} (T_{m,n-1} - T_{m,n}) - k \frac{\Delta x}{\Delta y} (T_{m,n} - T_{m,n+1}) = 0$$

Assuming $\Delta x = \Delta y$, we obtain:

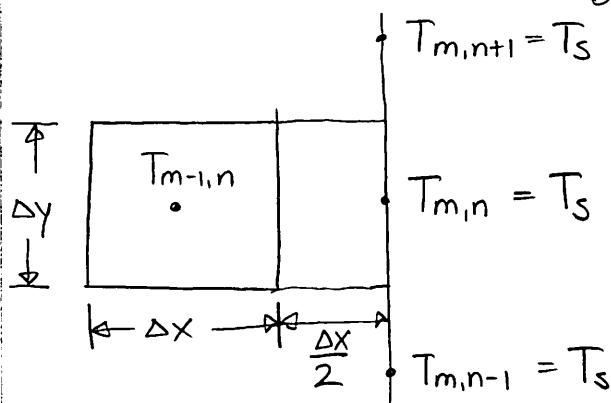
$$\frac{T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1}}{4} = T_{m,n}$$

\Rightarrow The nodal temp. is the average of its neighbours.

We would apply this equation for all interior nodes.

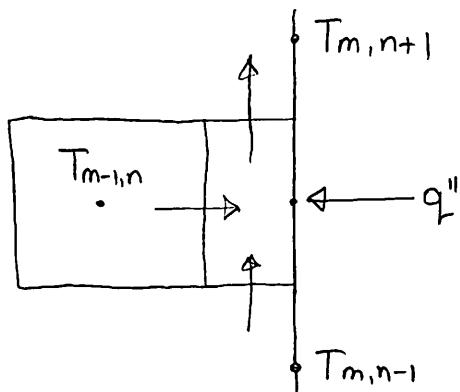
Boundary Conditions (Applied to all surface or boundary nodes)

- ① Fixed Temperature - set node temperature to equal the boundary condition.



For all other B.C.'s
see Table 4.2, pg. 246
of Textbook.

- ② Uniform Heat Flux (q'')



Applying an energy balance:

$$\dot{E}_{in} - \dot{E}_{out} = 0$$

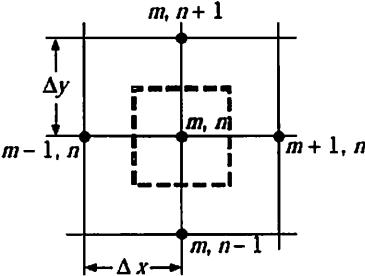
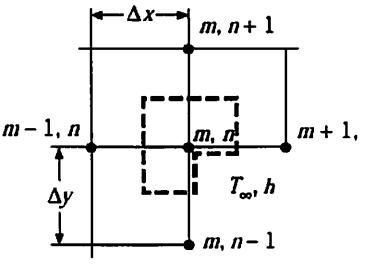
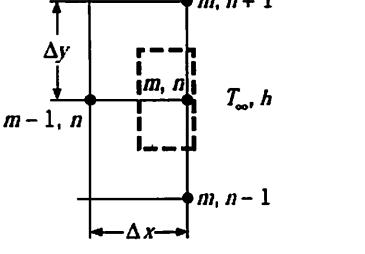
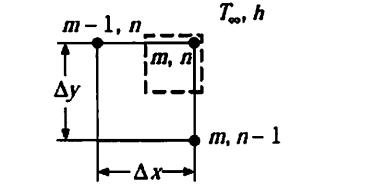
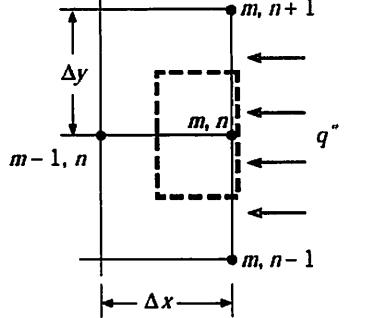
$$k \frac{\Delta y}{\Delta x} (T_{m-1,n} - T_{m,n}) + \frac{k \Delta x}{2 \Delta y} (T_{m,n-1} - T_{m,n}) - k \frac{\Delta x}{2 \Delta y} (T_{m,n} - T_{m,n+1}) + q'' \Delta y = 0$$

If $\Delta x = \Delta y$:

$$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} + \frac{2q'' \Delta x}{k} - 4T_{m,n} = 0$$

Special case ($q''=0$): $T_{m,n} = \frac{1}{4}(2T_{m-1,n} + T_{m,n-1} + T_{m,n+1})$ (88)

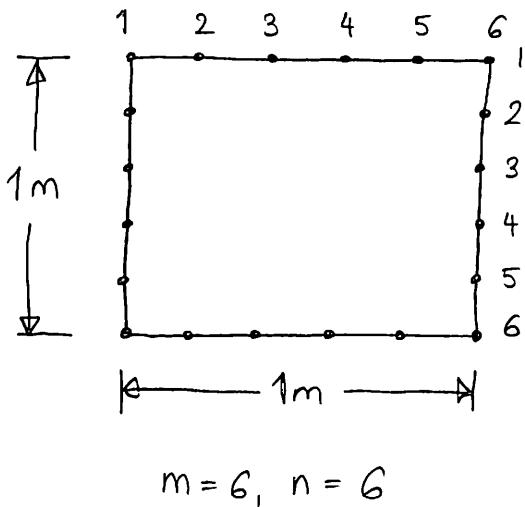
Table 4.2 of Incropera

Configuration	Finite-Difference Equation for $\Delta x = \Delta y$
	$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$
Case 1. Interior node	
	$2(T_{m-1,n} + T_{m,n+1}) + (T_{m+1,n} + T_{m,n-1}) + 2\frac{h\Delta x}{k} T_\infty - 2\left(3 + \frac{h\Delta x}{k}\right) T_{m,n} = 0$
Case 2. Node at an internal corner with convection	
	$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\Delta x}{k} T_\infty - 2\left(\frac{h\Delta x}{k} + 2\right) T_{m,n} = 0$
Case 3. Node at a plane surface with convection	
	$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\Delta x}{k} T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right) T_{m,n} = 0$
Case 4. Node at an external corner with convection	
	$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2q''\Delta x}{k} - 4T_{m,n} = 0$
Case 5. Node at a plane surface with uniform heat flux	

^{a,b}To obtain the finite-difference equation for an adiabatic surface (or surface of symmetry), simply set h or q'' equal to zero.

Solution Methodology

- ① Discretize the solution domain



$$M = \# \text{ of nodes in } x\text{-direction}$$

$$N = \# \text{ of nodes in } y\text{-direction}$$

$$\Delta x = \frac{1}{M-1} = 0.2 \text{ m}$$

$$\Delta y = \frac{1}{N-1} = 0.2 \text{ m}$$

$$\begin{aligned} \# \text{ of interior nodes} &= 16 \\ \# \text{ of exterior nodes} &= 20 \end{aligned} \quad \left. \right\} 36 \text{ total nodes}$$

- ② Form finite volume equations for all internal nodes

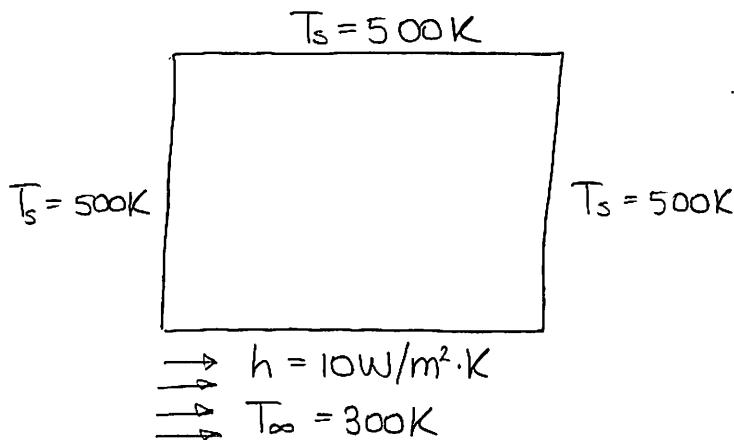
$$2 \leq m \leq M-1, \quad 2 \leq n \leq N-1$$

- ③ Form finite volume equations for all boundary nodes

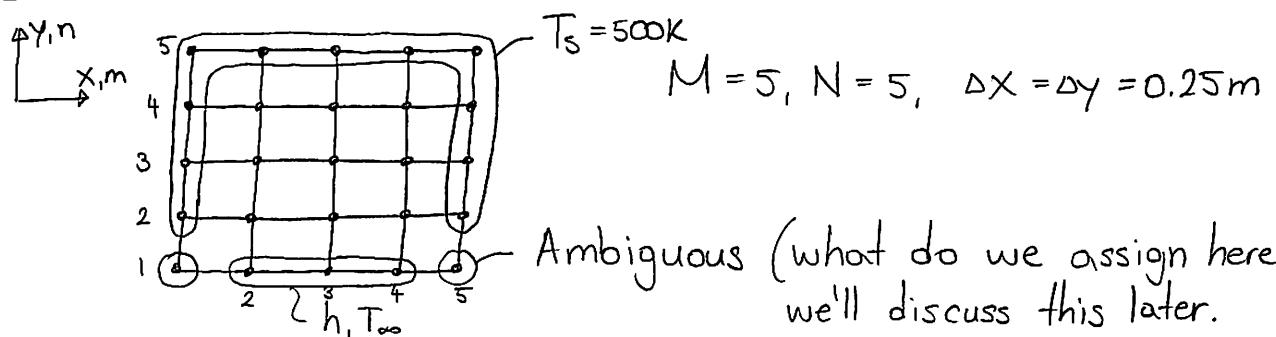
- ④ Solve $M \times N$ equations and unknowns using:

- i) Gaussian elimination (direct)
- ii) Gauss-Seidel method (indirect and iterative)
- iii) Software (i.e. Excel, Matlab, Maple, etc...)

Example] A $1 \times 1\text{m}$ square concrete column, $k = 1\text{ W/m}\cdot\text{K}$. Find the temperature distribution in the column.



① Discretize the solution domain:



② Interior nodes: $2 \leq m \leq 4, 2 \leq n \leq 4$

$$T_{m,n} = \frac{1}{4} (T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1})$$

↳ 9 equations

③ Exterior nodes:

For: $m=1, 2 \leq n \leq 5$

$n=5, 1 \leq m \leq 5$

$m=5, 2 \leq n \leq 5$

$n=1, 2 \leq m \leq 4 \Rightarrow$ Convective boundary condition

$T_{m,n} = T_s = 500\text{K}$

→ 11 equations

For the convective boundary condition, look at Table 4.2, eq. 4.42^a

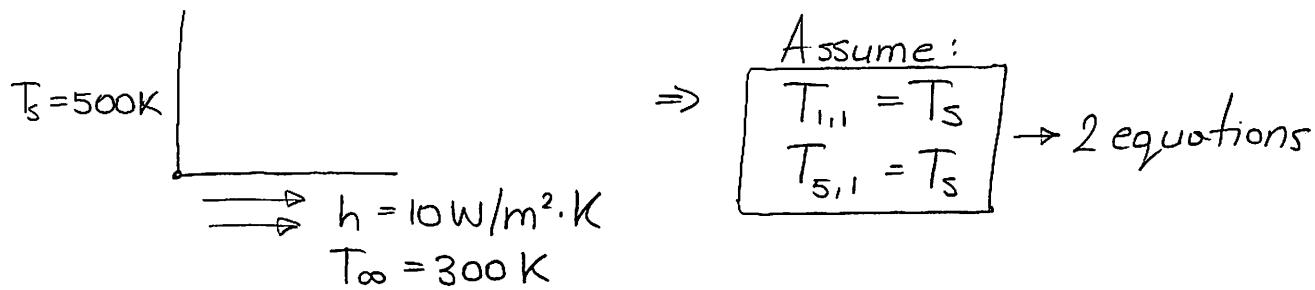
$$(2T_{m,n+1} + T_{m-1,n} + T_{m+1,n}) + \frac{2h\Delta Y}{k} T_{\infty} - 2\left(\frac{h\Delta Y}{k} + 2\right) T_{m,n} = 0$$

Back substituting h & T_{∞} , & solving for $T_{m,n}$:

$$T_{m,n} = \frac{1}{q} (2T_{m,n+1} + T_{m-1,n} + T_{m+1,n} + 1500)$$

↳ 3 equations

But what about the corners?



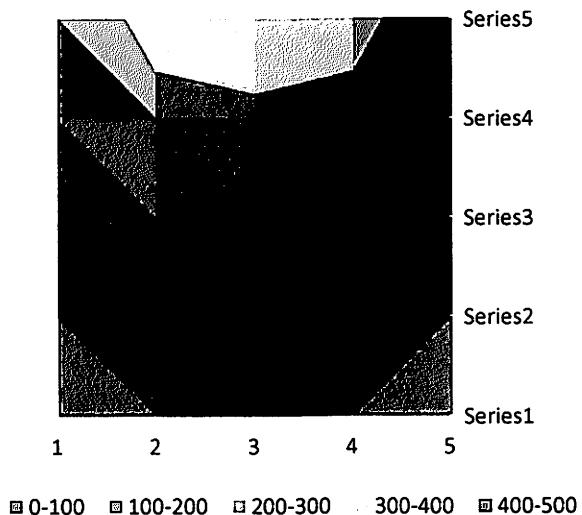
- ④ Assemble the equation set : 25 equations, 25 unknowns
- ⑤ Solve the set of equations (we will use Excel)

$$N = M = 5$$

500	500	500	500	500
500	489.3047	485.1538	489.3047	500
500	472.0651	462.0058	472.0651	500
500	436.9498	418.7393	436.9498	500
500	356.9946	339.052	356.9946	500

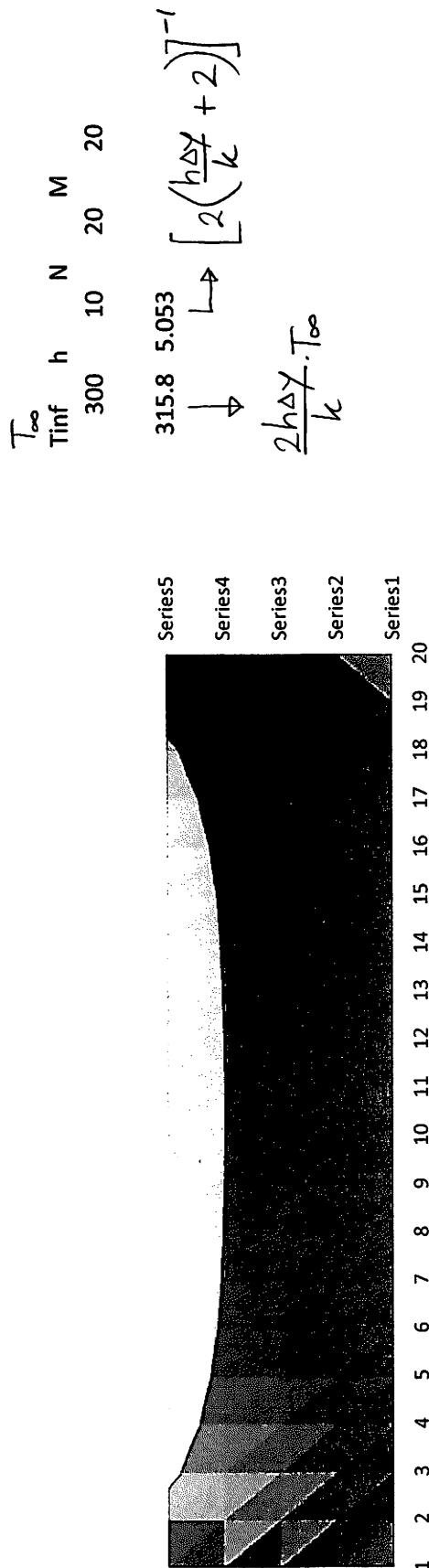
T_{∞}	T_{inf}	h	N	M
	300	10	5	5

$$\frac{2h\Delta y}{k} T_{\infty} \downarrow \rightarrow \left[2 \left(\frac{h\Delta y}{k} + 2 \right) \right]^{-1}$$



)
 $N = M = 20$

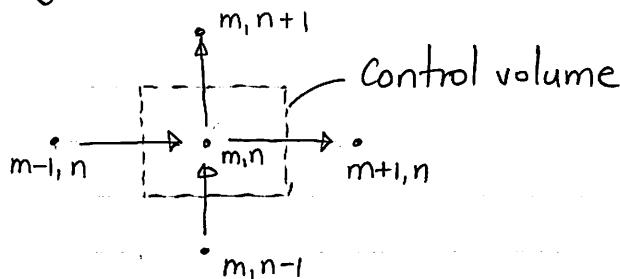
500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500
500	489.2	480.8	475	471.3	469.2	467.9	467.2	466.8	466.6	466.8	467.2	467.9	469.2	471.3	475	480.8	489.2	500	500
500	476.2	458.9	447.8	441.2	437.4	435.2	434	433.3	433	433.3	434	435.2	437.4	441.2	447.8	458.9	476.2	500	500
500	456.6	430.6	416.3	408.4	404	401.5	400.2	399.5	399.2	399.2	399.5	400.2	401.5	404	408.4	416.3	430.6	456.6	500
500	419.6	390.9	378.3	372	368.6	366.8	365.8	365.3	365.1	365.1	365.3	365.8	366.8	368.6	372	378.3	390.9	419.6	500



300-400 400-500

Transient Heat Conduction Numerical Solution

Looking back at our interior node:



Performing an energy balance on our control volume:

$$\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{stored} \quad ①$$

$$\dot{E}_{stored} = \rho A C \frac{\partial T}{\partial t} \quad ②$$

Approximating ② as a linear function (assuming unit depth)

$$\dot{E}_{stored} \approx \rho \Delta x \Delta y C \left(\frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t} \right)$$

where: Δt = time step $T_{m,n}^{p+1}$ = new value $T_{m,n}^p$ = old value

For our other terms:

$$\dot{E}_{in}, \dot{E}_{out} = -k \Delta x (\text{or } \Delta y) \cdot \frac{T_{m,n} ? - T_{m-1,n} ?}{\Delta y (\text{or } \Delta x)}$$

where ? refers to when (time) the values are considered.

① Old values - explicit

② New values - implicit

③ Combination - Crank-Nicholson scheme

The general approach is:

$$T_{m,n}^? = f T_{m,n}^{P+1} + (1-f) T_{m,n}^P$$

for $f = 0$ = explicit (old values)

$f = 1$ = implicit (new values)

$f = \frac{1}{2}$ = Crank - Nicolson scheme (combination)

Now we can back substitute & solve:

$$\dot{E}_{in} - \dot{E}_{out} = \dot{E}_{STORED}$$

$$k \frac{\Delta Y}{\Delta X} (T_{m-1,n}^? - T_{m,n}^?) - k \frac{\Delta Y}{\Delta X} (T_{m,n}^? - T_{m+1,n}^?)$$

$$+ k \frac{\Delta X}{\Delta Y} (T_{m,n-1}^? - T_{m,n}^?) - k \frac{\Delta X}{\Delta Y} (T_{m,n}^? - T_{m,n+1}^?) = \rho \Delta X \Delta Y C \left(\frac{T_{m,n}^{P+1} - T_{m,n}^P}{\Delta t} \right)$$

Dividing both sides by $\Delta X \Delta Y \rho C$; $\alpha = \frac{k}{\rho C}$

$$\underbrace{\frac{1}{\alpha} \frac{T_{m,n}^{P+1} - T_{m,n}^P}{\Delta t}}_{\frac{\partial T}{\partial t}} = \underbrace{\frac{T_{m+1}^? - 2T_{m,n}^? + T_{m-1,n}^?}{(\Delta X)^2}}_{\frac{\partial^2 T}{\partial x^2}} + \underbrace{\frac{T_{m,n+1}^? + 2T_{m,n}^? + T_{m,n-1}^?}{(\Delta Y)^2}}_{\frac{\partial^2 T}{\partial y^2}}$$

(Linearisation
of these terms)

① Explicit ($f=0$)

$$\frac{1}{\alpha} \frac{T_{m,n}^{P+1} - T_{m,n}^P}{\Delta t} = \frac{T_{m+1} - 2T_{m,n} + T_{m-1,n}}{(\Delta X)^2} + \frac{T_{m,n+1} + 2T_{m,n} + T_{m,n-1}}{(\Delta Y)^2}$$

↳ Assuming $\Delta X = \Delta Y$ and 1D (x-only)

$$T_{m,n}^{p+1} = T_{m,n}^p + \underbrace{\frac{\alpha \Delta t}{(\Delta x)^2} (\dots)}_{\text{Fourier \#}}^p$$

$$\text{Fourier \#} = \frac{\alpha \Delta t}{(\Delta x)^2}$$

Expanding this expression, we obtain:

$$T_{m,n}^{p+1} = F_0 (T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1})^p + (1 - 4F_0) T_{m,n}^p$$

\hookrightarrow 2D conduction

$$T_m^{p+1} = F_0 (T_{m+1}^p + T_{m-1}^p) + (1 - 2F_0) T_m^p \rightarrow 1D \text{ conduction}$$

For other cases, look at Table 5.3 of text.

Stability

Although the explicit scheme is fast & nice, it suffers from instability in certain conditions. (Oscillations)

It can be shown mathematically that as long as the coefficient on the $T_{m,n}^p$ term is positive or zero, stability is maintained. I.e.:

1D conduction	$1 - 2F_0 \geq 0$
2D conduction	$1 - 4F_0 \geq 0$
3D conduction	$1 - 6F_0 \geq 0$

$$F_0 = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2} \text{ (for 1D)}$$

For small Δx , Δt must be very small. Computation expensive.

② Implicit ($f=1$)

$$\frac{1}{\alpha} \frac{T_{m,n}^{p+1} - T_{m,n}^p}{\Delta t} = \left(\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2} \right)^{p+1}$$

For $\Delta x = \Delta y$

$$(1 + 4F_0) T_{m,n}^{p+1} - F_0 (T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1})^{p+1} = T_{m,n}^p$$

Table 5.3 of Incropera

Configuration	(a) Explicit Method		(b) Implicit Method	
	Finite-Difference Equation	Stability Criterion	Finite-Difference Equation	Stability Criterion
	$\begin{aligned} T_{m,n}^{t+1} = & Fo(T_{m+1,n}^t + T_{m-1,n}^t \\ & + T_{m,n+1}^t + T_{m,n-1}^t) \\ & + (1 - 4Fo)T_{m,n}^t \end{aligned} \quad (5.76)$	$Fo \leq \frac{1}{4} \quad (5.80)$	$\begin{aligned} (1 + 4Fo)T_{m,n}^{t+1} - & Fo(T_{m+1,n}^{t+1} + T_{m-1,n}^{t+1} \\ & + T_{m,n+1}^{t+1} + T_{m,n-1}^{t+1}) = T_{m,n}^t \end{aligned}$	
1. Interior node				
	$\begin{aligned} T_{m,n}^{t+1} = & \frac{2}{3}Fo(T_{m+1,n}^t + 2T_{m-1,n}^t \\ & + 2T_{m,n+1}^t + T_{m,n-1}^t + 2Bi T_x) \\ & + (1 - 4Fo - \frac{4}{3}Bi Fo)T_{m,n}^t \end{aligned} \quad (5.85)$	$Fo(3 + Bi) \leq \frac{3}{4} \quad (5.86)$	$\begin{aligned} (1 + 4Fo(1 + \frac{1}{3}Bi))T_{m,n}^{t+1} - & \frac{2}{3}Fo \cdot \\ & (T_{m+1,n}^{t+1} + 2T_{m-1,n}^{t+1} + 2T_{m,n+1}^{t+1} + T_{m,n-1}^{t+1}) \\ = & T_{m,n}^t + \frac{4}{3}Bi Fo T_x \end{aligned}$	
2. Node at interior corner with convection				
	$\begin{aligned} T_{m,n}^{t+1} = & Fo(2T_{m-1,n}^t + T_{m,n+1}^t \\ & + T_{m,n-1}^t + 2Bi T_x) \\ & + (1 - 4Fo - 2Bi Fo)T_{m,n}^t \end{aligned} \quad (5.87)$	$Fo(2 + Bi) \leq \frac{1}{2} \quad (5.88)$	$\begin{aligned} (1 + 2Fo(2 + Bi))T_{m,n}^{t+1} - & \\ & - Fo(2T_{m-1,n}^{t+1} + T_{m,n+1}^{t+1} + T_{m,n-1}^{t+1}) \\ = & T_{m,n}^t + 2Bi Fo T_x \end{aligned}$	
3. Node at plane surface with convection ^a				
	$\begin{aligned} T_{m,n}^{t+1} = & 2Fo(T_{m-1,n}^t + T_{m,n-1}^t + 2Bi T_\infty) \\ & + (1 - 4Fo - 4Bi Fo)T_{m,n}^t \end{aligned} \quad (5.89)$	$Fo(1 + Bi) \leq \frac{1}{4} \quad (5.90)$	$\begin{aligned} (1 + 4Fo(1 + Bi))T_{m,n}^{t+1} - & \\ & - 2Fo(T_{m-1,n}^{t+1} + T_{m,n-1}^{t+1}) \\ = & T_{m,n}^t + 4Bi Fo T_\infty \end{aligned}$	
4. Node at exterior corner with convection				

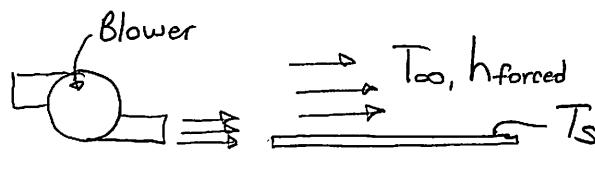
^aTo obtain the finite-difference equation and/or stability criterion for an adiabatic surface (or surface of symmetry), simply set Bi equal to zero.

Convection Heat Transfer

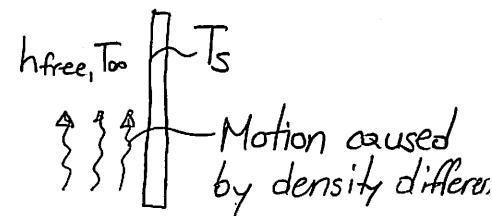
Convection is the transfer of thermal energy due to both conduction and by bulk "carrying" of the energy through the velocity of the fluid.

Four Categories

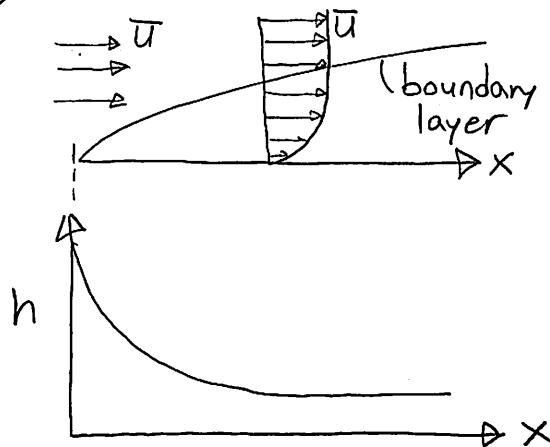
① Forced vs. Free (Natural)



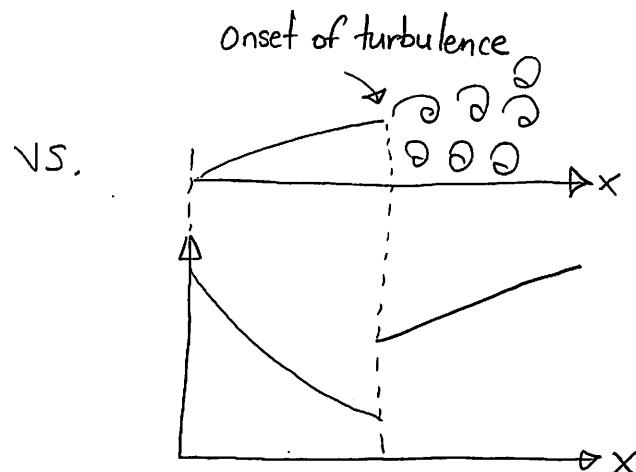
vs.



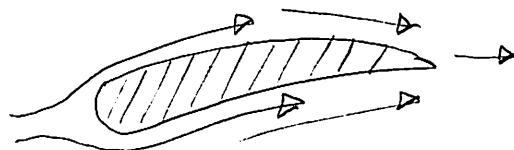
② Laminar vs. Turbulent



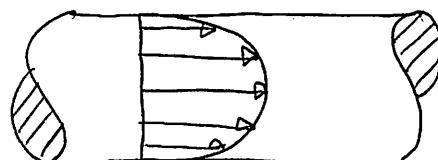
vs.



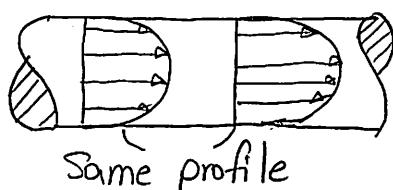
③ External vs. Internal



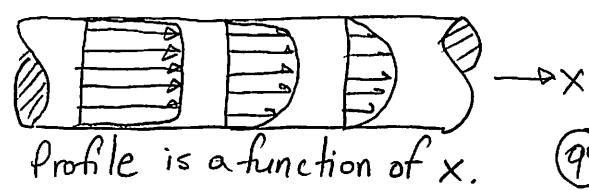
vs.



④ Fully Developed vs. Developing



vs.



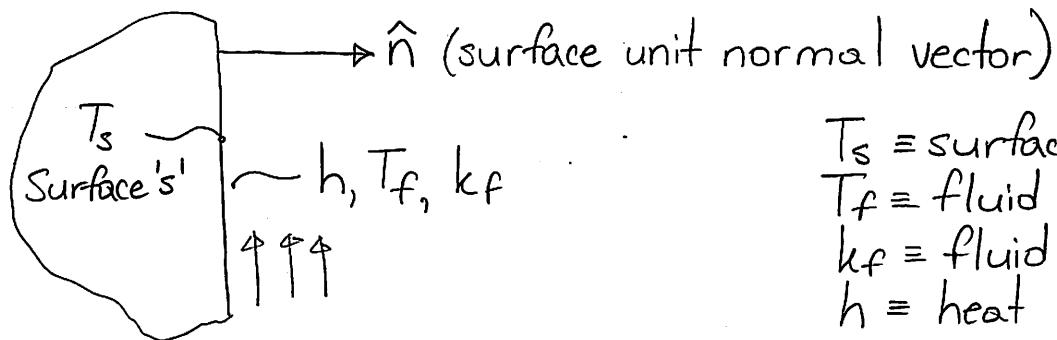
The Assumptions

- 1) Steady-State
- 2) Constant properties except where density differences cause the flow (Boussinesq approximation)
- 3) Incompressible

The Equations

- 1) Conservation of mass
- 2) Conservation of momentum (Navier-Stokes)
- 3) Conservation of energy (Heat equation for a fluid flow)

We've already seen convection before :



T_s = surface temperature
 T_f = fluid temperature
 k_f = fluid thermal cond.
 h = heat transfer coeff.

We know that :

$$q'' = h(T_s - T_f) \quad ① \quad (\text{heat flux})$$

We can also write that :

$$q'' = -k_f \frac{\partial T_f}{\partial \hat{n}} \Big|_s \quad ② \quad (\text{heat flux by Fourier's Law in fluid})$$

Let's non dimentionalize our problem :

$$\Theta = \frac{T - T_f}{T_s - T_f} ; \quad n^* = \frac{\hat{n}}{L}$$

Back substituting into ① & ② & equating the two:

$$-k_f \frac{T_s - T_f}{L} \cdot \left. \frac{\partial \Theta}{\partial n^*} \right|_s = h (T_s - T_f)$$

$$\left. - \frac{\partial \Theta}{\partial n^*} \right|_s = \frac{hL}{k_f} = Nu \Rightarrow \text{Nusselt Number}$$

$$Nu = \frac{hL}{k_f} = \frac{\text{convective heat transfer rate in fluid}}{\text{conductive heat transfer rate in fluid}}$$

→ Note, very similar to Biot number (Bi) but

$$Bi = \frac{hL}{k_s} \leftarrow \text{solid thermal conductivity}$$

So Nu is a measure of the heat transfer enhancement you have due to bulk fluid motion when compared to conduction only.

Note, if $Nu = 1$, no convection effects & conduction only.

$$Nu = - \left. \frac{\partial \Theta}{\partial n^*} \right|_s = 1 \Rightarrow -k_f \underbrace{\frac{T_s - T_f}{L}}_{\text{Conduction only}} = \underbrace{h(T_s - T_f)}_{\text{convection}}$$

Since $Nu \sim \left. \frac{\partial \Theta}{\partial n^*} \right|_s \sim \Theta$, it typically depends on:

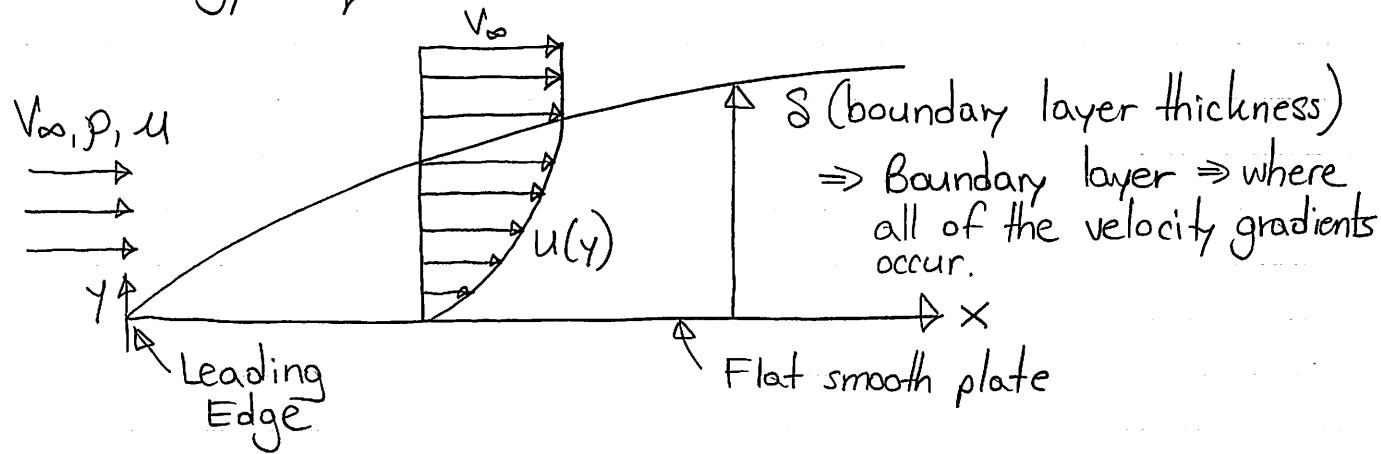
- 1) Flow conditions (laminar, turbulent)
- 2) Fluid properties
- 3) Geometry
- 4) Boundary conditions

External Flow

Assuming:

- 1) $\rho, u, C = \text{constant}$
- 2) Incompressible

We need to solve the coupled fluid flow (momentum) and energy equations to solve for h .



For a 2D laminar flow, flat plate, & steady

X-momentum:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

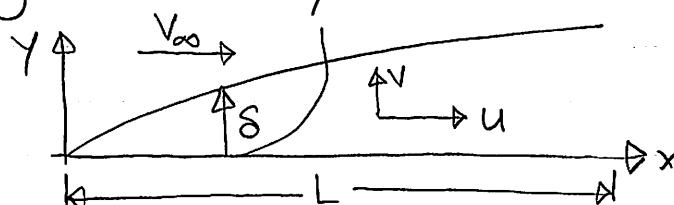
y-momentum:

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$$

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Looking more closely at our boundary layer :



Let's non-dimensionalize

$$\text{Let: } \bar{U} = \frac{U}{V_\infty}, \quad \bar{V} = \frac{V}{V_s}, \quad \bar{P} = \frac{P}{\rho V_\infty^2}$$

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{\delta}$$

We know all of these parameters except for V_s .
Using continuity: $\bar{U} \sim V_\infty, \bar{x} \sim L, \bar{y} \sim \delta$ ("~" means scales as)

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \Rightarrow \frac{V_\infty}{L} + \frac{V_s}{\delta} = 0 \Rightarrow V_s \sim \left(\frac{\delta}{L}\right) V_\infty$$

Note, when I use "~", it means scales as or is on the same order of magnitude as. Very powerfull method!

We know for boundary layers that $\delta \ll L$, hence $V_s \ll V_\infty$

So if we back substitute into our x -momentum

$$V_\infty \bar{U} \frac{V_\infty}{L} \frac{\partial \bar{U}}{\partial \bar{x}} + \left(\frac{\delta}{L}\right) V_\infty \bar{V} \frac{V_\infty}{\delta} \frac{\partial \bar{U}}{\partial \bar{y}} = - \frac{1}{\rho} \frac{\partial V_\infty^2}{L} \frac{\partial \bar{P}}{\partial \bar{x}} + \left[\frac{V_\infty \partial \bar{U}}{L^2 \partial \bar{x}^2} + \frac{V_\infty \partial \bar{U}}{\delta^2 \partial \bar{y}^2} \right]$$

$$\frac{V_\infty^2}{L} \bar{U} \frac{\partial \bar{U}}{\partial \bar{x}} + \frac{V_\infty^2}{L} \bar{V} \frac{\partial \bar{U}}{\partial \bar{y}} = - \frac{V_\infty^2}{L} \frac{\partial \bar{P}}{\partial \bar{x}} + \frac{V_\infty V}{L^2} \left[\frac{\partial^2 \bar{U}}{\partial \bar{x}^2} + \underbrace{\left(\frac{L^2}{\delta^2} \right)}_{\gg 1, \text{ so dominate}} \frac{\partial^2 \bar{U}}{\partial \bar{y}^2} \right]$$

Divide both sides by $\frac{V_\infty^2}{L}$

$$\bar{U} \frac{\partial \bar{U}}{\partial \bar{x}} + \bar{V} \frac{\partial \bar{U}}{\partial \bar{y}} = - \frac{\partial \bar{P}}{\partial \bar{x}} + \underbrace{\frac{U}{V_\infty L} \frac{\partial^2 \bar{U}}{\partial \bar{y}^2}}$$

Note, these are all ~ 1 .

$$\frac{1}{Re_L} \Rightarrow Re_L = \frac{V_\infty L}{V} = \frac{\rho V_\infty L}{U}$$

Where: $Re_L = \text{Reynolds Number} = \frac{\text{inertial force}}{\text{viscous force}}$

$$\bar{U} \frac{\partial \bar{U}}{\partial \bar{x}} + \bar{V} \frac{\partial \bar{U}}{\partial \bar{y}} = - \frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{Re_L} \frac{\partial^2 \bar{U}}{\partial \bar{y}^2} \quad (1)$$

Non-dimensional
x-momentum equation

Let's now look at y-momentum:

$$U \frac{\partial V}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{1}{\rho} \frac{\partial P}{\partial Y} + \nu \left[\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right] \Rightarrow \text{Non-dimensionalize this}$$

$$V_\infty \bar{U} \left(\frac{\delta}{L} \right) V_\infty \cdot \frac{1}{L} \frac{\partial \bar{V}}{\partial \bar{x}} + \left(\frac{\delta}{L} \right) V_\infty \bar{V} \left(\frac{\delta}{L} \right) V_\infty \frac{1}{\delta} \frac{\partial \bar{V}}{\partial \bar{y}} = - \frac{\rho V_\infty^2 \partial \bar{P}}{\delta \delta^2 \partial \bar{y}} + \nu \left[\left(\frac{\delta}{L} \right) V_\infty \frac{1}{L^2} \frac{\partial^2 \bar{V}}{\partial \bar{x}^2} + \left(\frac{\delta}{L} \right) V_\infty \frac{1}{\delta^2} \frac{\partial^2 \bar{V}}{\partial \bar{y}^2} \right]$$

Collecting all terms and multiplying both sides by δ

$$V_\infty^2 \left(\frac{\delta^2}{L^2} \right) \bar{U} \frac{\partial \bar{V}}{\partial \bar{x}} + V_\infty^2 \left(\frac{\delta^2}{L^2} \right) \bar{V} \frac{\partial \bar{V}}{\partial \bar{y}} = - V_\infty^2 \frac{\partial \bar{P}}{\partial \bar{y}} + \nu V_\infty \left[\left(\frac{\delta^2}{L^3} \right) \frac{\partial^2 \bar{V}}{\partial \bar{x}^2} + \frac{1}{L} \frac{\partial^2 \bar{V}}{\partial \bar{y}^2} \right]$$

Divide both sides by V_∞^2 :

$$\left(\frac{\delta^2}{L^2} \right) \bar{U} \frac{\partial \bar{V}}{\partial \bar{x}} + \left(\frac{\delta^2}{L^2} \right) \bar{V} \frac{\partial \bar{V}}{\partial \bar{y}} = - \frac{\partial \bar{P}}{\partial \bar{y}} + \frac{\nu}{V_\infty L} \left[\left(\frac{\delta^2}{L^2} \right) \frac{\partial^2 \bar{V}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{V}}{\partial \bar{y}^2} \right]$$

Since $\left(\frac{\delta}{L} \right)^2 \ll 1$ (Boundary Layer approx), many of our terms drop out.

$$\frac{\partial \bar{P}}{\partial \bar{y}} = \frac{1}{Re_L} \frac{\partial^2 \bar{V}}{\partial \bar{y}^2} \quad (2)$$

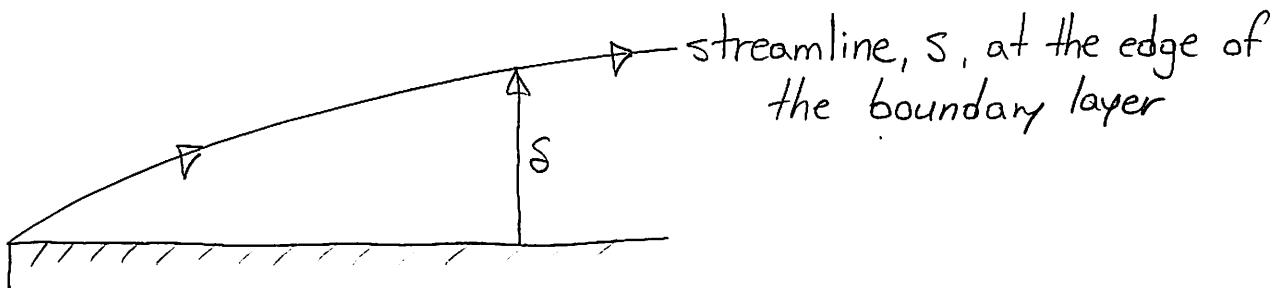
Non-dimensional y-momentum

It turns out you can use scaling analysis to show that

$$\frac{\partial^2 \bar{V}}{\partial \bar{y}^2} \ll \frac{\partial^2 \bar{U}}{\partial \bar{y}^2} \Rightarrow \frac{\partial \bar{P}}{\partial \bar{y}} \approx 0 \Rightarrow \boxed{P = f(x) \text{ only}}$$

↳ Derived in ME521 (Convective Heat Transfer)

Now our life becomes much easier. We can use one more trick up our sleeves:



At the streamline, our x-momentum equation becomes:

$$u \frac{\partial u}{\partial x} + v \underbrace{\frac{\partial u}{\partial y}}_{0 \text{ at } s \text{ (no shear)}} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \underbrace{\left(\frac{\partial^2 u}{\partial y^2} \right)}_{0 \text{ at } s \text{ (no shear)}}$$

At s , $u = V_\infty$

$$V_\infty \frac{\partial V_\infty}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad \xrightarrow{\text{integrate}} \boxed{p + \frac{1}{2} \rho V_\infty^2 = \text{constant}}$$

\hookrightarrow Bernoulli equation

And since we know $p \neq f(y)$, this is also true inside the boundary layer.

We also know that: $V_\infty = \text{constant}$, so $\frac{\partial V_\infty}{\partial x} = 0$

$$\therefore \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \Rightarrow \boxed{p(x) = \text{constant}} \quad \text{or} \quad \boxed{\frac{\partial p}{\partial x} = \frac{\partial \bar{p}}{\partial x} = 0}$$

Note, this is only true for a flat plate. Not true for cylinders or spheres where $V_\infty \neq \text{constant}$.

Now we can re-cast our x-momentum equation:

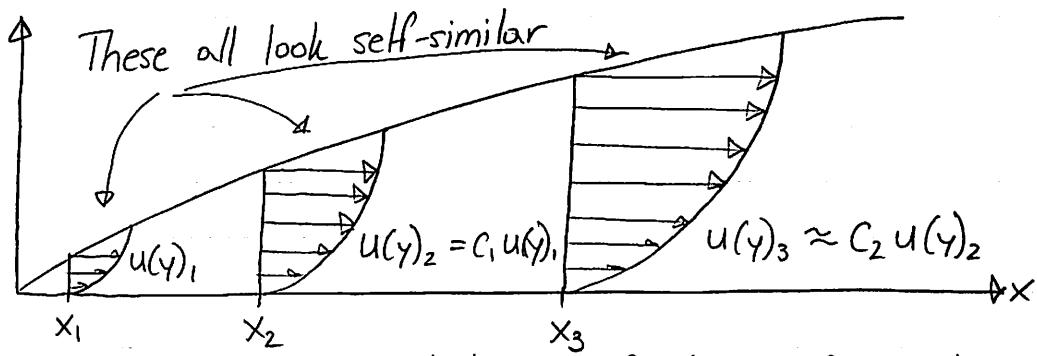
$$\boxed{u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2}}$$

Our boundary conditions are:

- 1) $U = V = 0$ at $y = 0$
- 2) $U = V_\infty$ at $y \rightarrow \infty$

Just like in transient conduction, we want to convert our simpler but still complex PDE into an ODE.

Cannot use separation of variables (infinite domain, $y \rightarrow \infty$)
Let's try the similarity solution:



→ A function ϕ that is a function of η only.

$$\frac{U}{V_\infty} = \phi(\eta); \text{ where } \eta = \frac{y}{\delta}$$

↳ Scales from 0 to 1

↳ Scales from 0 to 1

But how do we define δ ?

Let's look back at our dimensionless momentum equation:

$$\overline{U} \frac{\partial \overline{U}}{\partial \overline{x}} + \overline{V} \frac{\partial \overline{U}}{\partial \overline{y}} = \underbrace{\frac{1}{Re_L} \frac{\partial^2 \overline{U}}{\partial \overline{y}^2}}$$

We know these are both on the order 1. Also:

$$\overline{U} \sim V_\infty$$

$$\frac{\overline{Y}}{\overline{Y}} = \frac{Y}{L} \sim \frac{\delta}{L} \Rightarrow \frac{\partial \overline{Y}}{\partial \overline{Y}} \sim \frac{\delta}{L}$$

$$\frac{V}{V_\infty L} \cdot V_\infty \cdot \left(\frac{L}{\delta} \right)^2 \sim 1 \Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}} \text{ or } \frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}}$$

So we end up with: $\delta \sim \sqrt{\frac{Ux}{V_\infty}}$ or $n = y \sqrt{\frac{V_\infty}{Ux}}$

→ We'll see a much easier way to get this later.

So what do we do about V ?

Using continuity:

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0 \Rightarrow \bar{U} = \phi(n) = \frac{y}{V_\infty}$$

$$\frac{\partial \bar{U}}{\partial x} = \frac{\partial \bar{U}}{\partial n} \cdot \frac{\partial n}{\partial x}$$

$$\frac{\partial \bar{U}}{\partial x} = -\frac{1}{2x} n \phi' \Rightarrow \text{Back substitute into continuity}$$

$$\frac{\partial \bar{V}}{\partial y} = \frac{1}{2x} n \phi' \quad d\eta = dy \sqrt{\frac{V_\infty}{Ux}}$$

$$\bar{V} = \frac{1}{2x} \int_0^n \phi' n dy = \underbrace{\frac{1}{2x} \sqrt{\frac{XU}{V_\infty}}}_{\frac{1}{2\sqrt{R_{ex}}} \downarrow} \int_0^n \phi' n d\eta$$

We need to solve this integral:

$$\int_0^n n \phi' d\eta = uv - \int v du \quad (\text{Integration by parts})$$

$$= n\phi - \int_0^n \phi d\eta \quad \int u dv = uv - \int v du$$

$$\text{let } \int_0^n \phi d\eta = F(n) \quad u = n, v = \phi$$

Now we have: $\bar{V} = \frac{1}{2\sqrt{R_{ex}}} (nF' - F)$

Back substituting into our initial PDE

$$\bar{U} \frac{\partial \bar{U}}{\partial \bar{x}} + \bar{V} \frac{\partial \bar{U}}{\partial \bar{y}} = \frac{1}{Re_L} \frac{\partial^2 \bar{U}}{\partial \bar{y}^2}$$

\uparrow

$\bar{U} = F'$ $\frac{\partial \bar{U}}{\partial \bar{x}} = \frac{\partial \bar{U}}{\partial \eta} \cdot \frac{\partial \eta}{\partial \bar{x}} = F'' \cdot \frac{\partial \eta}{\partial \bar{x}}$... (Back substitute the rest)

Simplifying, we will obtain:

$$\boxed{F''' + \frac{1}{2} F F'' = 0} \quad \text{where } F = \int_0^2 \phi d\eta, \quad \eta = y \sqrt{\frac{V_\infty}{x U}}$$

↳ ODE!

Our boundary conditions now become:

- 1) At the wall ($y=0, \eta=0$), $F' = F = 0$
- 2) Outside the boundary layer ($y \rightarrow \infty, \eta \rightarrow \infty$), $F' = 1$

Rewriting our equation in terms of \bar{U}, F , and η :

$$\boxed{\frac{\partial^2 \bar{U}}{\partial \eta^2} + \frac{1}{2} F \frac{\partial \bar{U}}{\partial \eta} = 0} \quad ①$$

To solve this ODE, we can assume an infinite series sol.:

$$F = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + \dots$$

$$F' = a_1 + 2a_2 \eta + 3a_3 \eta^2 + \dots$$

$$F'' = 2a_2 + 6a_3 \eta + \dots$$

$$F''' = 6a_3 + \dots$$

Back substitute into ① and solve for the coefficients

$$\underbrace{(\dots)}_0 n^0 + \underbrace{(\dots)}_0 n' + \underbrace{(\dots)}_0 n^2 + \underbrace{(\dots)}_0 n^3 + \dots + \underbrace{(\dots)}_0 n^n = 0$$

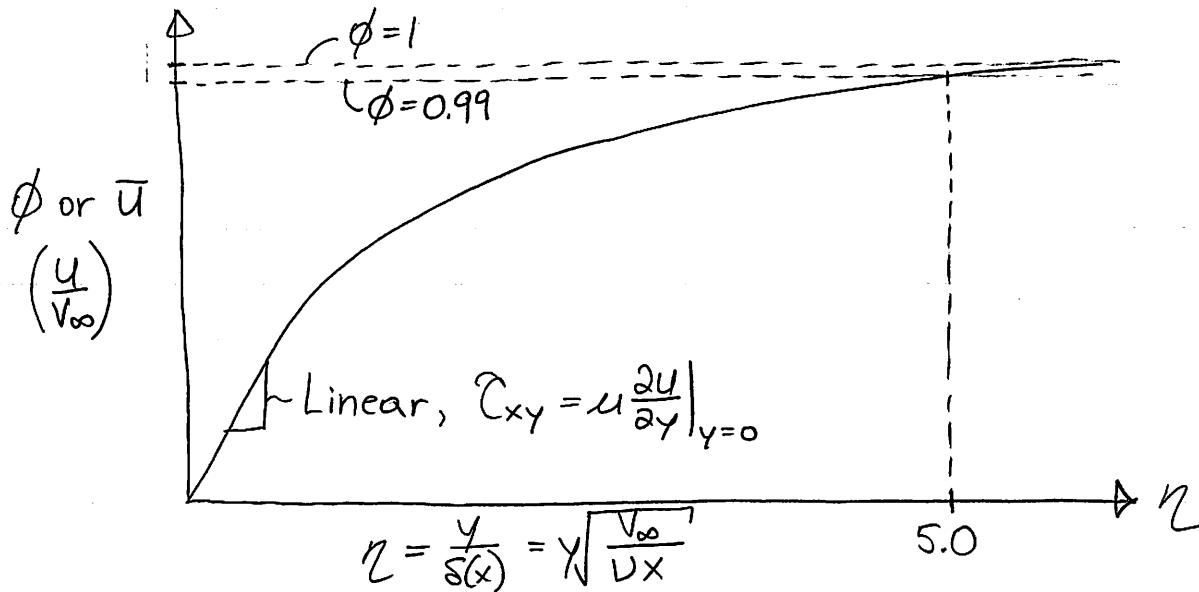
You obtain a recursion formula that relates your constants:

$$F = \frac{\alpha_2 n^2}{2!} - \frac{\alpha_2 n^5}{2 \cdot 5!} + \frac{11}{4} \frac{\alpha_2^3 n^8}{8!} + \dots$$

$\alpha_2 = 0.332 \Rightarrow$ Heinrich Blasius solved this in 1911
for his PhD work with Ludwig Prandtl.

$$\text{So } F = \int_0^n \phi dn, \quad F' = \phi = \bar{U} = \frac{U}{V_\infty}$$

We can now plot our result:



From our solution, we can solve for the hydrodynamic b.l. thick.

$$5.0 = S \sqrt{\frac{V_\infty}{U x}} = \frac{S}{x} \sqrt{R e_x} \Rightarrow S(x) = \frac{5x}{\sqrt{R e_x}}$$

↳ Boundary layer thickness

This is awesome because now we can calculate all kinds of interesting & usefull things:

Shear Stress:

$$\begin{aligned}\tau(x) &= \mu \frac{\partial u}{\partial y} \Big|_{y=0} = \rho U V_\infty \frac{\partial \bar{u}}{\partial y} \Big|_{y=0} \\ &= \rho U V_\infty \frac{\partial \bar{u}}{\partial z} \Big|_{z=0} \cdot \frac{\partial \eta}{\partial y} \Big|_{y=0} \\ \tau(x) &= \rho U V_\infty \alpha_2 \sqrt{\frac{V_\infty}{xU}}\end{aligned}$$

Typically, you're used to seeing it in terms of a skin friction coefficient, C_D

Reynolds number

$$C_{D,x} = \frac{\tau(x)}{\frac{1}{2} \rho V_\infty^2} = \frac{2 \alpha_2}{\sqrt{Re_x}} = \frac{0.664}{Re_x^{1/2}}$$

$$Re_x = \frac{\rho V_\infty x}{\mu}$$

↳ Skin friction coefficient for a flat plate in laminar flow.

Typically we want the average shear:

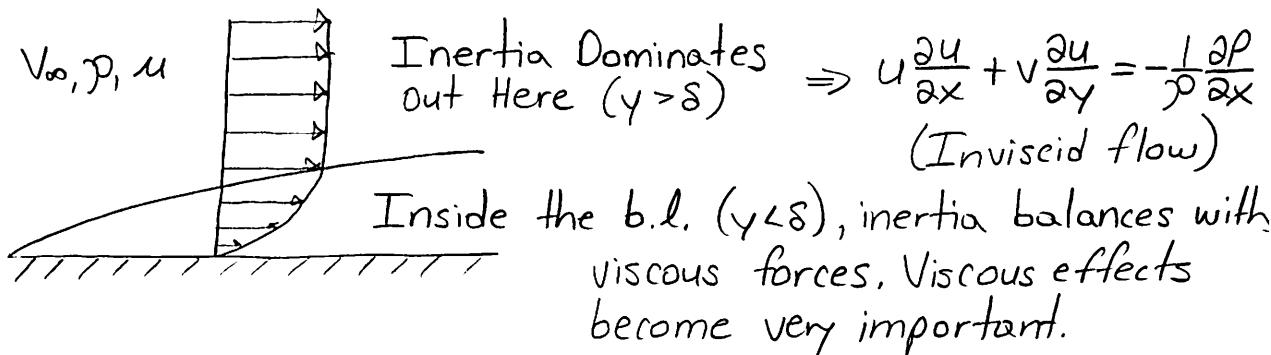
$$\bar{\tau} = \frac{1}{L} \int_0^L \tau(x) dx \Rightarrow \tau(x) = C \cdot \frac{1}{\sqrt{x}}$$

$$= C \cdot \frac{1}{L} \int_0^L \frac{dx}{\sqrt{x}} = \frac{2C}{\sqrt{L}}$$

$$C_D = \frac{\bar{\tau}}{\frac{1}{2} \rho V_\infty^2} = \frac{1.328}{Re_L^{1/2}}, \quad Re_L = \frac{\rho V_\infty L}{\mu}$$

↳ Plate averaged skin friction coefficient

Note, there is a much easier way to solve for all this.
Use scaling:



Inertia \sim Viscosity

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} \sim V \frac{\partial^2 U}{\partial y^2}$$

$$V_\infty \frac{V_\infty}{L} + V_\infty \left(\frac{\delta}{L}\right) \cdot \frac{V_\infty}{\delta} \sim V \frac{V_\infty}{\delta^2}$$

$$\frac{V_\infty^2}{L} \sim V \frac{V_\infty}{\delta^2}$$

$$\delta^2 \sim \frac{UL}{\rho V_\infty} \Rightarrow \delta \sim \sqrt{\frac{UL}{V_\infty}} \cdot \left(\frac{L}{L}\right) \Rightarrow \boxed{\delta \sim \frac{L}{\sqrt{Re_L}}} \quad \text{OMG! So easy!}$$

Note, this scaling result is great for giving you a sense of orders of magnitude, however not as accurate as analytical solutions.

Example Calculate the boundary layer thickness on a 737 jet.

$$V_\infty = 400 \text{ miles/hour} = 177 \text{ m/s} \quad (\text{Plane speed})$$

$$\nu_{air} = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$$

$L = 5 \text{ m}$ (Length of the wing in the fuselage direction)

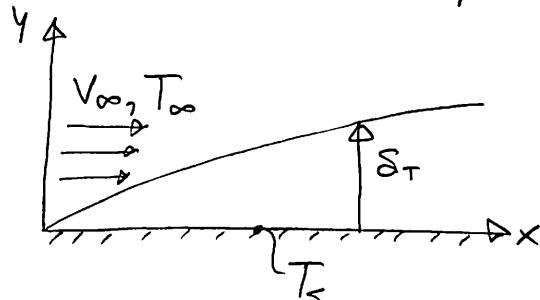
$$Re_L = \frac{V_\infty L}{\nu} = 5.9 \times 10^7 \Rightarrow \boxed{\delta = \frac{5L}{Re_L} = 3.25 \text{ mm}} \Rightarrow \text{Less than 1cm thick!}$$

Very difficult to observe.

Heat Transfer

Back from our interesting hydrodynamic excursion, we are interested in the convective heat transfer from the flat plate:

$$h = \frac{q''|_{y=0}}{\Delta T} = \frac{q''|_{y=0}}{T_s - T_\infty} = ?$$



Note, here we have the thermal boundary layer thickness, δ_T .

S = thickness where the velocity gradient exists $\Leftrightarrow \delta_T$ = thickness where the temperature gradient exists.

To solve for this second boundary layer, we need to solve the energy equation for the fluid flow.

$$\underbrace{U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y}}_{\text{Convection of energy}} = \underbrace{\alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{Conduction of energy}} \Rightarrow \text{Note, we always had } \nabla^2 T = 0 \text{ in conduction (R.H.S.)}$$

For the $T_s = \text{constant}$ case, $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$

Boundary conditions: 1) $T(y=0) = T_s$
2) $T(y \rightarrow \infty) = T_\infty$

Non-dimensionalizing : Let $\Theta = \frac{T - T_s}{T_\infty - T_s}$

$$U \frac{\partial \Theta}{\partial x} + V \frac{\partial \Theta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial y^2}$$

Note the similarity of the boundary layer equations

Hydrodynamic (δ)	Thermal (δ_T)
$U \frac{\partial \bar{U}}{\partial x} + V \frac{\partial \bar{U}}{\partial y} = V \frac{\partial^2 \bar{U}}{\partial y^2}$	$U \frac{\partial \theta}{\partial x} + V \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2}$
B.C.'s: $\bar{U}(y=0) = 0$	$\theta(y=0) = 0$
$\bar{U}(y \rightarrow \infty) = 1$	$\theta(y \rightarrow \infty) = 1$
$\left. \frac{\partial \bar{U}}{\partial y} \right _{y \rightarrow \infty} = 0$	$\left. \frac{\partial \theta}{\partial y} \right _{y \rightarrow \infty} = 0$

So we can find an analogy to our fluids solution

We can first define a usefull property called the Prandtl number

$$\boxed{Pr = \frac{V}{\alpha} = \frac{\text{viscous (momentum) diffusion rate}}{\text{thermal (heat) diffusion rate}}}$$

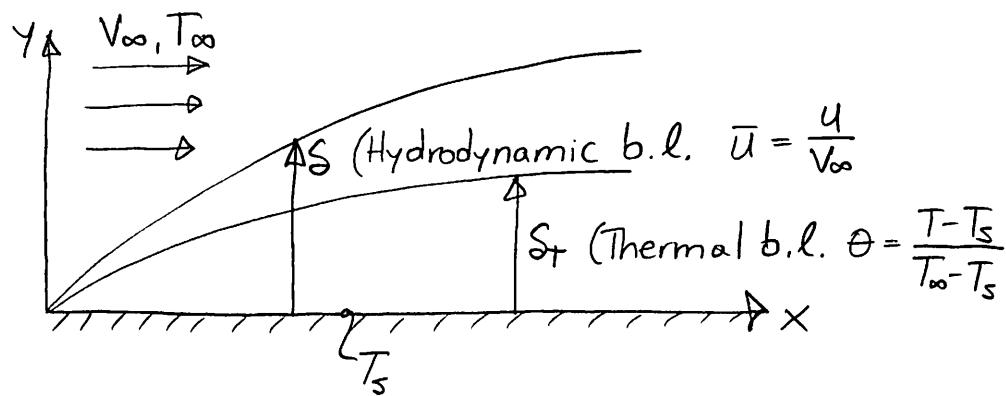
If $Pr = 1$, $\boxed{\theta = \bar{U}, \quad \delta = \delta_T}$ \Rightarrow Hydrodynamic & thermal boundary layers overlap.

However, in real life, most fluids aren't $Pr = 1$.

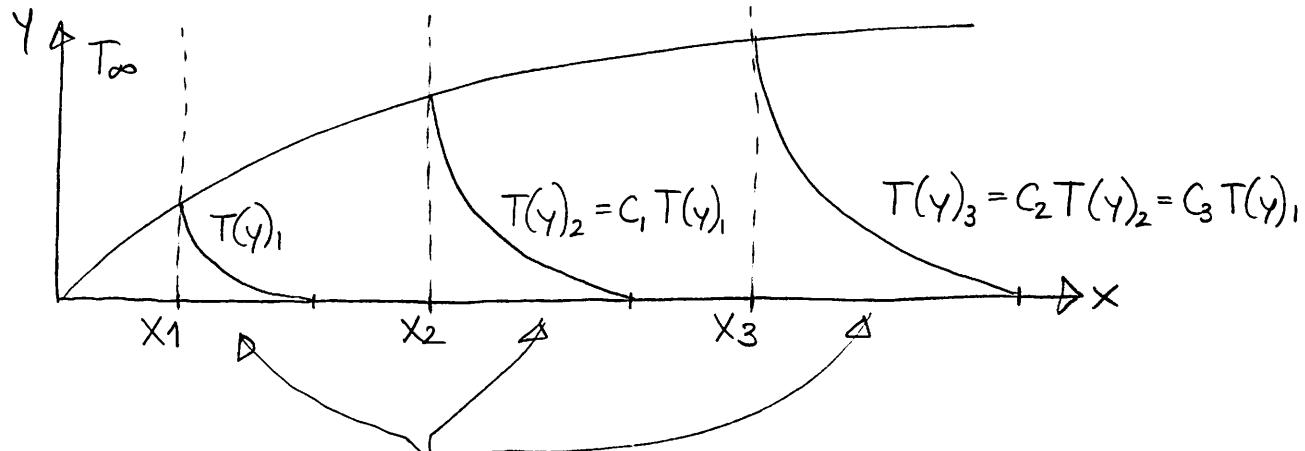
$$\begin{aligned}
 Pr &\approx 0.001 \text{ for liquid metals} \\
 &\approx 0.015 \text{ for mercury} \\
 &\approx 0.7 - 0.8 \text{ for air} \\
 &\approx 4 - 5 \text{ for refrigerants} \\
 &\approx 7 \text{ for water} \\
 &\approx 100 - 40,000 \text{ for engine oil} \\
 &\approx 1 \times 10^{25} \text{ for the earth's mantle}
 \end{aligned}
 \quad \left. \begin{array}{l} \delta < \delta_T \\ \delta > \delta_T \end{array} \right\}$$

If $U \neq \infty$:

Note, the case
I drew here
is for $\text{Pr} > 1$.



OK, so let's solve the thermal b.l. equation for $\alpha \neq U$
We will use the same similarity approach:



All three profiles look self-similar in nature.

Assume: $\Theta = f(\eta)$, $\eta = y \sqrt{\frac{U_x}{U_\infty}}$ (same similarity variable as in the hydrodynamic case)

$$\left. \begin{aligned} \frac{\partial \Theta}{\partial x} &= \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \\ \frac{\partial \Theta}{\partial y} &= \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} \\ \frac{\partial^2 \Theta}{\partial y^2} &= \frac{\partial^2 \Theta}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial y} \right)^2 \end{aligned} \right\}$$

Back substitute these into our energy equation PDE:

$$U \frac{\partial \Theta}{\partial x} + V \frac{\partial \Theta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial y^2} \Rightarrow U \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} + V \frac{\partial \Theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial y} \right)^2$$

Note, we also know that: $F = \int_0^\eta \phi d\eta$, $\bar{U} = \phi(\eta) = \frac{U}{V} = F$

Back substituting and doing some algebra, we obtain:

$$\frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} F \cdot \text{Pr} \frac{\partial \Theta}{\partial \eta} = 0$$

Using a trick: $\text{Pr}^{2/3} \cdot F(\eta) = F(\eta^*)$; $\eta^* = \eta \text{Pr}^{1/3}$
 $d\eta^* = \text{Pr}^{1/3} d\eta$

Now our equation becomes:

F at η^* , not F times η^* !

$$\frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} \underbrace{F(\eta^*)}_{\text{F at } \eta^*} \text{Pr}^{1/3} \frac{\partial \Theta}{\partial \eta} = 0$$

$$\frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) \text{Pr}^{2/3} \cdot \underbrace{\frac{\partial \Theta}{\partial \eta}}_{\partial \eta^*} = 0$$

Multiplying through by $\text{Pr}^{-2/3}$

$$\underbrace{\frac{1}{\text{Pr}^{2/3}} \left(\frac{\partial^2 \Theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) \text{Pr}^{2/3} \cdot \frac{\partial \Theta}{\partial \eta^*} = 0 \right)}_{\partial \eta^{*2}}$$

Note, this works because
 $F = \frac{a_2 \eta^2}{2} \dots (\text{H.O.T.})$

If we only use the lead term:

$$F(\eta) = \frac{a_2 \eta^2}{2}$$

$$F(\eta^*) = \frac{a_2 \text{Pr}^{2/3} \eta^{*2}}{2}$$

$$F(\eta^*) = \text{Pr}^{2/3} F(\eta)$$

Now our PDE becomes identical to our hydrodynamic ODE

$$\boxed{\frac{\partial^2 \Theta}{\partial \eta^{*2}} + \frac{1}{2} F(\eta^*) \frac{\partial \Theta}{\partial \eta^*} = 0}$$

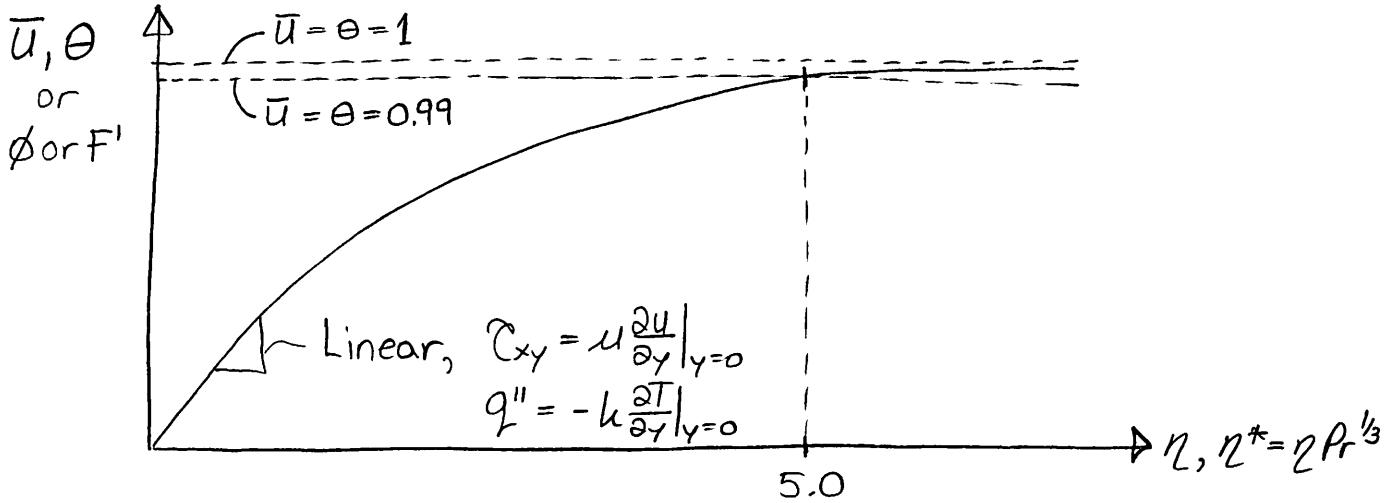
$$\Rightarrow \Theta(\eta^*) \equiv \bar{U}(\eta)$$

Before, we had: $F''' + \frac{1}{2} FF'' = 0$

B.C.'s : 1) $\Theta(\eta^* = 0) = 0$ 2) $\Theta(\eta^* \rightarrow \infty) = 1$

$$\Theta = \frac{T - T_s}{T_\infty - T_s}$$

So we can now use the exact same Blasius solution plot that we had before.



$$\delta \sqrt{\frac{V_\infty}{U_x}} = \delta (y = \delta) = 5.0 \quad (\text{Hydrodynamic b.l.})$$

$$\Pr^{1/3} \delta_T \sqrt{\frac{V_\infty}{U_x}} = \delta^* (y = \delta_T) = 5.0 \quad (\text{Thermal b.l.})$$

Taking the ratio of our two solutions above:

$$\frac{\delta}{\delta_T} = \Pr^{1/3} \Rightarrow \text{Makes sense since the only difference is } V \text{ & } \alpha \Rightarrow \Pr = \frac{V}{\alpha}$$

Heat Transfer (q'' = heat flux)

$$q'' \Big|_{y=0} = -k \frac{\partial T}{\partial y} \Big|_{y=0} \Rightarrow \Theta = \frac{T - T_s}{T_\infty - T_s}, \quad 2\Theta = \frac{\partial T}{T_\infty - T_s} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Back substitute}$$

$$\delta^* = \delta \Pr^{1/3} = y \sqrt{\frac{V_\infty}{U_x}} \cdot \Pr^{1/3}$$

$$2\delta^* = 2y \left(\frac{V_\infty}{U_x} \right)^{1/2} \Pr^{1/3}$$

$$q'' \Big|_{y=0} = -k (T_\infty - T_s) \left(\frac{V_\infty}{U_x} \right)^{1/2} \Pr^{1/3} \frac{\partial \Theta}{\partial \delta^*} \Big|_{\delta^*=0}$$

$$= k \frac{(T_s - T_\infty)}{x} \underbrace{\left(\frac{V_\infty x}{U} \right)^{1/2}}_{Re_x^{1/2}} \Pr^{1/3} \underbrace{F''(0)}_{\alpha_2 = 0.332}$$

$$Nu_x = 0.332 Re_x^{1/2} \Pr^{1/3}$$

↳ Local Nusselt number for a flat plate in laminar flow.

$$q'' \Big|_{y=0} = \frac{k \Delta T}{x} Re_x^{1/2} \Pr^{1/3} \alpha_2, \quad Nu_x = \frac{h x}{k} = \frac{q'' \Big|_{y=0} \cdot x}{\Delta T} = \alpha_2 Re^{1/2} \Pr^{1/3}$$

Some observations & notes

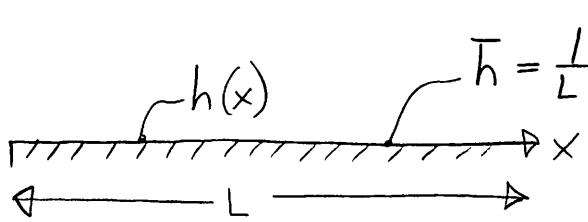
All of our previous results are valid for the following conditions:

- 1) Re_x or $Re_L < 5.0 \times 10^5$ (Laminar flow)
- 2) $Ma = \frac{V_\infty}{\text{speed of sound}} < 0.3$ (Incompressible flow)
- 3) $Ec = \frac{V_\infty}{C_p(T_s - T_\infty)} \ll 1$ (Viscous dissipation heating is negligible)
 $Ec \equiv Eckert \#$
- 4) We have assumed constant properties. Evaluate all properties ($\rho, \mu, Pr, \text{etc.} \dots$) at the film temp. (T_f)

$$T_f = \frac{T_s + T_\infty}{2} \Rightarrow \text{Use Table A.4 for gases}$$

A.5 for liquids

- 5) We have been working mostly with h in the class. Many times, it's easier to work with averaged quantities



$$\bar{h} = \frac{1}{L} \int_0^L h(x) dx \quad (\text{Average heat transfer coefficient})$$

Also when we write: $Nu_x = \frac{h_x}{k_f}$ (Local Nusselt number)

$$\overline{Nu}_L = \frac{\bar{h}L}{k_f} \quad (\text{Averaged Nusselt number})$$

- 6) $h \propto \frac{1}{\sqrt{x}}$ or $\bar{h} \propto \frac{1}{\sqrt{L}}$, $\Rightarrow Nu_x \propto \sqrt{x}$

Thus $h \rightarrow \infty$ and $Nu_x \rightarrow 0$ as $x \rightarrow 0$. Of course $h \rightarrow \infty$ does not really occur in reality, since the b.l. model breaks down at the leading edge ($x=0$).

7) Different Boundary Conditions

So far, we've only been dealing with $T_s = \text{constant}$
 What if we have uniform wall heat flux:

$$\overline{h} = \frac{\overline{q''}}{\Delta T}$$

$\underbrace{T_s = \text{constant}}$

$$\overline{h} = \frac{\overline{q''}}{\Delta T}$$

$\underbrace{q''|_{y=0} = \text{constant}}$

Uniform Wall Temperature (T_s)

$$\overline{h} = \frac{\overline{q''}}{\Delta T} = \frac{1}{\Delta T} \left[\frac{1}{L} \int_0^L q'' dx \right] = \frac{1}{L} \int_0^L h(x) dx$$

h as a function of x ,
not h times x .

Uniform Wall Heat Flux (q'')

$$\overline{h} = \frac{\overline{q''}}{\Delta T} = \frac{\overline{q''}}{\frac{1}{L} \int_0^L \Delta T(x) dx}$$

ΔT as a function of x , not ΔT times x

Nusselt Number Rules:

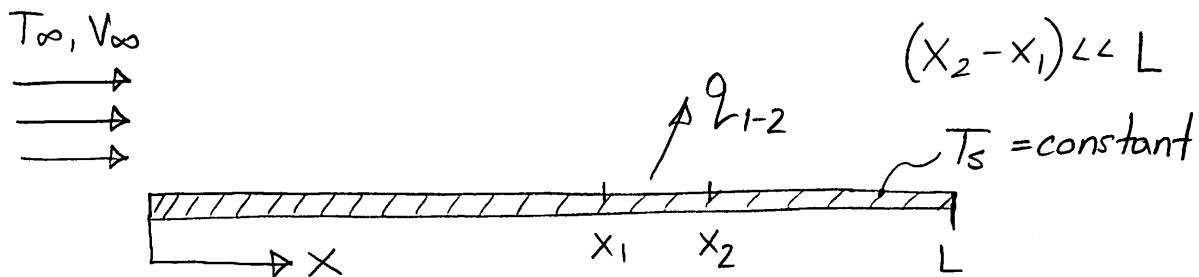
$$\overline{Nu}_L = \frac{\overline{h} L}{k} \neq \frac{1}{L} \int_0^L Nu(x) dx$$

For a flat plate with $T_s = \text{constant}$

$$\overline{h} = \frac{1}{L} \int_0^L \underbrace{h(x)}_{\frac{k}{x} Nu_x} dx = \frac{0.332 k Pr^{1/3}}{L} \cdot \sqrt{\frac{V_\infty}{D}} \int_0^L \frac{\sqrt{x}}{x} dx$$

$$\boxed{\overline{h} = 0.664 \left(\frac{k}{L} \right) Re_L^{1/2} Pr^{1/3}}$$

Example | Parallel flow over a flat plate. We need to analyze a short span of interest:



Find three different expressions for q_{1-2} (heat transfer)

① The best method:

$$q_{1-2} = \bar{h}_{1-2} (x_2 - x_1) (T_s - T_\infty)$$

Solve for the local heat transfer coefficient since $(x_2 - x_1) \ll L$

$$\boxed{\bar{h}_{1-2} \approx h_{\bar{x}} = h \left(x = \frac{x_2 - x_1}{2} \right)} \Rightarrow \text{Best solution for this problem}$$

② Approximation #1: Local coefficients

$$\boxed{\bar{h}_{1-2} \approx [h_{x_1} + h_{x_2}] / 2}$$

③ Approximation #2: Average coefficients for x_1 & x_2

$$q_{1-2} = q_{0-2} - q_{0-1}$$

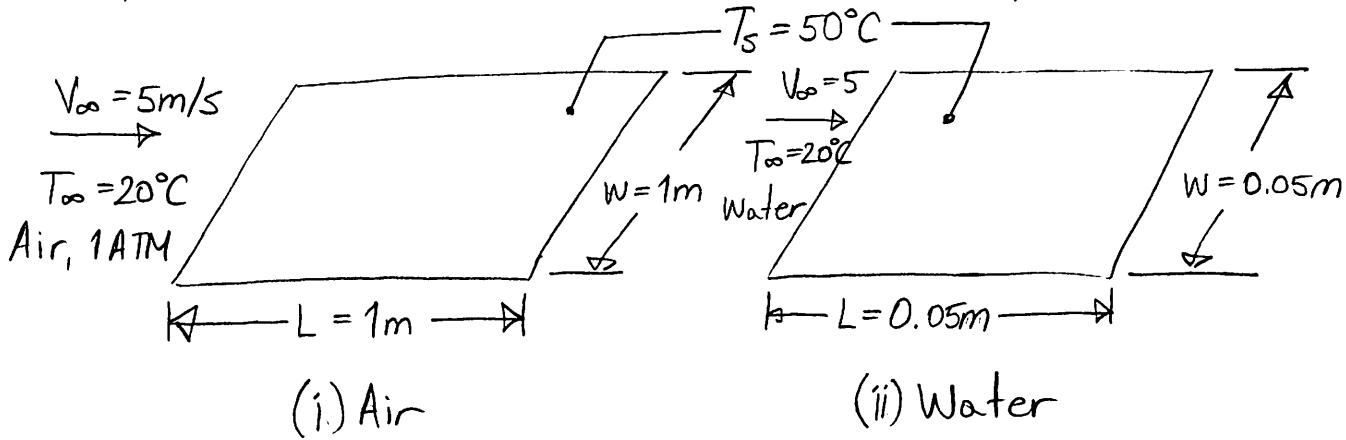
$$\bar{h}_{1-2} (x_2 - x_1) = \bar{h}_2 x_2 - \bar{h}_1 x_1,$$

$$\boxed{\bar{h}_{1-2} = \bar{h}_2 \frac{x_2}{x_2 - x_1} - \bar{h}_1 \frac{x_1}{x_2 - x_1}}$$

$$\text{where } \bar{h}_2 = \frac{1}{x_2 - x_1} \int_{x_0}^{x_2} h dx$$

$$\bar{h}_1 = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} h dx$$

Example Air and Water flow over 2 flat plates:



Determine : (a) Average convective heat transfer coefficient
 (b) Convective heat transfer rate
 (c) Drag force on the plate

Assumptions:

- 1) Steady state
- 2) B.L. assumptions are valid
- 3) Constant properties

(a) Let's begin by computing our Reynolds number for each case:

$$Re_{L,Air} = \frac{V_\infty L}{V_{air}} = \frac{(5 \text{ m/s})(1 \text{ m})}{1.669 \times 10^{-5} \text{ m}^2/\text{s}} = 3 \times 10^5 < 5 \times 10^5 \text{ (Laminar)}$$

evaluated at $T_f = \frac{50^\circ\text{C} + 20^\circ\text{C}}{2} = 35^\circ\text{C}$

$$Re_{L,water} = \frac{V_\infty L}{V_{water}} = \frac{(5 \text{ m/s})(0.05 \text{ m})}{7.291 \times 10^{-7} \text{ m}^2/\text{s}} = 3.43 \times 10^5 < 5 \times 10^5 \text{ (Laminar)}$$

So both flows are laminar, therefore we can use our previous solutions.

$$\overline{Nu}_L = \frac{\overline{h}L}{k_{fluid}} = 0.664 Re_L^{1/2} Pr^{1/3}$$

Using Tables A.4 & A.5 in the Textbook to look up \Pr

$$\Pr_{\text{Air}} = 0.706$$

$$k_{\text{Air}} = 26.9 \times 10^{-3} \text{ W/m}\cdot\text{K}$$

$$\rho_{\text{Air}} = 1.135 \text{ kg/m}^3$$

$$\Pr_{\text{water}} = 4.85$$

$$k_{\text{water}} = 0.625 \text{ W/m}\cdot\text{K}$$

$$\rho_{\text{water}} = 994 \text{ kg/m}^3$$

$$\overline{Nu}_{L,\text{Air}} = \frac{\overline{h}_{\text{Air}} (1\text{m})}{(26.9 \times 10^{-3} \text{ W/m}\cdot\text{K})} = 0.664 (3 \times 10^5)^{1/2} (0.706)^{1/3}$$

$$\boxed{\overline{h}_{\text{Air}} = 8.71 \text{ W/m}^2 \cdot \text{K}}$$

$$\overline{Nu}_{L,\text{water}} = \frac{\overline{h}_{\text{water}} (0.05\text{m})}{(0.625 \text{ W/m}\cdot\text{K})} = 0.664 (3.43 \times 10^5)^{1/2} (4.85)^{1/3}$$

$$\boxed{\overline{h}_{\text{water}} = 8228.21 \text{ W/m}^2 \cdot \text{K}}$$

Note the Huge! difference.

The smaller boundary layer length for water jacks up the \overline{h} .

$$(b) q_{\text{Air}} = \overline{h}_{\text{Air}} \cdot w \cdot L \cdot (T_s - T_\infty)$$

$$= (8.71 \text{ W/m}^2 \cdot \text{K})(1\text{m})(1\text{m})(50^\circ\text{C} - 20^\circ\text{C})$$

$$\boxed{q_{\text{Air}} = 174.2 \text{ W}}$$

$$q_{\text{water}} = \overline{h}_{\text{water}} \cdot w \cdot L \cdot (T_s - T_\infty)$$

$$= (8228.21 \text{ W/m}^2 \cdot \text{K})(0.05\text{m})(0.05\text{m})(50^\circ\text{C} - 20^\circ\text{C})$$

$$\boxed{q_{\text{water}} = 617.1 \text{ W}}$$

Even though the water plate area is 400X smaller than the air plate, the heat transfer is 3.5X larger!

This is why when you fall in a frozen lake at 0°C you get hypothermia in minutes, but chilling outside at 0°C is ok for hours. (121)

(c) Now we can solve for shear force (drag)

$$\overline{C}_{D,Air} = \frac{\overline{C}}{\frac{1}{2} \rho V_\infty^2} = \frac{1.328}{Re_L^{1/2}} \quad (\text{pg. 110 of notes})$$

$$= \frac{1.328}{(3 \times 10^5)^{1/2}} = 0.00242$$

$$F_{\text{drag}, \text{Air}} = \overline{C} \cdot w \cdot L = \overline{C}_{D,Air} \frac{1}{2} \rho_{\text{Air}} V_\infty^2 w \cdot L$$

$$= (0.00242) \left(\frac{1}{2}\right) (1.135 \text{ kg/m}^3) (5 \text{ m/s})^2 (1 \text{ m}^2)$$

$$F_{\text{drag}, \text{Air}} = 0.034 \text{ N}$$

$$F_{\text{drag}, \text{WATER}} = \overline{C} \cdot w \cdot L = \overline{C}_{D,WATER} \cdot \frac{1}{2} \rho_{\text{WATER}} \cdot V_\infty^2 \cdot w \cdot L$$

$$= (0.0023) \left(\frac{1}{2}\right) (998 \text{ kg/m}^3) (5 \text{ m/s})^2 (0.05 \text{ m}) (0.05 \text{ m})$$

$$F_{\text{drag}, \text{WATER}} = 0.072$$

So even though the water plate area is 400x smaller, the drag force on the plate is 2x larger than the air case!

This example shows the tradeoff of using air & water.

$$\overline{h}_{\text{WATER}} \gg \overline{h}_{\text{Air}}$$

but

Pressure drop in the flow
↓

$$F_{\text{drag}, \text{WATER}} \gg F_{\text{drag}, \text{air}} \quad \text{or} \quad \Delta P_{\text{WATER}} \gg \Delta P_{\text{air}}$$

So you need way more energy to drive your flow for water.

EXTERNAL FLOW CORRELATIONS

FOR CONVECTION

(For isothermal surfaces)

Table 7.7 in your edition, pg. 484 (Seventh Edition)

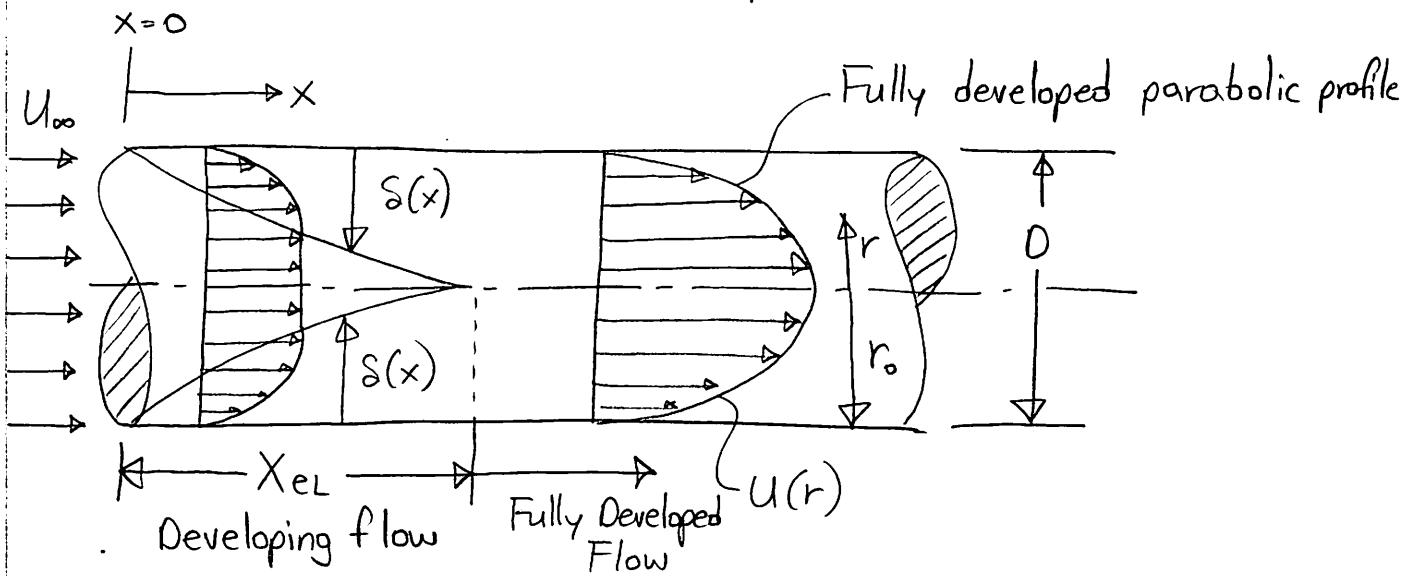
TABLE 7.9 Summary of convection heat transfer correlations for external flow^{a, b}

Correlation		Geometry	Conditions ^c
$\delta = 5x Re_x^{-1/2}$	(7.19)	Flat plate	Laminar, T_f
$C_{f,x} = 0.664 Re_x^{-1/2}$	(7.20)	Flat plate	Laminar, local, T_f
$Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}$	(7.23)	Flat plate	Laminar, local, $T_f, Pr \geq 0.6$
$\delta_t = \delta Pr^{-1/3}$	(7.24)	Flat plate	Laminar, T_f
$\bar{C}_{f,x} = 1.328 Re_x^{-1/2}$	(7.29)	Flat plate	Laminar, average, T_f
$\bar{Nu}_x = 0.664 Re_x^{1/2} Pr^{1/3}$	(7.30)	Flat plate	Laminar, average, $T_f, Pr \geq 0.6$
$Nu_x = 0.565 Pe_x^{1/2}$	(7.32)	Flat plate	Laminar, local, $T_f, Pr \leq 0.05, Pe_x \geq 100$
$C_{f,x} = 0.0592 Re_x^{-1/5}$	(7.34)	Flat plate	Turbulent, local, $T_f, Re_x \leq 10^8$
$\delta = 0.37x Re_x^{-1/5}$	(7.35)	Flat plate	Turbulent, $T_f, Re_x \leq 10^8$
$Nu_x = 0.0296 Re_x^{4/5} Pr^{1/3}$	(7.36)	Flat plate	Turbulent, local, $T_f, Re_x \leq 10^8, 0.6 \leq Pr \leq 60$
$\bar{C}_{f,L} = 0.074 Re_L^{-1/5} - 1742 Re_L^{-1}$	(7.40)	Flat plate	Mixed, average, $T_f, Re_{x,c} = 5 \times 10^5, Re_L \leq 10^8$
$\bar{Nu}_L = (0.037 Re_L^{4/5} - 871) Pr^{1/3}$	(7.38)	Flat plate	Mixed, average, $T_f, Re_{x,c} = 5 \times 10^5, Re_L \leq 10^8, 0.6 \leq Pr \leq 60$
$\bar{Nu}_D = C Re_D^m Pr^{1/3}$ (Table 7.2)	(7.52)	Cylinder	Average, $T_f, 0.4 \leq Re_D \leq 4 \times 10^5, Pr \geq 0.7$
$\bar{Nu}_D = C Re_D^m Pr^n (Pr_l/Pr_g)^{1/4}$ (Table 7.4)	(7.53)	Cylinder	Average, $T_\infty, 1 \leq Re_D \leq 10^6, 0.7 \leq Pr \leq 500$
$\bar{Nu}_D = 0.3 + [0.62 Re_D^{1/2} Pr^{1/3} \times [1 + (0.4/Pr)^{2/3}]^{-1/4} \times [1 + (Re_D/282,000)^{5/8}]^{4/5}]$	(7.54)	Cylinder	Average, $T_f, Re_D Pr \geq 0.2$
$\bar{Nu}_D = 2 + (0.4 Re_D^{1/2} + 0.06 Re_D^{2/3}) Pr^{0.4} \times (\mu/\mu_s)^{1/4}$	(7.56)	Sphere	Average, $T_\infty, 3.5 \leq Re_D \leq 7.6 \times 10^4, 0.71 \leq Pr \leq 380$
$\bar{Nu}_D = 2 + 0.6 Re_D^{1/2} Pr^{1/3}$	(7.57)	Falling drop	Average, T_∞
$\bar{Nu}_D = 1.13 C_1 C_2 Re_{D,\max}^m Pr^{1/3}$ (Tables 7.5, 7.6)	(7.60), (7.61)	Tube bank ^d	Average, $\bar{T}_f, 2000 \leq Re_{D,\max} \leq 4 \times 10^4, Pr \geq 0.7$
$\bar{Nu}_D = CC_2 Re_{D,\max}^m Pr^{0.36} (Pr_l/Pr_g)^{1/4}$ (Tables 7.7, 7.8)	(7.64), (7.65)	Tube bank ^d	Average, $\bar{T}, 1000 \leq Re_D \leq 2 \times 10^6, 0.7 \leq Pr \leq 500$

Some Definitions

$$Re_x = \frac{\rho V_\infty x}{\mu} ; \quad Pr = \frac{\nu}{\alpha} ; \quad Re_0 = \frac{\rho V_\infty D}{\mu} \quad \text{Pipe outer diameter}$$

$$Nu_x = \frac{h x}{k_f} ; \quad U = \frac{\mu}{\rho} ; \quad Nu_0 = \frac{h D}{k_f} ; \quad \bar{Nu}_0 = \frac{\bar{h} D}{k_f}$$

Internal Flow - Fully Developed Flow in Tubes

x_{el} = entrance length or developing length. Here, the velocity profile varies with radial position r , and axial location, x .

We can estimate the x_{el} by using our Blasius solution to see when our boundary layers begin to overlap:

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}$$

We can estimate that $\delta = \frac{D}{2}$ when b.l.'s merge

$$\frac{D}{2x_{el}} \underset{\text{scales with}}{\sim} \frac{5.0}{\sqrt{Re_x}}$$

$$\frac{D}{x_{el}} \sim \frac{10}{\sqrt{Re_x}} = \frac{10}{\sqrt{\frac{\rho U_\infty x_{el}}{\mu}}} = \frac{10}{\sqrt{\frac{\rho U_\infty x_{el} \cdot D}{\mu D}}} = \frac{10}{\sqrt{\frac{\rho U_\infty D}{\mu}} \cdot \sqrt{\frac{x_{el}}{D}}}$$

$$\sqrt{\frac{D}{x_{el}}} \sim \frac{10}{\sqrt{Re_0}} \Rightarrow \boxed{\frac{x_{el}}{D} \sim \frac{Re_0}{100} \sim 0.01 Re_0}; Re_0 = \frac{\rho U_\infty D}{\mu}$$

Experimentally: $\boxed{\frac{x_{el}}{D} = 0.05 Re_0}$: If $x > x_{el}$, flow is fully developed. (124)

Fully Developed Region ($x > x_{el}$)

We need to solve for the hydrodynamic properties of the flow to help us understand the heat transfer:

Applying the axial momentum equation (x -momentum)

$$\rho \left(\frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + \frac{u_\phi}{r} \frac{\partial u_x}{\partial \phi} + u_x \frac{\partial u_x}{\partial x} \right) = - \frac{\partial P}{\partial x} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \phi^2} + \frac{\partial^2 u_x}{\partial x^2} \right] + \rho g_x = 0$$

For fully developed flow: $\frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial \phi} = 0$
 $u_\phi = u_r = 0$

For flow in a horizontal tube: $g_x = 0$

Most of our terms drop out and we are left with:

$$-\frac{\partial P}{\partial x} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) = 0 \quad ①$$

B.C.'s: $\left. \frac{\partial u_x}{\partial r} \right|_{r=0} = 0$ (Symmetry) ②

$$u_x(r=r_0) = 0 \quad (\text{no slip}) \quad ③$$

From here on in, I will use $u_x = u$ (dropping x subscript)

Integrating eq. ① twice & applying ② & ③:

$$u(r) = \frac{r_0^2}{4U} \left(-\frac{\partial P}{\partial x} \right) \left(1 - \frac{r^2}{r_0^2} \right) \Rightarrow \text{Fully developed velocity profile in a round pipe. (Laminar flow)}$$

Solving for our average velocity:

$$\bar{U} = \frac{1}{\pi r_0^2} \int_0^{r_0} u \cdot 2\pi r dr = \frac{r_0^2}{4U} \left(-\frac{\Delta P}{2x} \right) \int_0^1 (1-\lambda) d\lambda ; \lambda = \frac{r^2}{r_0^2}$$

$$\boxed{\bar{U} = \frac{r_0^2}{8U} \left(-\frac{\Delta P}{2x} \right)}$$

$$\boxed{u(r) = 2\bar{U} \left(1 - \frac{r^2}{r_0^2} \right)}$$

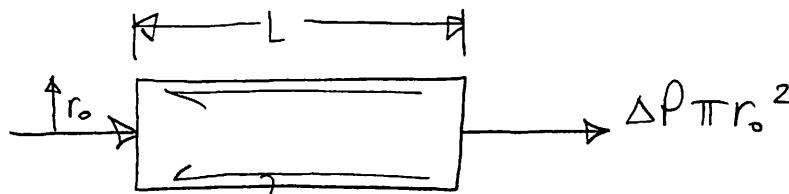
\Rightarrow Fully developed laminar flow in a round pipe.

Typically, we need to know friction, pressure drop, or force.

$$f = \frac{\Delta P}{\left(\frac{L}{D}\right) \frac{1}{2} \rho \bar{U}^2} = \text{Tube friction factor. Easy way to calculate pressure loss in tubes of length } L, \text{ diameter } D.$$

$$C_{fx} = \frac{C_x}{\frac{1}{2} \rho \bar{U}^2} = \text{Local tube friction coefficient. Related to the friction factor, } f.$$

Drawing a force balance diagram on a segment of our fluid in the pipe. $\sum F_x = 0$ since $u(r) = f(x)$ = fully developed



$$C_o (2\pi/\lambda L) = \Delta P \pi r_0^2 = \frac{\Delta P D}{2}$$

$$4C_o = \frac{\Delta P}{\left(\frac{L}{D}\right)}$$

We can say: $4C_f = f$ \Rightarrow Let's now solve for friction factor

$$f = \frac{\Delta P}{\left(\frac{L}{D}\right) \frac{1}{2} \rho \bar{U}^2} \quad ①$$

But we've just solved for laminar flow that: $\bar{U} = \frac{r_0^2}{8\mu} \left(-\frac{\partial P}{\partial x} \right)$

$$\frac{8\mu \bar{U}}{r_0^2} = -\frac{\partial P}{\partial x} \approx \frac{\Delta P}{L} \quad ②$$

↓ substitute here

$$\text{From } ① : \frac{\Delta P}{L} \cdot \frac{2D}{\rho \bar{U}^2} = f = \frac{8\mu \bar{U}}{r_0^2} \cdot \frac{2D}{\rho \bar{U}^2}$$

$$\frac{16\mu D}{r_0^2 \rho \bar{U}} = f = \frac{16\mu D}{\left(\frac{D}{2}\right)\left(\frac{D}{2}\right) \rho \bar{U}} = \frac{64\mu}{\rho \bar{U} D} = \frac{64}{Re_0}$$

$f = \frac{64}{Re_0}$ ⇒ Pipe friction factor for laminar flow in a smooth round tube. Darcy-Weisbach equation.

We can also say the following:

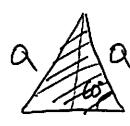
$$f \cdot Re_0 = \text{constant}$$

For non-circular channels/pipes:

$$D_h = \text{hydraulic diameter} = \frac{4A}{P}; \quad A = \text{cross sectional area}$$

$P = \text{wetted perimeter}$

Example]



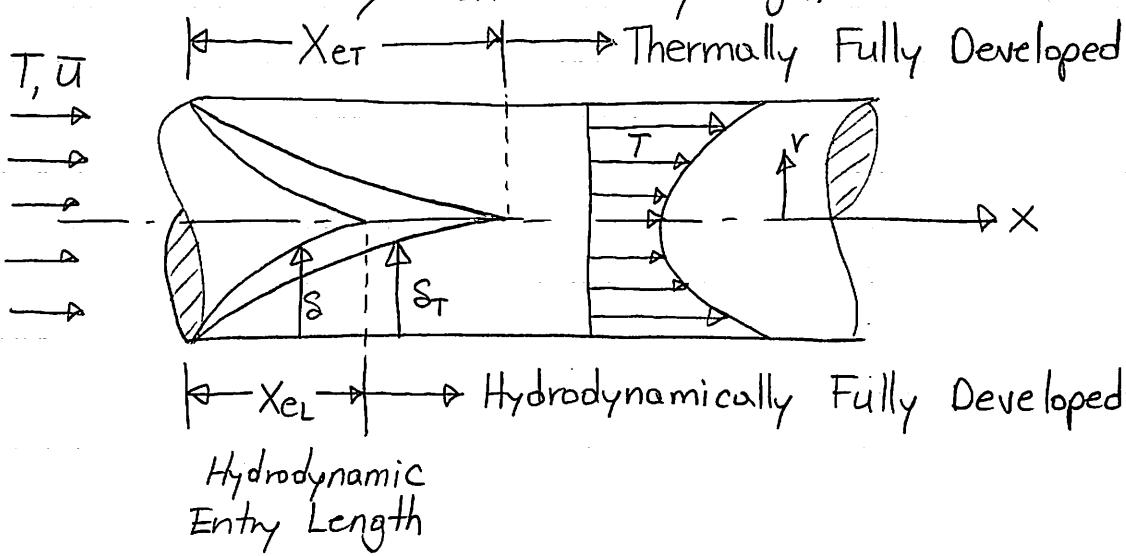
$$A = \frac{a \cdot a \sin 60^\circ}{2}$$

$$P = 3a$$

$$D_h = \frac{4a^2 \sin 60^\circ}{3a} = 1.15a$$

Heat Transfer in Internal Flows

How does the thermal boundary layer or entry length develop?



For fully developed flow, both the Thermal entry length (X_{eT}) and hydrodynamic entry length (X_{eL}) must be passed; $X > X_{eT} \& X_{eL}$

$$\frac{X_{eT}}{D} = 0.017 Re_0 \cdot Pr \Rightarrow \text{Experimentally obtained Thermal Entry Length}$$

So how about heat transfer in the fully developed region:
i.e. $X > X_{eL} \& X > X_{eT}$

$$h = \frac{q''_{\text{wall}}}{\Delta T}$$

From energy balance at the wall (conduction = convection)

$$h \Delta T = k_f \frac{\Delta T}{s_T} \Rightarrow h \sim \frac{k_f}{s_T}$$

For fully developed flow (overlapping thermal boundary layers):

$$s_T \sim \frac{D}{2}$$

$$h \sim \frac{2k_f}{D}$$

But we know that: $Nu_0 = \frac{hD}{k_f} = \frac{2k_f(D)}{k_f} = \frac{2k_f(D)}{k_f}$

$Nu_0 \approx 2$ \Rightarrow We expect a constant Nusselt #!

It turns out this scaling is pretty good. If we solve the coupled momentum and energy equations for internal smooth round tube pipe flow, we will obtain the following:

$$\overline{Nu}_0 = \frac{\bar{h}D}{k_f} = 4.364$$

\Rightarrow Constant wall heat flux
Laminar flow, round tube
 $Re_0 < 2300$; $Re_0 = \frac{\rho \bar{U} D}{\mu}$

$$\overline{Nu}_0 = \frac{\bar{h}D}{k_f} = 3.66$$

\Rightarrow Constant wall temperature
Laminar flow, round tube
 $Re_0 < 2300$

Bulk Fluid Temperature

It is important to note that for internal flows, heat transfer is computed by using the bulk fluid temperature: (T_b)

$$\bar{h} = \frac{q''|_{\text{wall}}}{T_w - T_b} ; \quad T_w = \text{wall temperature} \\ T_b = \text{bulk fluid temperature}$$

T_b can be thought of as if we allow the pipe fluid to uniformly mix & come to an equilibrium temperature:

$$T_b = \frac{1}{A \bar{U}} \int_A u(r) \cdot T dA$$

where $A = \text{cross sectional area}$
 $\bar{U} = \text{average fluid velocity}$

TABLE 8.1 Nusselt numbers and friction factors for fully developed laminar flow in tubes of differing cross section

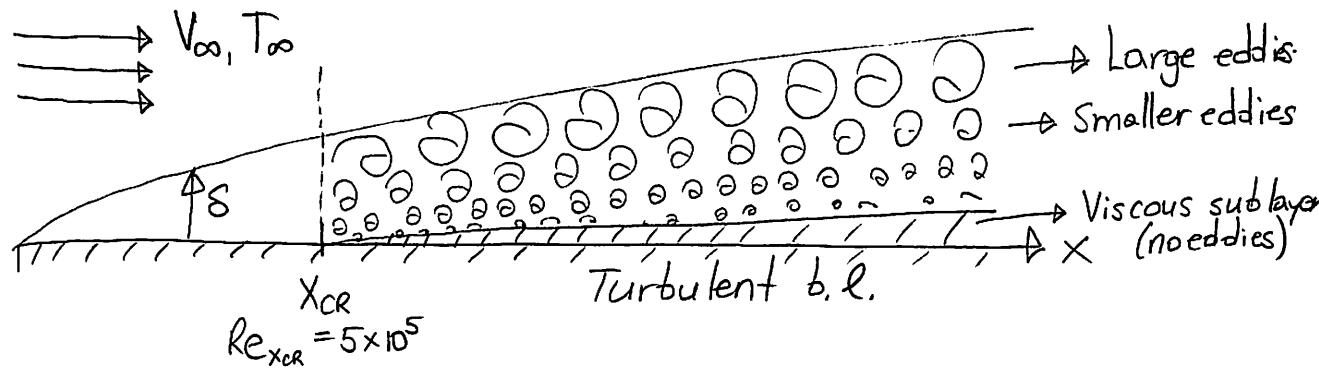
Cross Section	$\frac{b}{a}$	$Nu_D \equiv \frac{hD_h}{k}$		
		(Uniform q''_s)	(Uniform T_s)	Re_{D_h}
	—	4.36	3.66	64
	1.0	3.61	2.98	57
	1.43	3.73	3.08	59
	2.0	4.12	3.39	62
	3.0	4.79	3.96	69
	4.0	5.33	4.44	73
	8.0	6.49	5.60	82
	∞	8.23	7.54	96
Heated Insulated	∞	5.39	4.86	96
	—	3.11	2.49	53

Used with permission from W. M. Kays and M. E. Crawford, *Convection Heat and Mass Transfer*, 3rd ed. McGraw-Hill, New York, 1993.

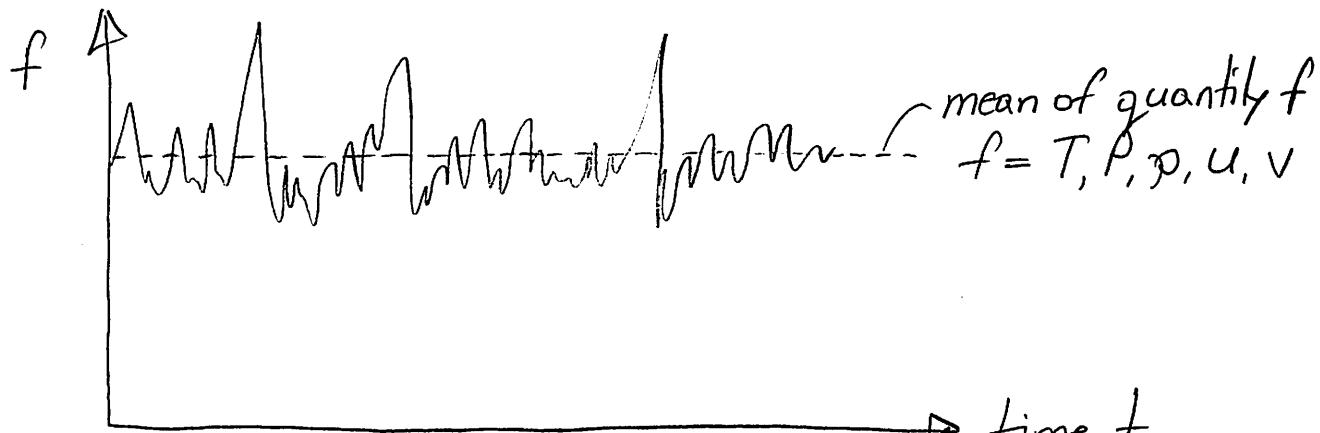
Turbulent Flow

Turbulence in a fluid can be seen as a spectrum of coexisting vortices (eddies) in which kinetic energy from larger ones is dissipated to successively smaller ones until the very smallest of these vortices are damped out by viscous shear.

For a flat plate, external flow:



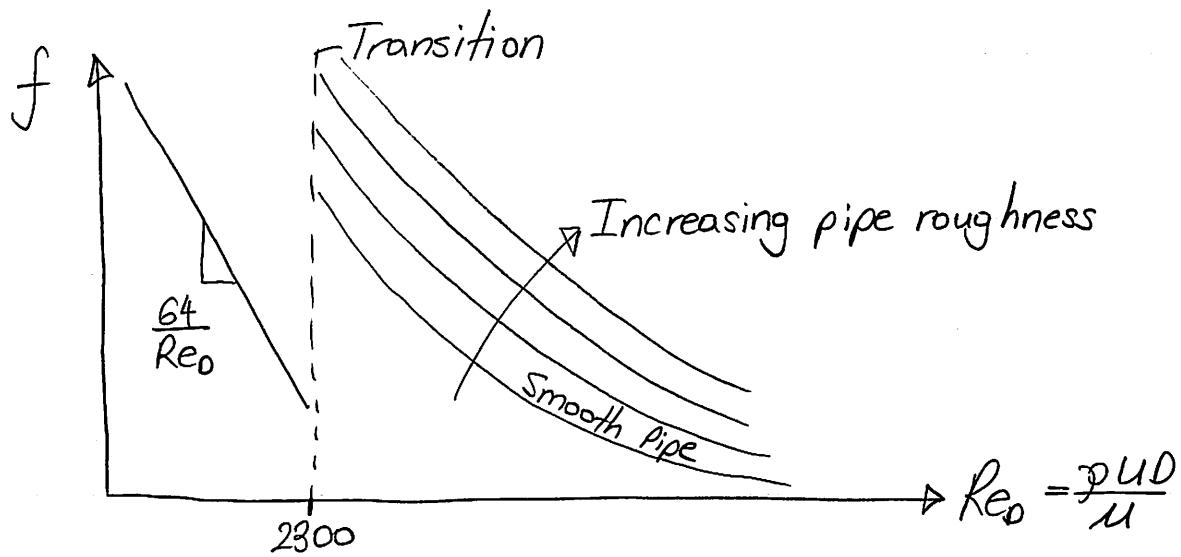
We handle this chaps by time averaging. Turbulent flow causes fluctuations of the velocity components, pressure, temperature, and in compressible flow, density.



By defining $f = \bar{f} + f'$, we can back substitute into our momentum equations & solve (ME 420)

We handle the mathematics in ME 420 & 521, for 320, we learn to use the results: (correlations)

One of the best & useful results is the empirical Moody chart:



Good correlation to know (if you have a smooth pipe)

$$f = (0.79 \ln(Re_0) - 1.64)^{-2}$$

$10^4 < Re_0 < 10^6$, smooth pipe

$$f = \frac{\Delta P}{\left(\frac{L}{D}\right) \frac{1}{2} \rho U^2}$$

$$C_f = \frac{C_0}{\frac{1}{2} \rho U^2}$$

$$C_f = \frac{f}{4}$$

(for smooth pipes)

L = pipe length, D = pipe diameter, U = average flow velocity
 C_0 = average shear stress at the pipe wall

You may ask, why friction & pressure drop, this isn't fluid mechanics. Well, momentum transfer & heat transfer are intricately linked, hence to solve for heat transfer, we need fluid mechanics, as we will see in the Nusselt # correlation

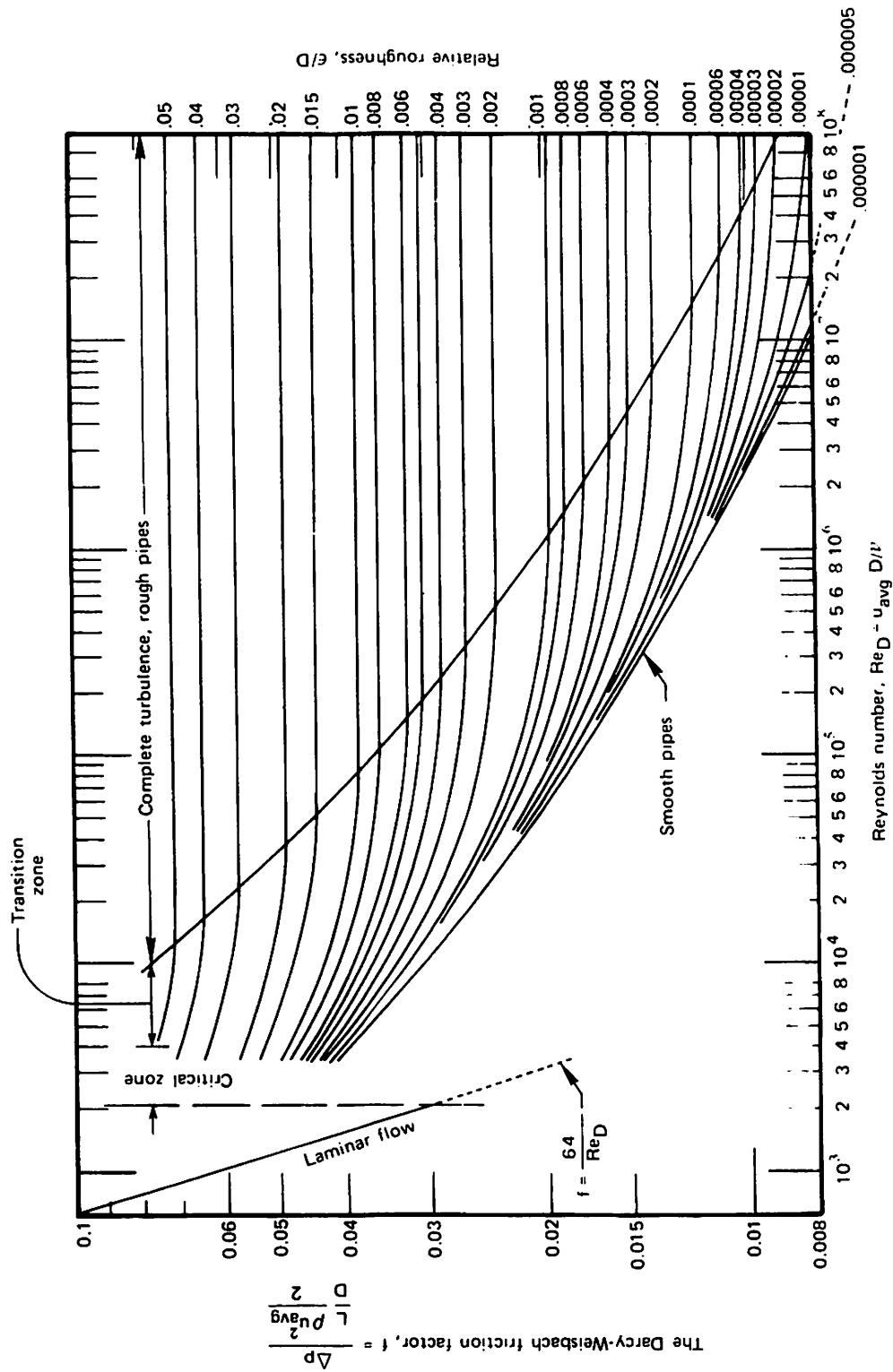


Figure 7.6 Pipe friction factors.

* Adapted from Lienhard & Lienhard, "A Heat Transfer Textbook"

Turbulent Flow Heat Transfer (Internal flow)

Some very good correlations to know are:

$$\overline{Nu_0} = \frac{\overline{hD}}{k_f} = 0.023 Re_0^{0.8} Pr^{0.4}$$

$\Rightarrow Dittus - Boelter eq.$
 $\Rightarrow Re_0 > 10,000$

For a more accurate result, use:

$$Nu_0 = \frac{\overline{hD}}{k_f} = \frac{(f/8)(Re_0 - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)}$$

$\Rightarrow Gnielinski correlation$
 $3000 < Re_0 < 5 \times 10^6$

External Flow (Turbulent heat transfer)

Remember, $Re_{cr} = \frac{V_\infty X_{cr}}{V} = 5 \times 10^5$ (Turbulent transition)

Some useful correlations: ($C_{o,x}$ = local shear stress at x)

$$C_{f,x} = \frac{C_{o,x}}{\frac{1}{2} \rho V_\infty^2} = 0.0592 Re_x^{-1/5}$$

$10^5 < Re_x < 10^7$
Local skin friction coefficient at x.

$$C_{f,x} = \frac{C_{o,x}}{\frac{1}{2} \rho V_\infty^2} = 0.026 Re_x^{-1/7}$$

$10^7 < Re_x < 10^9$

Note, these external flow correlations are also on pg. 123 of the notes (Table 7.7)

For even greater accuracy:

$$C_{f,x} = \frac{0.455}{\ln(0.06 Re_x)^2}$$

$10^5 < Re_x < 10^9$
 \hookrightarrow White correlation

Typically, we need averaged skin friction coefficient for the whole plate:

$$C_{0,x} = \frac{1}{2} \rho V_\infty^2 C_{f,x} = \frac{1}{2} \rho V_\infty^2 \cdot 0.0592 Re_x^{-0.2}$$

$$\overline{C}_{av,L} = \frac{1}{L} \int_0^L C_{0,x} dx = \frac{C_{0,L}}{0.8}$$

$$C_{f,av} = \overline{C}_f = \frac{C_{f,L}}{0.8} = \frac{0.0592 Re_L^{-0.2}}{0.8}$$

$$\boxed{\overline{C}_f = 0.074 Re_L^{-0.2}} \Rightarrow \text{Plate averaged skin friction coefficient.}$$

External Heat Transfer

$$Nu_x = \frac{h_x}{h_f} = 0.029 Re_x^{0.8} Pr^{1/3}$$

\Rightarrow Local Nusselt #
Analytical Result

$$Nu_x = \frac{h_x}{h_f} = 0.029 Re_x^{0.8} Pr^{0.43}$$

\Rightarrow Local Nusselt #
Experimental Result
Whitaker Correlation

As we can see, the analytical & experimental results match pretty well.

$$\overline{Nu}_L = \frac{\overline{h} L}{h_f} = 0.036 Re_L^{0.8} Pr^{0.43}$$

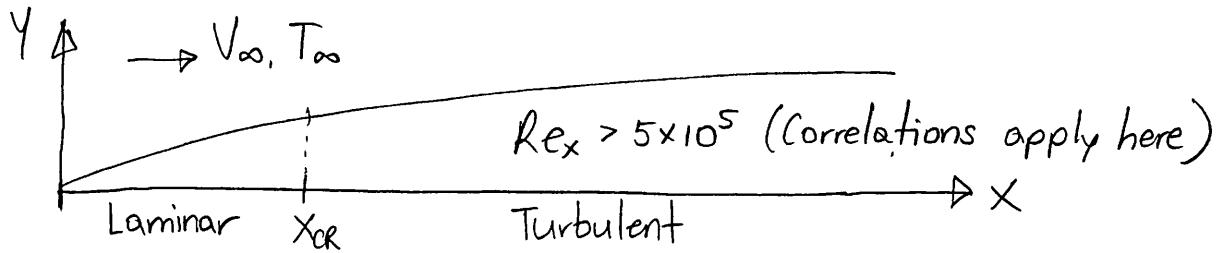
$0.7 < Pr < 400$
 $5 \times 10^5 < Re_L < 3 \times 10^7$

For a more accurate correlation:

$$Nu_x = \frac{\left(\frac{C_{f,x}}{2}\right) Re_x Pr}{1 + 12.7 \left(\frac{C_{f,x}}{x}\right)^{1/2} (Pr^{2/3} - 1)}$$

\Rightarrow White Correlation
 $0.5 < Pr < 2000$
 $5 \times 10^5 < Re_x < 10^7$

Note, so far, everything we've been covering has been in the turbulent region of the plate. But there is still a laminar section. How do we handle this?



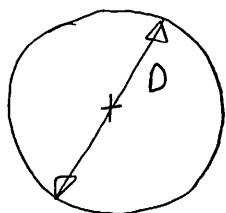
Mixed flow correlations exist (pg. 123 of notes)

$$\overline{C_f L} = 0.074 \text{Re}_L^{-1/5} - 1742 \text{Re}_L^{-1} \quad \text{Re}_L < 10^8$$

$$\overline{Nu_L} = \frac{\overline{h} L}{k_f} = (0.037 \text{Re}_L^{4/5} - 871) \text{Pr}^{1/3} \quad \begin{matrix} \text{Re}_L < 10^8 \\ 0.6 < \text{Pr} < 60 \end{matrix}$$

Cylinders & Spheres (External flow)

V_{∞}, T_{∞}



$$\overline{Nu_0} = \frac{\overline{h} D}{k_f} = \frac{1}{0.8237 - \ln(\text{Re}_0 \text{Pr})^{1/2}}$$

↳ Cylinder, $\text{Re}_0 \text{Pr} < 0.2$

$$\overline{Nu_0} = \frac{\overline{h} D}{k_f} = 0.3 + \frac{0.62 \text{Re}_0^{1/2} \text{Pr}^{1/3}}{\left[1 + (0.4/\text{Pr})^{2/3}\right]^{1/4}}$$

↳ Cylinder, $\text{Re}_0 > 10^4$

$$\overline{Nu_0} = 2 + (0.4 \text{Re}_0^{1/2} + 0.06 \text{Re}_0^{2/3}) \text{Pr}^{0.4} \quad \begin{matrix} \Rightarrow \text{Sphere} \\ 3.5 < \text{Re}_0 < 8 \times 10^4 \\ 0.7 < \text{Pr} < 380 \end{matrix}$$

For rest of correlations, see pg. 123 of notes.

TABLE 8.4 Summary of convection correlations for flow in a circular tube^{a,b,c}

Correlation	Conditions
$f = 64/Re_D$	(8.19) Laminar, fully developed
$Nu_D = 4.36$	(8.53) Laminar, fully developed, uniform q''_s
$Nu_D = 3.66$	(8.55) Laminar, fully developed, uniform T_s
$\overline{Nu}_D = 3.66 + \frac{0.0668 Gz_D}{1 + 0.04 Gz_D^{2/3}}$	(8.57) Laminar, thermal entry (or combined entry with $Pr \geq 5$), uniform T_s , $Gz_D = (D/x) Re_D Pr$
$\overline{Nu}_D = \frac{\frac{3.66}{\tanh[2.264 Gz_D^{-1/3} + 1.7 Gz_D^{-2/3}]} + 0.0499 Gz_D \tanh(Gz_D^{-1})}{\tanh(2.432 Pr^{1/6} Gz_D^{-1/6})}$	(8.58) Laminar, combined entry, $Pr \geq 0.1$, uniform T_s , $Gz_D = (D/x) Re_D Pr$
$\frac{1}{\sqrt{f}} = -2.0 \log \left[\frac{el/D}{3.7} + \frac{2.51}{Re_D \sqrt{f}} \right]$	(8.20) ^c Turbulent, fully developed
$f = (0.790 \ln Re_D - 1.64)^{-2}$	(8.21) ^c Turbulent, fully developed, smooth walls, $3000 \leq Re_D \leq 5 \times 10^6$
$Nu_D = 0.023 Re_D^{4/5} Pr^n$	(8.60) ^d Turbulent, fully developed, $0.6 \leq Pr \leq 160$, $Re_D \geq 10,000$, $(L/D) \geq 10$, $n = 0.4$ for $T_s > T_m$ and $n = 0.3$ for $T_s < T_m$
$Nu_D = 0.027 Re_D^{4/5} Pr^{1/3} \left(\frac{\mu}{\mu_s} \right)^{0.14}$	(8.61) ^d Turbulent, fully developed, $0.7 \leq Pr \leq 16,700$, $Re_D \geq 10,000$, $L/D \geq 10$
$Nu_D = \frac{(f/8)(Re_D - 1000) Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)}$	(8.62) ^d Turbulent, fully developed, $0.5 \leq Pr \leq 2000$, $3000 \leq Re_D \leq 5 \times 10^6$, $(L/D) \geq 10$
$Nu_D = 4.82 + 0.0185(Re_D Pr)^{0.827}$	(8.64) Liquid metals, turbulent, fully developed, uniform q''_s , $3.6 \times 10^3 \leq Re_D \leq 9.05 \times 10^5$, $3 \times 10^{-3} \leq Pr \leq 5 \times 10^{-2}$, $10^3 \leq Re_D Pr \leq 10^4$
$Nu_D = 5.0 + 0.025(Re_D Pr)^{0.8}$	(8.65) Liquid metals, turbulent, fully developed, uniform T_s , $Re_D Pr \geq 100$

^aThe mass transfer correlations may be obtained by replacing Nu_D and Pr by Sh_D and Sc , respectively.

^bProperties in Equations 8.53, 8.55, 8.60, 8.61, 8.62, 8.64, and 8.65 are based on T_m ; properties in Equations 8.19, 8.20, and 8.21 are based on $T_f = (T_s + T_m)/2$; properties in Equations 8.57 and 8.58 are based on $\bar{T}_m = (T_{m,i} + T_{m,o})/2$.

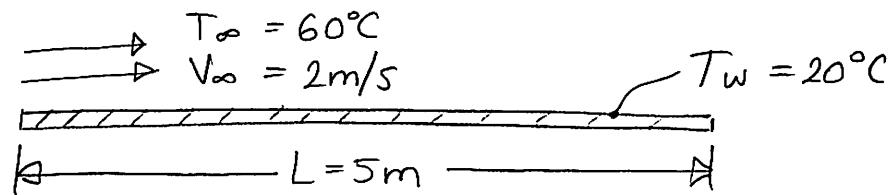
^cEquation 8.20 pertains to smooth or rough tubes. Equation 8.21 pertains to smooth tubes.

^dAs a first approximation, Equations 8.60, 8.61, or 8.62 may be used to evaluate the average Nusselt number \overline{Nu}_D over the entire tube length, if $(L/D) \geq 10$. The properties should then be evaluated at the average of the mean temperature, $\bar{T}_m = (T_{m,i} + T_{m,o})/2$.

*For tubes of noncircular cross section, $Re_D = D_h u_m / \nu$, $D_h = 4A_c/P$, and $u_m = m/\rho A_c$. Results for fully developed laminar flow are provided in Table 8.1. For turbulent flow, Equation 8.60 may be used as a first approximation.

NOTE* For a summary of external flow correlations, go to page # 123 of the notes (Table 7.7 or 7.9)

Example] Engine oil at 60°C flows over a 5m long flat plate (20°C) with velocity of 2m/s. Determine total drag force and heat transfer per unit width.



$$T_f = \bar{T} = \frac{T_w + T_\infty}{2} = 40^{\circ}\text{C}$$

$$\rho_{\text{oil}} = 876 \text{ kg/m}^3$$

$$k_{\text{oil}} = 0.144 \text{ W/m}\cdot\text{K}$$

$$\Pr = 2870$$

$$\nu_{\text{oil}} = 242 \times 10^{-6} \text{ m}^2/\text{s}$$

Let's check our flow regime:

$$Re_L = \frac{V_\infty L}{\nu_{\text{oil}}} = 4.13 \times 10^4 < 5 \times 10^5 \text{ (Laminar)}$$

We know from pg. 123 of notes:

$$\overline{C_f} = \frac{\overline{C_o}}{\frac{1}{2} \rho V_\infty^2} = 1.328 Re_L^{-0.5} = 0.00653$$

$$F_D = \overline{C_f} \cdot A \cdot \frac{\rho V_\infty^2}{2} = (0.00653)(5 \text{ m} \cdot 1 \text{ m}) \frac{(876 \text{ kg/m}^3)(2 \text{ m/s})^2}{2}$$

$F_D = 57.2 \text{ N}$

For heat transfer:

$$\overline{Nu}_L = \frac{\overline{h} L}{k_{oil}} = 0.664 Re_L^{1/2} Pr^{1/3} = 1918$$

$$\overline{h} = 55.2 \text{ W/m}^2 \cdot \text{K}$$

$$Q = \overline{h} A (T_\infty - T_w) = 11.04 \text{ kW}$$

Now what if we increased the flow rate by 15x.
Do we expect friction & heat transfer to increase by 15x?

$$Re_L = \frac{V_\infty L}{D_{oil}} = 6.2 \times 10^5 > Re_{cr} \quad (\text{Turbulent flow})$$

$$\overline{C_f} = 0.074 Re_L^{-1/5} - 1742 Re_L^{-1} \quad (Re_L < 10^8) \Rightarrow \text{Mixed flow}$$

$$\overline{C_f} = 0.00233$$

$$F_D = \overline{C_f} \cdot A \cdot \frac{\rho V_\infty^2}{2} = (0.00233)(5 \text{ m}^2) \frac{(876 \text{ kg/m}^3)(30 \text{ m/s})^2}{2}$$

$$F_D = 4.59 \text{ kN}$$

For heat transfer:

$$\overline{Nu}_L = \frac{\overline{h} L}{k_{oil}} = (0.037 Re_L^{4/5} - 871) Pr^{1/3} = 10255.2$$

$$\overline{h} = 295.3 \text{ W/m}^2 \cdot \text{K}$$

$$Q = \overline{h} A (T_\infty - T_w) = 59.1 \text{ kW}$$

$$\frac{F_{D,TURB}}{F_{D,LAM}} = 80.2$$

$$\frac{Q_{TURB}}{Q_{LAM}} = 5.4$$

What if we picked a different working fluid such as water

For water: $\rho_w = 1000 \text{ kg/m}^3$

$$k_w = 0.6 \text{ W/m}\cdot\text{K}$$

$$\Pr = 7$$

$$V_w = 0.658 \times 10^{-6} \text{ m}^2/\text{s}$$

$$Re_L = \frac{V_\infty L}{V_w} = \frac{(2 \text{ m/s})(5 \text{ m})}{(0.658 \times 10^{-6} \text{ m}^2/\text{s})} = 1.52 \times 10^7 \quad (\text{Turbulent!})$$

$$\overline{C_f} = 0.074 Re_L^{-1/5} - 1742 Re_L^{-1} = 0.00259$$

$$F_0 = \overline{C_f} \cdot A \cdot \frac{\rho_w V_\infty^2}{2} = (0.00259)(5 \text{ m}^2) \frac{(1000 \text{ kg/m}^3)(2 \text{ m/s})^2}{2}$$

$$F_0 = 25.9 \text{ N}$$

For heat transfer:

$$\overline{Nu}_L = \frac{\overline{h} L}{k_w} = (0.037 Re_L^{4/5} - 871) \Pr^{1/3} = 35593.8$$

$$\overline{h} = \frac{\overline{Nu}_L \cdot k_w}{L} = 4271.3 \text{ W/m}^2 \cdot \text{K}$$

$$Q = \overline{h} A (T_w - T_\infty) = 0.854 \text{ MW} \quad !!!$$

$$\frac{F_{0,oil}}{F_{0,w}} = 2.2$$

$$; \quad \frac{Q_{oil}}{Q_w} = 0.013 \text{ or } 1.3\% \text{ only!}$$

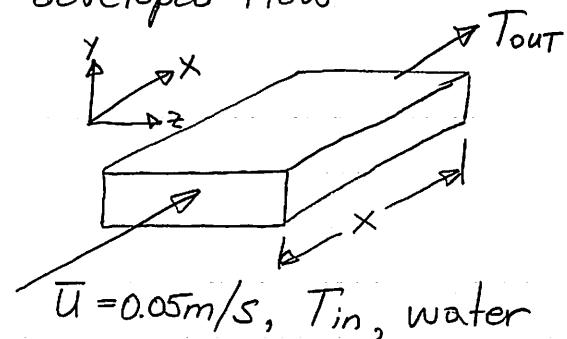
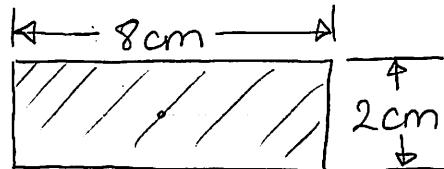
This simple analysis shows the importance of selecting the appropriate working fluid.

You may ask yourself, why would anybody use high Pr oils for thermal applications. Isn't water way better?

The answer is NO! Oils have 2 main advantages:

- 1) Temperature range (Typically -40°C to 300°C).
- 2) Low volatility. Water has a relatively high vapor pressure, meaning it will evaporate easily.
- 3) Corrosion resistance and enhanced lubrication.
- 4) Oil is a dielectric fluid (electrically insulating). Water is also a dielectric, however dissolved ions can readily make it into an electrolyte.

Example Heat exchange to a liquid flowing in a channel with $T_w = \text{constant}$ and fully developed flow:



Determine T_{out} as a function of x (length of channel)

First let's find the flow regime:

$$Re_0 = \frac{\rho \bar{U} D}{\mu} \Rightarrow \text{But what } D \text{ do we use?}$$

We need hydraulic diameter:

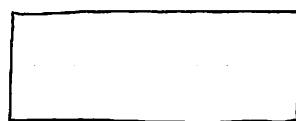
$$D_h = \frac{4A}{P} = \frac{4(8\text{cm})(2\text{cm})}{2(2\text{cm}) + 2(8\text{cm})} = \frac{64\text{cm}^2}{20\text{cm}} = 3.2\text{cm}$$

$$Re_{oh} = \frac{\rho_w D_h \bar{U}}{\mu_w} = \frac{(1000 \text{ kg/m}^3)(0.032\text{m})(0.05\text{m/s})}{(0.001 \text{ Pa}\cdot\text{s})} = 1600$$

$Re_{oh} = 1600 < 2300 \Rightarrow \text{LAMINAR Flow}$

So to solve for h , now we must look at table 8.1 on page of the class notes: (since non-circular cross section)

$$b = 8\text{cm}$$



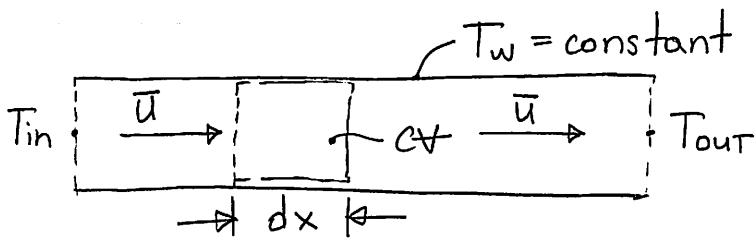
$$a = 2\text{cm}$$

$$\frac{b}{a} = \frac{8}{2} = 4.0, \quad Nu_0 = \frac{h D_h}{k_f} = 4.44$$

Uniform T_s

$$h = \frac{Nu_0 \cdot k_f}{D_h} = \frac{(4.44)(0.6 \text{ W/m}\cdot\text{K})}{0.032\text{m}} = 83.25 \text{ W/m}^2\cdot\text{K}$$

So now we can solve our problem. Let's look at a differential axial section of our channel:



Doing an energy balance on our control volume:

$$\underbrace{\rho A \bar{u} C dT_f}_m = \underbrace{h P dx}_{dA_{\text{surface}}} (T_w - T_f) ; \quad \begin{aligned} C &= \text{heat capacity} \\ A &= \text{cross-sectional area} \\ P &= \text{perimeter} \end{aligned}$$

We want to convert this equation into a homogeneous ODE
Let's non-dimensionalize:

$$\text{Let: } T^* = \frac{T_f - T_w}{T_{in} - T_w} ; \quad x^* = \frac{x}{L} ; \quad L = \text{Length of channel}$$

$$dT^* = \frac{dT_f}{T_{in} - T_w} ; \quad dx^* = \frac{dx}{L}$$

Back substitute into our ODE:

$$\rho A \bar{u} C (T_{in} - T_w) dT^* = h P L dx^* (-T^*) (T_{in} - T_w)$$

$$\rho A \bar{u} C dT^* = -h P L T^* dx^*$$

We can solve this homogeneous ODE with integration

$$\int_{T^*_{in}=0}^{T^*_{out}} \frac{dT^*}{T^*} = - \frac{hPL}{\rho A \bar{U}} \int_0^l dx^*$$

$$\ln\left(\frac{T^*}{T^*_{in}}\right) + \frac{hPLx^*}{\rho A \bar{U} C} = 0$$

$$T^* = T^*_{in} e^{-\frac{hPLx^*}{\rho A \bar{U} C}} \Rightarrow T^*_{in} = \frac{T_{in} - T_w}{T_{in} - T_w} = 1$$

$$T_f = T_w + (T_{in} - T_w) e^{-\frac{h\rho_x}{\rho A \bar{U} C}}$$

↳ Bulk fluid temperature as a function of x along the length of the rectangular channel.

Also remember: $T_f = T_b = \text{bulk fluid temperature}$

i.e. $T_b = \frac{1}{\rho A \bar{U} C} \int_A \rho c u(z, y) \cdot T(z, y) dA$

Note also: $dQ = h \rho dx (T_w - T_b)$

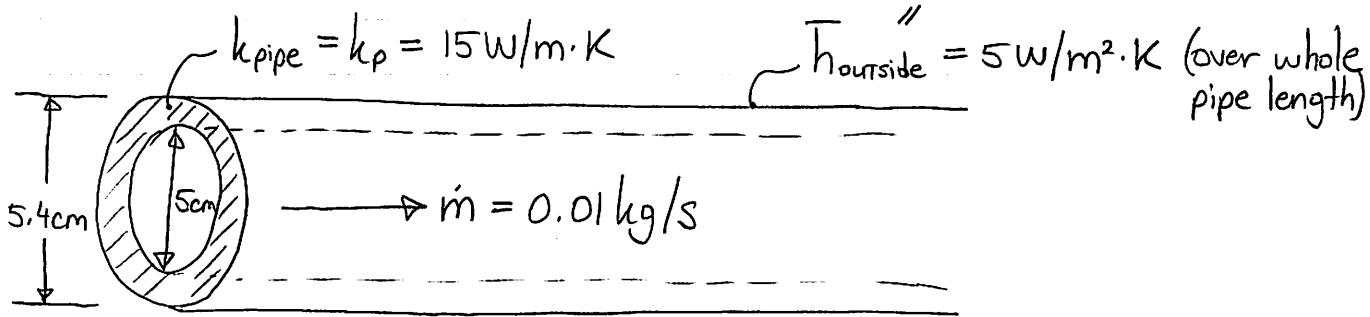
\uparrow $h = f(k_f) = f(T_b) \Rightarrow \text{fluid properties are temperature dependent}$

For problems like these (internal flow), evaluate properties at \bar{T}

$$\bar{T} = \frac{T_{in} + T_{out}}{2}$$

Example Let's do a comprehensive example that uses everything we've learned so far in class (conduction & convection)

Freezing water pipes in New England: \bar{h}_o



$$\sim T_\infty = -10^\circ\text{C}$$

$T_{in} = 1.5^\circ\text{C}$ (inlet pipe fluid bulk temperature)

- (a) Calculate the heat transfer coefficient on the inside of the pipe (\bar{h}_{in}). Assume fully developed flow (both thermally and hydrodynamically), and no ice is present.

First let's see our flow regime:

$$\dot{m} = \frac{\rho_w \pi D_{in}^2 \bar{U}}{4} \Rightarrow \text{from tables, } \rho_w = 1000 \text{ kg/m}^3$$

$$V_w = 1.5 \times 10^{-6} \text{ m}^2/\text{s}$$

$$k_w = 0.6 \text{ W/m}\cdot\text{K}$$

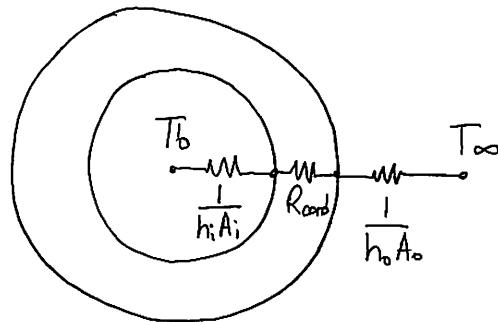
$$Re_0 = \frac{\bar{U} D_{in}}{V_w} = \frac{4\dot{m}}{\pi \rho_w D_{in} V_w} = 170 \text{ (LAMINAR FLOW)}$$

So we know $\overline{Nu}_0 = \text{constant}$, however, we don't have constant wall temperature or constant heat flux. As a rough approximation, we can use the average:

$$Nu_0 \approx \frac{1}{2} (3.657 + 4.364) = 4.01 \Rightarrow \boxed{\bar{h}_i = \frac{\overline{Nu}_0 \cdot k_w}{D_{in}} = 45.7 \text{ W/m}^2 \cdot \text{K}}$$

- (b) Using our calculated \bar{h}_i , determine the bulk water temp. T_b at the point at which water first freezes.

To solve this question, we need to understand the overall heat transfer from the outside air, to the bulk fluid.



Overall thermal resistance

$$\frac{1}{\pi D_{in} \bar{U}} = \frac{1}{\pi D_{in} \bar{h}_i} + \frac{\ln(D_{out}/D_{in})}{2\pi k_p} + \frac{1}{\pi D_{out} \bar{h}_o}$$

We can solve for the overall heat transfer coefficient \bar{U}

$$\bar{U} = 4.20 \text{ W/m}^2 \cdot \text{K}$$

Now we can solve for when water begins to freeze:

$$q = \bar{h}_i (T_b - T_{freeze}) = \bar{U} (T_b - T_\infty)$$

\uparrow
 $T_{freeze} = 0^\circ\text{C}$

Rearranging:

$$T_b = \frac{\bar{h}_i T_{freeze} - \bar{U} T_\infty}{\bar{h}_i - \bar{U}} = 1.20^\circ\text{C}$$

(c) At what point in the pipe length L does the water begin to freeze?

Applying an energy balance on a differential fluid element just like the previous example:

$$\dot{m}c \frac{dT_b}{dx} = \pi D_{in} U (T_\infty - T_b)$$

We already solved that:

$$\frac{T_b - T_\infty}{T_{in} - T_\infty} = \exp\left(-\frac{\pi D_{in} U L}{\dot{m}c}\right); \text{ note } T_{in} = \text{inlet fluid temperature}$$

From this we can solve for L given that $T_b = 1.20^\circ\text{C}$

$$T_\infty = -10^\circ\text{C}$$

$$T_{in} = 1.5^\circ\text{C}$$

$$C = 4200 \text{ J/kg}\cdot\text{K} \quad (\text{from Tables})$$

$$L = \frac{\dot{m}c}{\pi D_{in} U} \cdot \ln\left(\frac{T_{in} - T_\infty}{T_b - T_\infty}\right) = 1.47 \text{ m}$$

(d) Suppose the flow rate is increased to $\dot{m} = 1 \text{ kg/s}$. At a point $L = 10 \text{ m}$ from the entrance, determine the temperature on the inner wall of the pipe. Assume same conditions as before.

First let's check our flow regime again:

$$Re_{D_h} = \frac{4\dot{m}}{\pi \rho_w D_{in} V_w} = 1.7 \times 10^4 > 2300 \quad (\text{Turbulent})$$

So we need to use the Gnielinski correlation:

$$f = (0.79 \ln Re_0 - 1.64)^{-2} = 0.0273$$

$$\overline{Nu}_0 = \frac{\overline{h}_i D_{in}}{k_f} = \frac{(f/8)(Re_0 - 1000) Pr_w}{1 + 12.7(f/8)^{1/2} (Pr_w^{2/3} - 1)}$$

From Tables, $Pr_w \approx 11$

$$\overline{Nu}_0 = 153 = \frac{\overline{h}_i D_{in}}{k_f} \Rightarrow \overline{h}_i = 1610 \text{ W/m}^2 \cdot \text{K}$$

We can now re-calculate our overall heat transfer coefficient, U :

$$U = \left\{ \pi D_{in} \left[\frac{1}{\pi D_{in} \overline{h}_i} + \frac{\ln(D_{out}/D_{in})}{2\pi k_p} + \frac{1}{\pi D_{out} \overline{h}_o} \right] \right\}^{-1}$$

$$U = 5.38 \text{ W/m}^2 \cdot \text{K}$$

Now we use our energy balance equation to solve for T_b at $L = 10 \text{ m}$

$$T_b = T_\infty + (T_{in} - T_\infty) \exp \left[- \frac{\pi D_{in} U L}{m c} \right] = 1.48^\circ \text{C}$$

Finally, using thermal resistance to solve for inner surface temperature: $(T_{s,i})$

$$q = \overline{h}_i (T_b - T_{s,i}) = U (T_b - T_\infty)$$

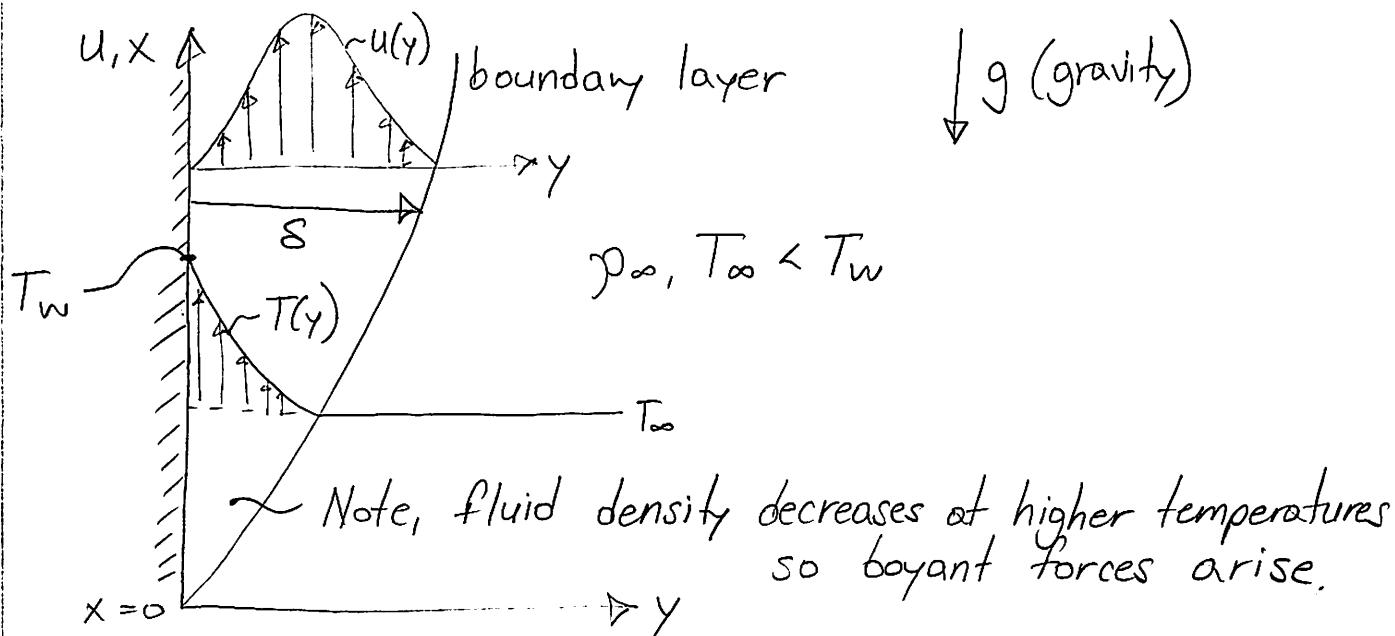
$$T_{s,i} = \frac{(\overline{h}_i - U) T_b - U T_\infty}{\overline{h}_i} = 1.44^\circ \text{C}$$

So we see that one mitigation strategy to prevent icing is to increase the flow rate.

Natural Convection

Unlike forced convection, in which the fluid motion driving force is external to the fluid, natural convection is driven by body forces acting directly on the fluid as a result of heating or cooling.

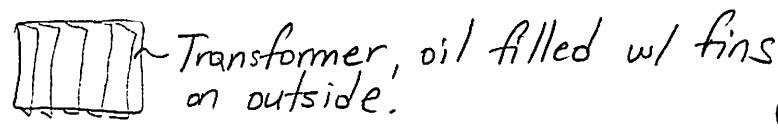
Simplest Case: Hot vertical plane wall



The flow is analogous to flow over a flat plate, but $u=0$ at $y=0$ and at $y \rightarrow \infty$.

Natural convection is all around us:

- 1) When humans are stationary, natural convection occurs & we lose heat to the air surrounding us.
- 2) Many heat sinks are designed for natural convection. i.e transformer cooling, home appliances, induction motors



To solve, we need to use the momentum & energy equations but now taking gravity into account.

X-momentum:

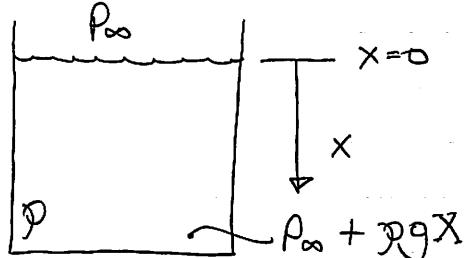
$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + v \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\text{Body force (gravity in-x)}} - g \quad ①$$

Body force (gravity in-x)

To help us solve, we know for a stationary fluid that:

$$\rho(x) = \rho_\infty + \rho g x$$

(Archimedes principle)



So we can say:

$$\frac{\partial p}{\partial x} = - \rho_\infty g \quad \begin{matrix} \downarrow \\ \text{density of fluid far away from wall} \end{matrix}$$

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = - \frac{\rho_\infty g}{\rho} \quad ②$$

$\begin{matrix} \nearrow \\ \text{density of fluid close to the wall} \end{matrix}$

Back substitute ② into ①

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \left(\frac{\rho_\infty - \rho}{\rho} \right) + v \frac{\partial^2 u}{\partial y^2} \quad ③$$

However we need the relation between density ρ and temperature, T . We can use the following:

$$\beta = \frac{1}{V} \frac{\partial V}{\partial T} \Big|_P ; \quad V \equiv \text{fluid specific volume}$$

$\beta \equiv \text{coefficient of volumetric thermal expansion}$

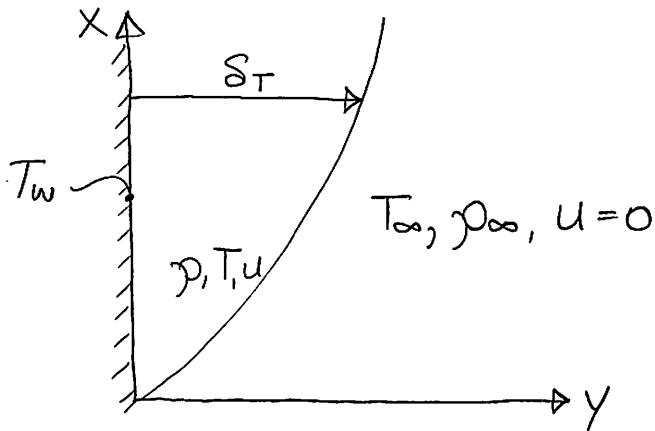
$$\beta = - \frac{1}{\rho} \frac{\partial \rho}{\partial T} \Big|_P \quad ④ \quad \Rightarrow \quad \frac{\rho_\infty - \rho}{\rho} = \beta(T - T_\infty)$$

So now our momentum equation becomes:

$$\underbrace{U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y}}_{\text{Inertia Forces}} = \underbrace{g\beta(T - T_\infty)}_{\text{Buoyancy Forces}} + \underbrace{V \frac{\partial^2 U}{\partial Y^2}}_{\text{Viscous Forces}}$$

Let's try to solve this using scaling analysis:

Looking at the problem as a transient conduction prob.



$$S_T \sim \sqrt{\alpha t} ; \quad \begin{aligned} t &= \text{time} \\ \alpha &= \text{fluid thermal diffusivity of the fluid} \end{aligned}$$

Speed of heat conduction (page 65 of the notes)

So what is t here? Well, we know $t = \frac{x}{u}$ ← location on plate
← flow speed

$$S_T \sim \sqrt{\frac{\alpha x}{u}} \quad \text{so} \quad u \sim \frac{\alpha x}{S_T^2} \quad \text{and} \quad y \sim S_T$$

So let's break down the scaling of the forces

Inertia, I	Viscosity, ν	Boyancy, B
$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \sim \frac{u^2}{x}$ $\sim \frac{\alpha^2 x}{\delta_T^4}$	$\nu \frac{\partial^2 u}{\partial y^2} \sim \nu \frac{u}{y^2}$ $\sim \frac{v \alpha x}{\delta_T^4}$	$g \beta (T - T_\infty)$ $= g \beta \Delta T$

Now if we take ratios of our forces to see which ones dominate: (focus on boyancy)

I/B	ν/B	B/B
$\frac{\alpha^2 x}{\delta_T^4 g \beta \Delta T}$	$\frac{v \alpha x}{\delta_T^4 g \beta \Delta T}$	1

Assuming that $\nu/B \sim o(1)$ (Inside the natural convection boundary layer). Think of the boundary layer definition.

$$\frac{v \alpha x}{\delta_T^4 g \beta \Delta T} \sim 1$$

Multiplying both sides by $\frac{x^3}{x^3}$

$$\frac{v \alpha}{g \beta \Delta T x^3} \left(\frac{x}{\delta_T} \right)^4 \sim 1$$

$$\boxed{\frac{x}{\delta_T} \sim \left(\frac{g \beta \Delta T x^3}{\alpha v} \right)^{1/4}} \Rightarrow (\text{Rayleigh number})^{1/4}$$

$$Ra_x = \frac{g\beta\Delta T x^3}{\alpha U} ; \quad \frac{x}{S_T} \sim Ra^{1/4}$$

$$Gr_x = \frac{g\beta\Delta T x^3}{U^2} \Rightarrow \text{Grashof number}$$

$$Ra = \underbrace{\frac{\text{Boycancy Force}}{\text{Viscous Force}}}_{Gr} \cdot \underbrace{\frac{\text{Momentum Diffusivity}}{\text{Thermal Diffusivity}}}_{Pr \text{ (Prandtl \#} = \frac{U}{\alpha})} = Gr \cdot Pr$$

If: $Ra < Ra_{crit}$, conduction dominates ($Gr \ll 1, Pr \ll 1$)
 $Ra > Ra_{crit}$, convection dominates ($Gr > 1, Pr > 1$)

Think of Rayleigh # & Grashof # as an analogy to Reynolds #, but here instead of inertia, we have boyancy.

Heat Transfer (Natural Convection)

$$\left. q'' \right|_{y=0} = - h_f \left. \frac{\partial T}{\partial y} \right|_{y=0} \sim h \frac{\Delta T}{S_T}$$

$$\left(\left. q'' \right|_{y=0} \right) \cdot \frac{x}{h_f} = \frac{h_f x}{h_f} = Nu_x \sim \frac{x}{S_T} \sim Ra^{1/4}$$

$$Nu_x \sim Ra_x^{1/4} \Rightarrow \text{Scaling tells us this is what we expect.}$$

Full analytical result:

$$Nu_x = 0.394 Ra_x^{1/4} \Rightarrow T_w = \text{constant}$$

$$\overline{Nu}_L = \frac{\overline{h}_L L}{h_f} = 0.525 Ra_L^{1/4} \Rightarrow \overline{Nu}_{L,exp} = 0.52 Ra_L^{1/4}$$

For cases where $T_w \neq \text{constant}$ & $q'' = \text{constant}$

We can define a modified Rayleigh number, Ra_x^*

$$\boxed{\text{Ra}_x^* = \frac{g\beta(q''/k_f)X^4}{V\alpha}} \Rightarrow \text{Constant heat flux Rayleigh \#}$$

$$\left(\frac{q''|_{y=0}}{\Delta T}\right) \cdot \frac{X}{k_f} = \frac{h_x X}{k_f} = \text{Nu}_x$$

$$\boxed{\text{Nu}_x = 0.503 \text{Ra}_x^{*1/5}} \Rightarrow \text{Constant heat flux natural convection, vertical wall Analytical result.}$$

In general: $\text{Ra}_L > 10^9 \Rightarrow$ Turbulent free convection

$$\boxed{\overline{\text{Nu}}_L = \frac{h_L}{k_f} = 0.1 \text{Ra}_L^{1/3}} ; 10^9 < \text{Ra}_L < 10^{13}$$

For laminar flow:

$$\boxed{\overline{\text{Nu}}_L = \frac{h_L}{k_f} = 0.59 \text{Ra}_L^{1/4}} ; 10^4 \leq \text{Ra}_L < 10^9$$

(Similar to what we derived)

Note, both correlations above are for constant vertical wall temperature ($T_w = \text{constant}$).

For other cases; i) Inclined walls

ii) Channels

iii) Enclosures

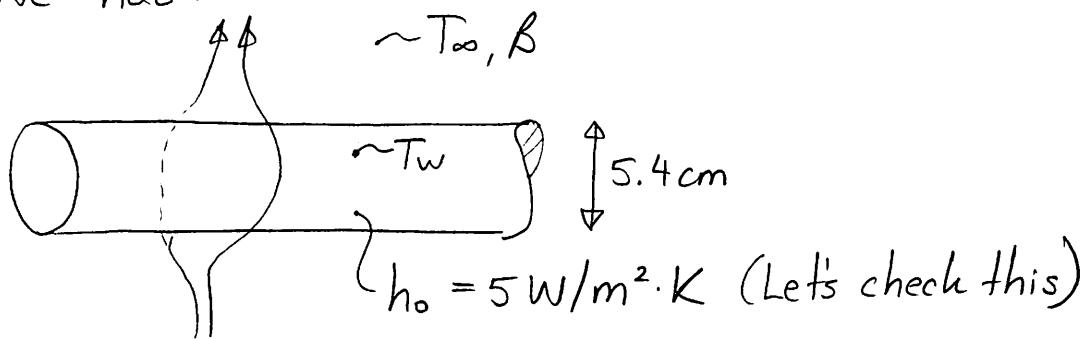
iv) Cylinders & Spheres

} See Chapter 9
of the textbook
Incropera.

* All properties evaluated at $\bar{T} = \frac{T_w + T_\infty}{2}$; $\beta = \frac{1}{T}$ for ideal gas. (154)

Example | Let's check the validity of our previous example with New England pipes.

We had:



First calculate the Rayleigh #:

$$Ra_0 = \frac{g\beta\Delta T D^3}{\alpha L} \quad \text{note, we use } D \text{ here instead of } L.$$

$$\text{For air at } \bar{T} = \frac{-10^\circ C + 1^\circ C}{2} = 4.5^\circ C \quad (\text{Table: A.4})$$

$$\rho_{\text{air}} = 1.2 \text{ kg/m}^3$$

$$\alpha_{\text{air}} = 20 \times 10^{-6} \text{ m}^2/\text{s}$$

$$\nu_{\text{air}} = 14 \times 10^{-6} \text{ m}^2/\text{s}$$

$$k_{\text{air}} = 24 \times 10^{-3} \text{ W/m}\cdot\text{K}$$

$$\beta = \frac{1}{T_0} = \frac{1}{263.15K} = 0.0038 \text{ K}^{-1}$$

$$Ra_0 = \frac{(9.81 \text{ N/kg})(0.0038 \text{ K}^{-1})(11^\circ C)(0.054 \text{ m})^3}{(20 \times 10^{-6} \text{ m}^2/\text{s})(14 \times 10^{-6} \text{ m}^2/\text{s})} = 2.3 \times 10^5$$

Looking at Eq. 9.33 of Textbook & corresponding Table 9.1

$$\overline{Nu}_0 = \frac{\overline{h}D}{kf} = 0.48 Ra_0^{1/4} = 0.48(2.3 \times 10^5)^{1/4} = 10.51$$

$$\overline{h} = \frac{\overline{Nu}_0 \cdot kf}{D} = \frac{(10.51)(0.024 \text{ W/m}\cdot\text{K})}{0.054 \text{ m}} = 4.67 \text{ W/m}^2\cdot\text{K} \quad (155)$$

Thermal Radiation

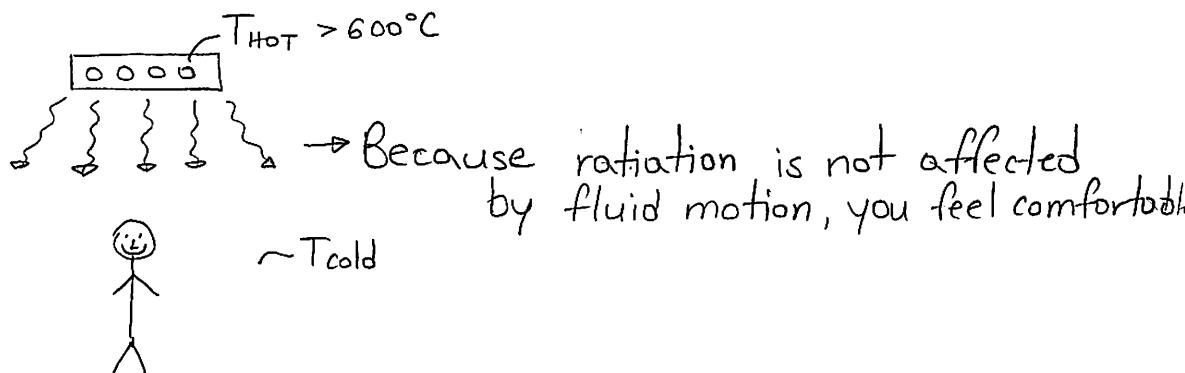
So far in ME 320, we've dealt with the following:

Conduction } Heat transfer by a temperature gradient in
Convection } a medium.

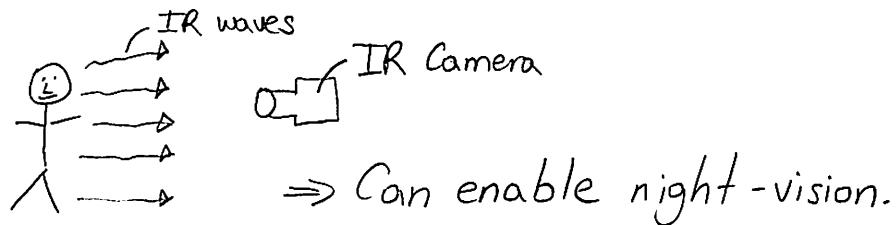
Radiation } - no medium required (can propagate in vacuum)
- a form of energy emitted by all matter
at a finite temperature.

Applications & Examples:

- 1) Global Energy Balance \rightarrow Earth receives its energy from the sun.
- 2) HVAC \rightarrow Heating, Ventilation & Air Conditioning
 \hookrightarrow Radiative heaters (bar or patio heaters)



- 3) Infrared Detection



Theory:

① Electromagnetic Theory (Maxwell) \Rightarrow Light is a wave

$$c = \lambda v = \frac{3.0 \times 10^8 \text{ m/s}}{n}$$

c = speed of light

n = medium index of refraction

$v = \frac{c}{\lambda}$ = frequency of light

λ = wavelength of light

② Quantum Theory (Planck) \Rightarrow Light is a particle (photon)

$$e = \frac{hc}{\lambda} = hv$$

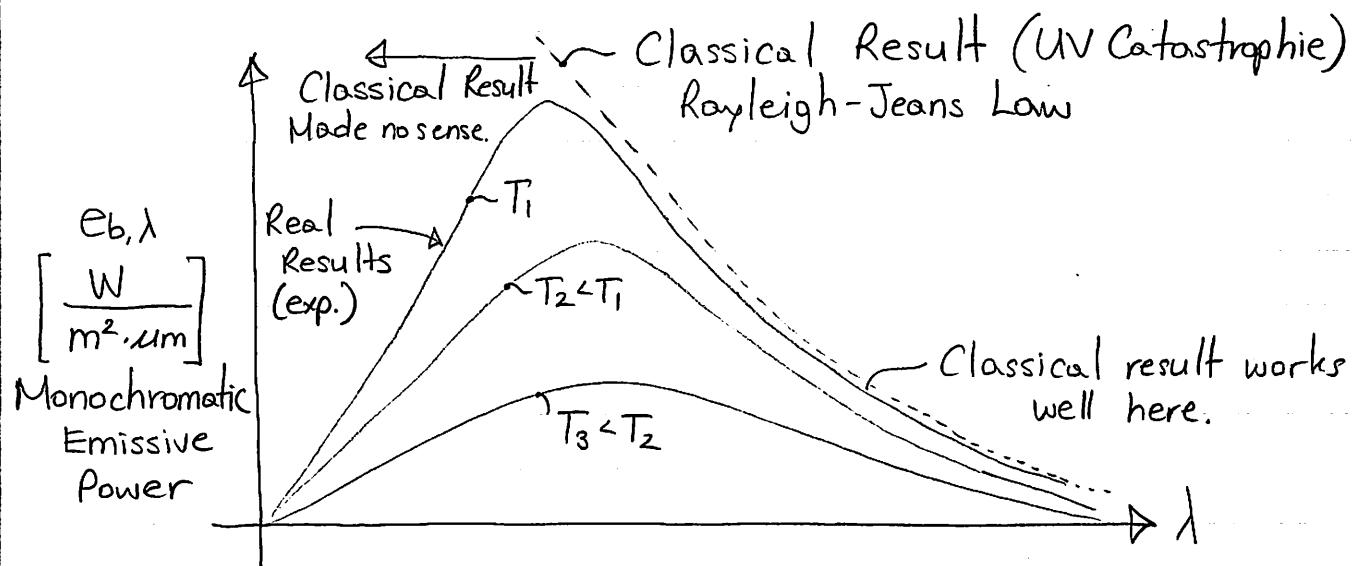
e = energy of a photon [J]

c = speed of light in a medium

λ = wavelength of light

$h = 6.6256 \times 10^{-34} \text{ J}\cdot\text{s}$ (Planck's const.)

These two theories came to a head in early 1900's.
Classical theory didn't make sense:



To solve this problem, Planck assumed light was made of particles, and applied kinetic theory. This was the basis for quantum theory.

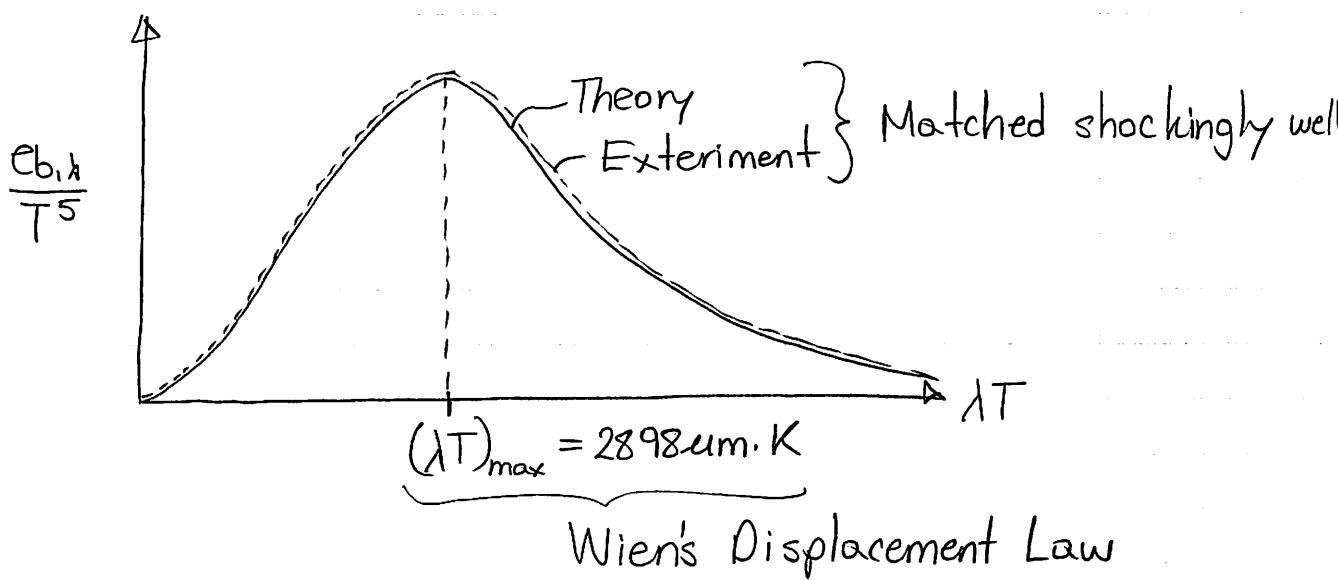
By treating light as particles & using statistical thermodynamics he derived:

$$e_{b,\lambda} = \frac{2\pi h c^2 \lambda^{-5}}{e^{\frac{hc}{k_B\lambda T}} - 1} = \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1}$$

Dividing both sides by T^5 :

$$\frac{e_{b,\lambda}}{T^5} = \frac{C_1 (\lambda T)^{-5}}{e^{C_2/\lambda T} - 1}$$

, k_B = Boltzmann constant
 $C_1 = 2\pi h c^2 = 3.742 \times 10^8 \frac{\text{W}\cdot\text{m}^4}{\text{m}^2}$
 $C_2 = \frac{hc}{k_B} = 1.4389 \times 10^4 \text{ um}\cdot\text{K}$



Visible wavelength range: $0.4 < \lambda < 0.7 \text{ um}$ (blue to red)
 Thermal radiation: $0.1 - 100 \text{ um}$

So as energy increases, (T^4), the peak emissive wavelength is shifted to lower wavelengths (visible)

Example: Hot IRON Bar \Rightarrow goes from no color \rightarrow red \rightarrow orange \rightarrow yellow \rightarrow white (158)

Total Emissive Power

If we want total power emitted by a radiating body over all wavelengths, we need to integrate over the whole spectrum:

$$\begin{aligned}
 e_b &= \int_0^\infty e_{b,\lambda} d\lambda = T^5 C_1 \int_0^\infty \frac{(\lambda T)^{-5}}{\exp(C_2/\lambda T) - 1} d\lambda \\
 &= T^4 \left[C_1 \int_0^\infty \frac{n^5}{e^{C_2/n} - 1} dn \right]; \quad n = \lambda T
 \end{aligned}$$

$\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4$

\hookrightarrow Stefan-Boltzmann constant

So our total emissive power is simply:

$$e_b = \sigma T^4 \Rightarrow T \text{ is in Kelvin!}$$

\hookrightarrow This is a heat flux [W/m²].

Note, the previous analysis is only valid for ideal emitters.

Ideal Emitter \Leftrightarrow "Blackbody"

- 1) Blackbodies absorb all incoming radiation
- 2) Radiation is independent of direction (Diffuse)
- 3) At a given temperature, T, no surface can emit more energy than a blackbody.

For non-blackbody surfaces (real surfaces), we can define a monochromatic emmisivity, ϵ_λ

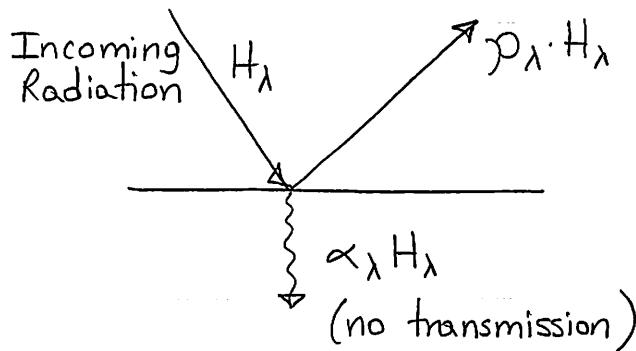
$$\epsilon_\lambda = \epsilon_\lambda \cdot e_{b,\lambda}$$

$$e = \int_0^\infty \epsilon_\lambda e_{b,\lambda} d\lambda = \epsilon \cdot e_b$$

$$\boxed{\epsilon = \frac{\int_0^\infty \epsilon_\lambda \cdot e_{b,\lambda} \cdot d\lambda}{\int_0^\infty e_{b,\lambda} d\lambda}}$$

⇒ Spectrally averaged emmisivity

In real life, we also have reflection (ρ) and absorption (α)



An energy balance:

$$\boxed{\rho_\lambda + \alpha_\lambda = 1}$$

In thermal equilibrium:

$$\boxed{\epsilon_\lambda = \alpha_\lambda} \text{ Kirchoff's Law}$$

Similarly for absorption:

$$\boxed{\alpha = \frac{\int_0^\infty \alpha_\lambda H_\lambda d\lambda}{\int_0^\infty H_\lambda d\lambda}}$$

⇒ Spectrally averaged absorptivity

From Kirchoff's Law we can say:

- 1) Good absorbers are good emitters
 - 2) Poor absorbers are poor emitters
- } Experimentally observed.

Note, if we have transmission (τ_λ) $\Rightarrow \alpha_\lambda + \rho_\lambda + \tau_\lambda = 1$

Example] A car accident injures a passenger at night and to help save bodyheat, the EMT's drape the passenger in an aluminum blanket. Assume $T_{\infty} = -50^{\circ}\text{C}$, and $E_{\text{blanket}} = 0.10$. How much heat loss is saved?

This is a very common technique to limit radiative losses, especially at night:

Assume your body temperature is at 37°C

$$e_b = \sigma T^4 = (5.67 \times 10^{-8}) [(37 + 273.15)^4 - (-50 + 273)^4]$$

$e_b = 384.05 \text{ W/m}^2$

\uparrow
 emission from atmosphere to you.

Now if we drape the aluminum blanket:

$$e_b = E_{\text{blanket}} \cdot \sigma T^4 = (0.10) (5.67 \times 10^{-8}) (310^4 - 223^4)$$

$e_b = 38.3 \text{ W/m}^2$

So how does this compare to convection? Lets assume $T_{\text{air}} \approx 5^{\circ}\text{C}$, and $h_{\text{conv}} = 20 \text{ W/m}^2 \cdot \text{K}$ (light breeze)

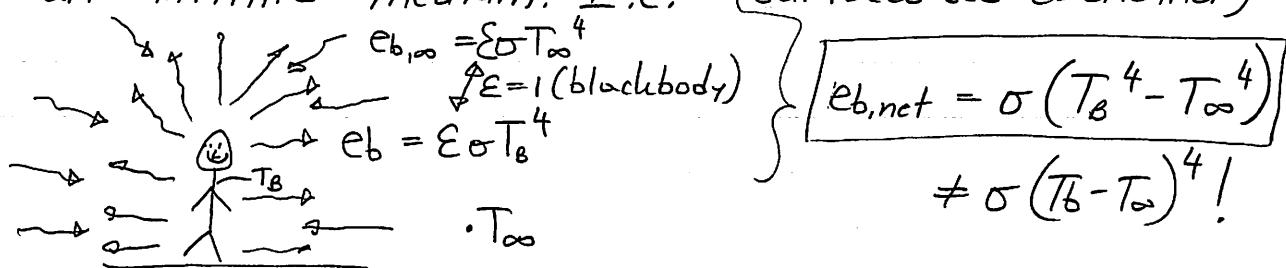
$$q''_{\text{conv}} = h \Delta T = (20 \text{ W/m}^2 \text{ K}) (37^{\circ}\text{C} - 5^{\circ}\text{C})$$

$q''_{\text{conv}} = 640 \text{ W/m}^2$

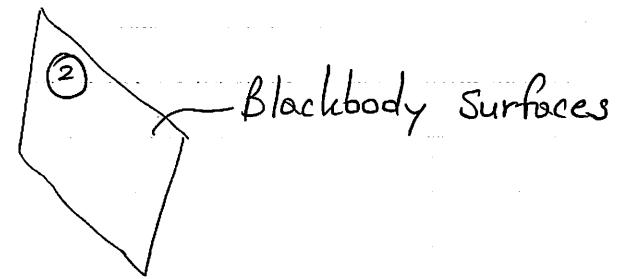
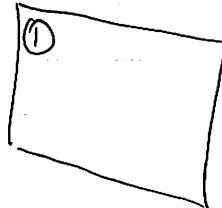
So we can see that both radiation & convection are very important mechanisms and that draping the reflective aluminum blanket lowers losses by about 35%.

View Factors and Blackbody Heat Exchange

So far, everything has been about blackbody radiation to an infinite medium. I.e.: (surfaces see each other)



But, what if we have finite surfaces:



$$\left. \begin{array}{l} q_{1 \rightarrow 2} = F_{12} A_1 \epsilon_{b1} \\ q_{2 \rightarrow 1} = F_{21} A_2 \epsilon_{b2} \end{array} \right\} \begin{array}{l} F_{ab} = \text{view factor} \\ = \text{fraction of radiation leaving surface "a" and reaching "b"} \end{array}$$

So our net exchange is:

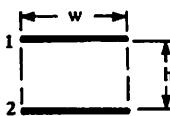
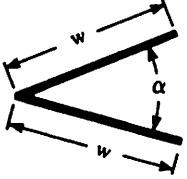
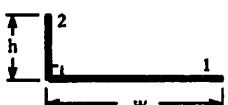
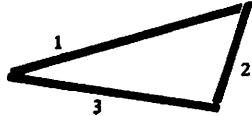
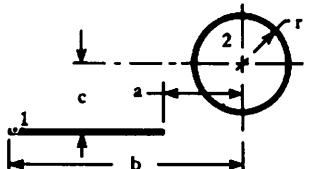
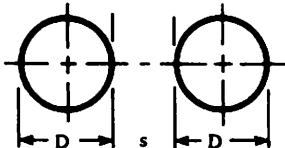
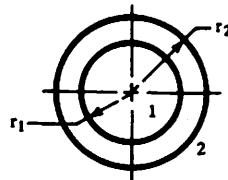
$$q_{12} = q_{1 \rightarrow 2} - q_{2 \rightarrow 1} = F_{12} A_1 \epsilon_{b1} - F_{21} A_2 \epsilon_{b2}$$

But if $T_1 = T_2$, $\epsilon_{b1} = \epsilon_{b2}$

$$q_{12} = \epsilon_b (A_1 F_{12} - A_2 F_{21}) = 0 \quad (\text{no net energy transfer in between surfaces at same } T)$$

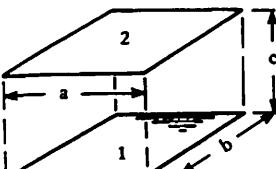
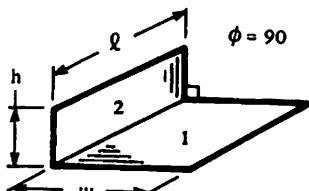
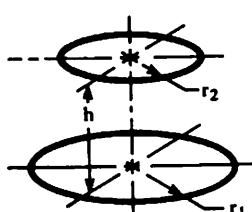
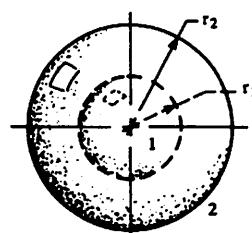
$A_1 F_{12} = A_2 F_{21} \Rightarrow \text{Reciprocity}$

Table 10.2 View factors for a variety of two-dimensional configurations (infinite in extent normal to the paper)

Configuration	Equation
1.	 $F_{1-2} = F_{2-1} = \sqrt{1 + \left(\frac{h}{w}\right)^2} - \left(\frac{h}{w}\right)$
2.	 $F_{1-2} = F_{2-1} = 1 - \sin(\alpha/2)$
3.	 $F_{1-2} = \frac{1}{2} \left[1 + \frac{h}{w} - \sqrt{1 + \left(\frac{h}{w}\right)^2} \right]$
4.	 $F_{1-2} = (A_1 + A_2 - A_3)/2A_1$
5.	 $F_{1-2} = \frac{r}{b-a} \left[\tan^{-1} \frac{b}{c} - \tan^{-1} \frac{a}{c} \right]$
6.	 Let $X = 1 + s/D$. Then: $F_{1-2} = F_{2-1} = \frac{1}{\pi} \left[\sqrt{X^2 - 1} + \sin^{-1} \frac{1}{X} - X \right]$
7.	 $F_{1-2} = 1, \quad F_{2-1} = \frac{r_1}{r_2}, \text{ and}$ $F_{2-2} = 1 - F_{2-1} = 1 - \frac{r_1}{r_2}$

Adapted From Lienhard & Lienhard "A Heat Transfer Textbook" (2012)

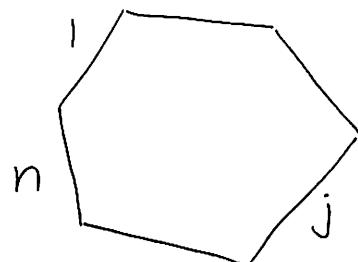
Table 10.3 View factors for some three-dimensional configurations

<i>Configuration</i>	<i>Equation</i>
1.	<p>Let $X = a/c$ and $Y = b/c$. Then:</p> $F_{1-2} = \frac{2}{\pi XY} \left\{ \ln \left[\frac{(1+X^2)(1+Y^2)}{1+X^2+Y^2} \right]^{1/2} - X \tan^{-1} X - Y \tan^{-1} Y + X \sqrt{1+Y^2} \tan^{-1} \frac{X}{\sqrt{1+Y^2}} + Y \sqrt{1+X^2} \tan^{-1} \frac{Y}{\sqrt{1+X^2}} \right\}$ 
2.	<p>Let $H = h/\ell$ and $W = w/\ell$. Then:</p> $F_{1-2} = \frac{1}{\pi W} \left\{ W \tan^{-1} \frac{1}{W} - \sqrt{H^2 + W^2} \tan^{-1} (H^2 + W^2)^{-1/2} + H \tan^{-1} \frac{1}{H} + \frac{1}{4} \ln \left\{ \left[\frac{(1+W^2)(1+H^2)}{1+W^2+H^2} \right]^{W^2} \left[\frac{H^2(1+H^2+W^2)}{(1+H^2)(H^2+W^2)} \right]^{H^2} \right\} \right\}$ 
3.	<p>Let $R_1 = r_1/h$, $R_2 = r_2/h$, and $X = 1 + (1 + R_2^2)/R_1^2$. Then:</p> $F_{1-2} = \frac{1}{2} \left[X - \sqrt{X^2 - 4(R_2/R_1)^2} \right]$ 
4.	<p>Concentric spheres:</p> $F_{1-2} = 1, \quad F_{2-1} = (r_1/r_2)^2, \quad F_{2-2} = 1 - (r_1/r_2)^2$ 

Now we can formulate our analysis as:

$$q_{r12} = \underbrace{A_1 F_{12} (e_{b1} - e_{b2})}_{A_1 F_{12} e_{b1} - A_2 F_{21} e_{b2}} \Leftrightarrow \begin{array}{c} q_{r12} \rightarrow \\ \text{---} \\ e_{b1} \quad \frac{1}{\underbrace{A_1 F_{12}}_{\text{Radiative resistance}}} \quad e_{b2} \end{array}$$

For enclosures:



$$\sum_{j=2}^n F_{ij} = 1$$

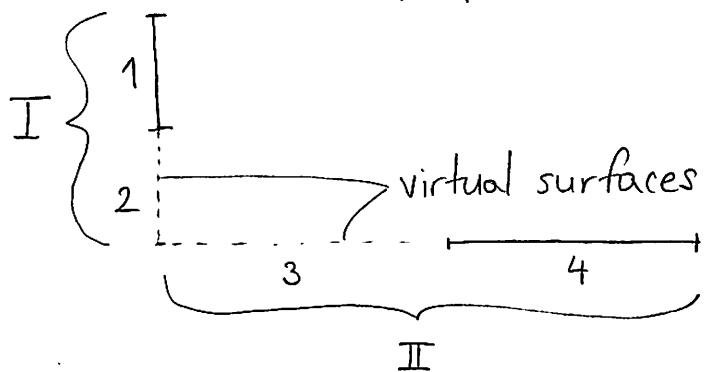
\Rightarrow True for flat or convex surfaces.

If concave surface

$$\sum_{j=2}^n F_{ij} \neq 1 \text{ since } F_{ii} \neq 0$$

(Surface sees itself)

Example Consider the following arrangement between two perpendicular plates 1 & 4



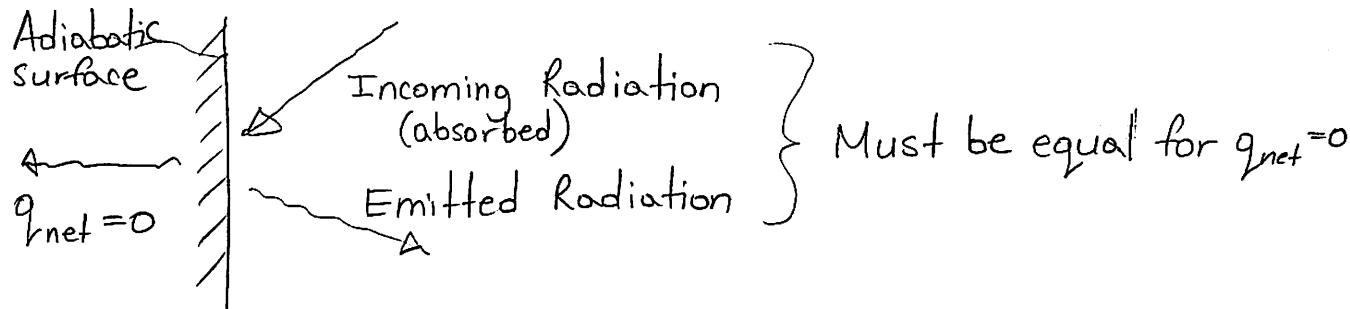
$$A_1 F_{14} = A_I F_{I4} - A_2 F_{24}$$

$$A_I F_{I4} = A_I F_{I\text{II}} - A_I F_{I3} \Rightarrow F_{I4} = F_{\text{II}} - F_{I3}$$

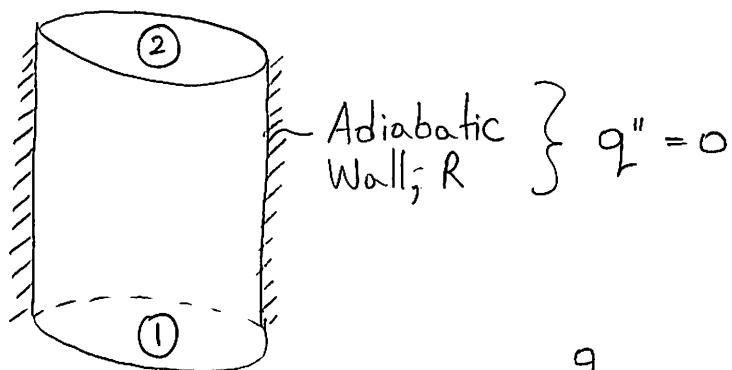
$$A_2 F_{24} = A_2 F_{2\text{II}} - A_2 F_{23} \Rightarrow F_{24} = F_{2\text{II}} - F_{23}$$

Can look these up
in view factor tables.

Adiabatic Surfaces (no heat transfer through that surface)



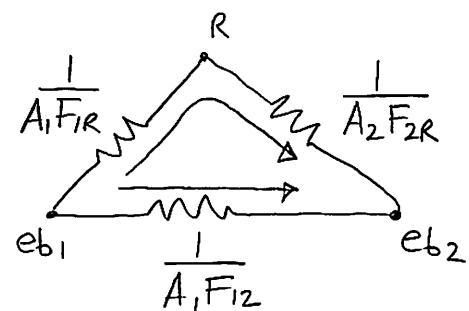
If we look at an enclosure: (furnace)



Previously we had:

$$e_{b1} \xrightarrow{\frac{1}{A_1 F_{12}}} e_{b2}$$

Now we have:



We can redraw this:

$$\left(\frac{1}{A_1 F_{1R}} + \frac{1}{A_2 F_{2R}} \right) \} R_{II}$$

$\Rightarrow R_{TOT} = \left[\frac{1}{R_I} + \frac{1}{R_{II}} \right]$

$\frac{1}{A_1 F_{12}} \} R_I$

(166)

$$\frac{1}{R_{TOT}} = \frac{1}{R_I} + \frac{1}{R_{II}}$$

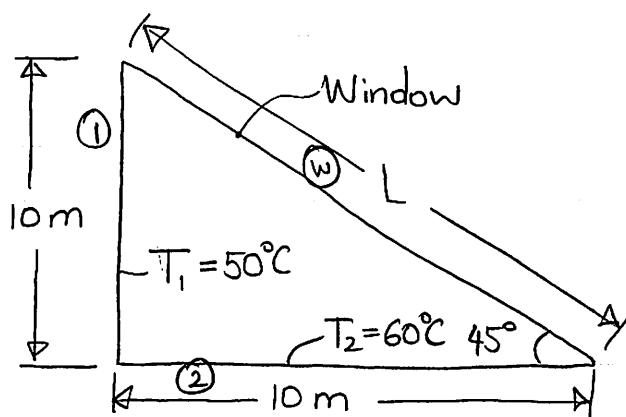
$$= A_1 F_{I2} + \frac{1}{\frac{1}{A_1 F_{IR}} + \frac{1}{A_2 F_{2R}}}$$

$$q_{I2} = \frac{e_{b1} - e_{b2}}{R_{TOT}}$$

$$q_{I2} = (e_{b1} - e_{b2}) \left[A_1 F_{I2} + \frac{1}{\frac{1}{A_1 F_{IR}} + \frac{1}{A_2 F_{2R}}} \right]$$

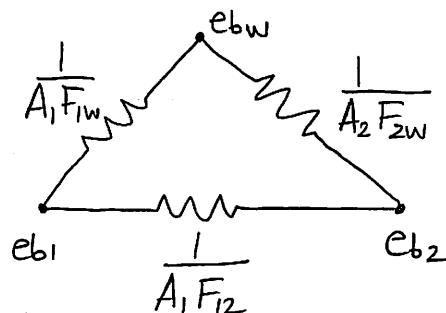
⇒ Units of Watts

Example | A solar greenhouse with a triangular architecture has the following design:



Assuming the widow is a perfect re-radiator, find the window temperature.

First, we can draw our radiative resistance network.



From symmetry:
 $A_1 = A_2$
 $F_{I2} = F_{2I}$
 $F_{IR} = F_{2R}$

$$q_{L12} = \frac{e_{b1} - e_{b2}}{R_{TOT}} = (e_{b1} - e_{b2}) \left[A_1 F_{12} + \frac{1}{\frac{1}{A_1 F_{1W}} + \frac{1}{A_2 F_{2W}}} \right]$$

Looking back at our view factor tables: (pg. 163 of notes, case 2)

$$F_{12} = F_{21} = 1 - \sin\left(\frac{\alpha}{2}\right); \quad \alpha = 90^\circ$$

$$F_{12} = F_{21} = 0.292$$

$$F_{1W} = 1 - F_{12} = 0.707 \quad (\text{since surface 1 is flat, i.e. } F_{11} = 0)$$

$$e_{b1} = \sigma T_1^4 = (5.67 \times 10^{-8})(50 + 273.15)^4 = 618.3 \text{ W/m}^2$$

$$e_{b2} = \sigma T_2^4 = (5.67 \times 10^{-8})(60 + 273.15)^4 = 698.46 \text{ W/m}^2$$

$$\begin{aligned} q_{L12} &= (e_{b1} - e_{b2}) \left[(10\text{m})(0.292) + \frac{1}{\frac{1}{(10\text{m})(0.707)} + \frac{1}{(10\text{m})(0.707)}} \right] \\ &= (e_{b1} - e_{b2})(6.455) \end{aligned}$$

$$\boxed{q_{L12} = -516.7 \text{ W/m}} \Rightarrow \text{Heat radiates from ② to ① actually}$$

Now we can solve for T_W . We first need to solve for heat transfer through each leg:



$$q_{L1} = (e_{b1} - e_{b2}) \left(\frac{1}{A_1 F_{12}} \right)^{-1} = -234.1 \text{ W/m}$$

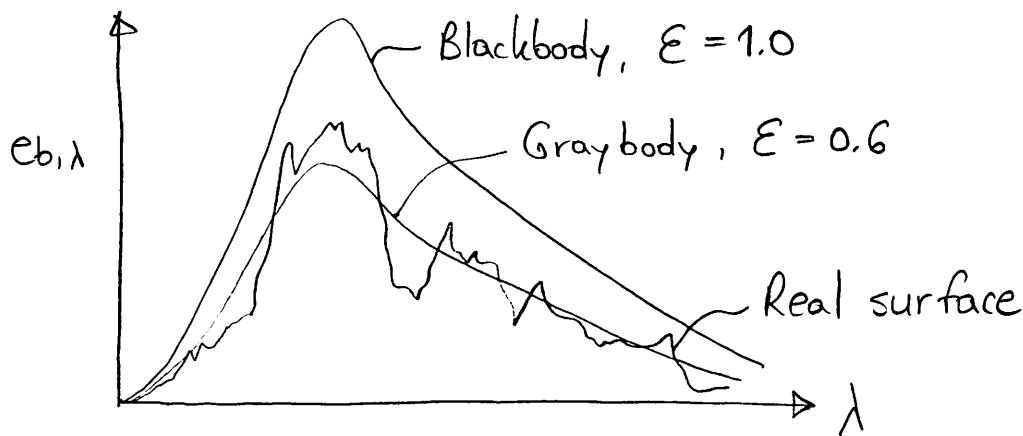
$$q_{L2} = q_{L12} - q_{L1} = -516.7 - (-234.1)$$

$$(e_{b1} - e_{bW}) \left(\frac{1}{A_1 F_{1W}} \right)^{-1} = q_{L2} \Rightarrow \boxed{T_W = 55^\circ C} = -282.6 \text{ W/m}$$

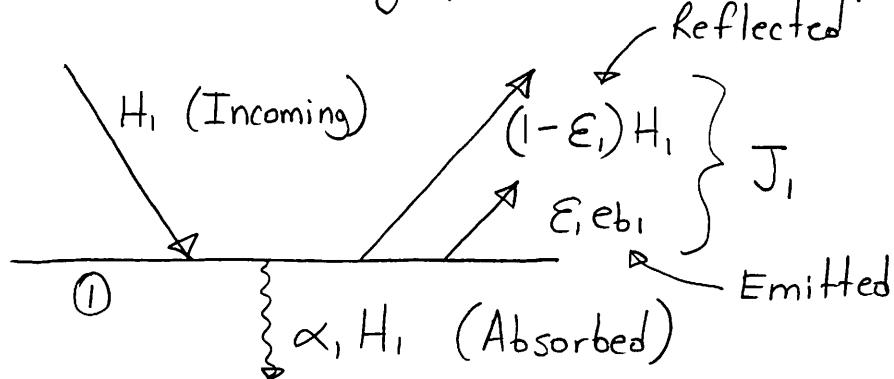
Gray Body Radiation

So far, we've only been dealing with cases where $\epsilon = 1$ or blackbodies. Most real surfaces fail to satisfy this.

For example:



If we look at a gray surface and apply an energy bal:



Here, we assume Kirchoff's law: $\epsilon = \alpha$

H_i = irradiance [W/m^2] (Incoming radiation)

J_i = radiosity [W/m^2] (Total radiant flux away from the surface, emitted + reflected)

Let's look at each component:

$$\text{Incoming} = H_i$$

$$\text{Absorbed} = \alpha_i H_i = \epsilon_i H_i$$

$$\text{Reflected} = \text{Incoming} - \text{Absorbed} = H_i - \alpha_i H_i = H_i(1-\epsilon_i)$$

$$\text{Emitted} = \epsilon_i e_{b1} = \epsilon_i \sigma T^4$$

So our radiosity (J_i) becomes:

$$J_i = (1-\epsilon_i)H_i + \epsilon_i e_{b1} \Rightarrow \text{Rearrange this}$$

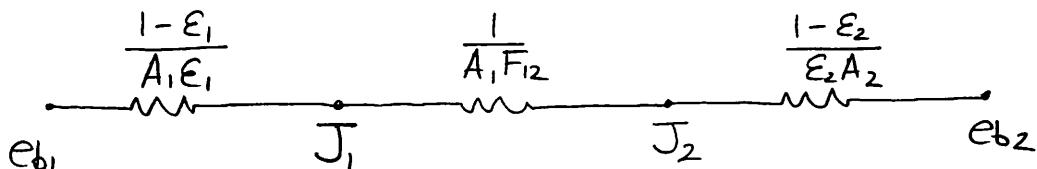
$$H_i = \frac{J_i - \epsilon_i e_{b1}}{1-\epsilon_i}$$

$$\underbrace{\frac{q_{i,\text{NET}}}{A_i}}_{\text{Net radiative energy leaving surface } i \text{ (heat flux)}} = (J_i - H_i) = J_i - \frac{J_i - \epsilon_i e_{b1}}{1-\epsilon_i}$$

$\underbrace{\text{Net radiative energy leaving surface } i \text{ (heat flux)}}$

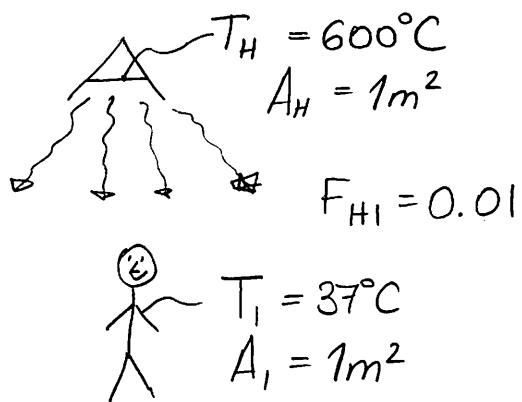
$$\frac{q_{i,\text{NET}}}{A_i} = \frac{J_i (1-\epsilon_i) - J_i + \epsilon_i e_{b1}}{1-\epsilon_i} = \frac{\epsilon_i}{1-\epsilon_i} (e_{b1} - J_i)$$

$$\boxed{\frac{q_{i,\text{NET}}}{A_i} = \frac{\epsilon_i}{1-\epsilon_i} (e_{b1} - J_i)} \Rightarrow \text{NET radiative heat flux leaving surface } i.$$



Note, if you set $\epsilon_1 = \epsilon_2 = 1$, you get the blackbody sol. (170)

Example | Radiative heater. Find the heat flux to your body assuming:

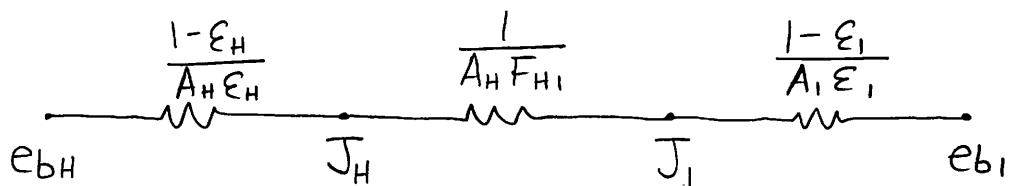


- (a) $\epsilon_H = 1.0, \epsilon_b = 1.0$ (blackbody)
- (b) $\epsilon_H = 0.8, \epsilon_b = 0.5$ (graybody)

(a) We can right away draw our thermal resistance diagram:

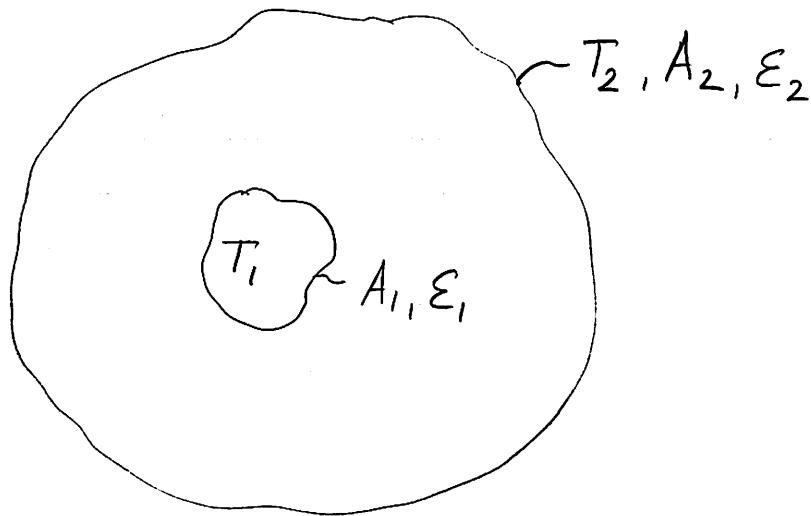
$$\frac{1}{A_H F_{H1}} \rightarrow e_{bH} \xrightarrow{q_{H1}} e_{b1} \Rightarrow q_{H1} = \frac{e_{bH} - e_{b1}}{\frac{1}{A_H F_{H1}}} = 320\text{W}$$

(b) Similar approach but more realistic:



$$q_{H1} = \frac{\overbrace{e_{bH} - e_{b1}}^{32431.65}}{\underbrace{\frac{1-\epsilon_H}{A_H \epsilon_H}}_{0.25} + \underbrace{\frac{1}{A_H F_{H1}}}_{100} + \underbrace{\frac{1-\epsilon_b}{A_b \epsilon_b}}_1} = 319\text{W}$$

This shows you the importance of relative contribution to resistances. In this case, view factor resistance dominates. (171)

Gray Body Enclosures

Assuming that body 2 is much larger than body 1:

$$e_{b1} \frac{1-\epsilon_1}{A_1\epsilon_1} J_1 \quad J_2 \quad \frac{1}{A_1 F_{12}} \quad \frac{1-\epsilon_2}{A_2\epsilon_2} e_{b2}$$

$$q_{1 \rightarrow 2} = \frac{e_{b1} - e_{b2}}{\frac{1-\epsilon_1}{A_1\epsilon_1} + \frac{1}{A_1 F_{12}} + \frac{1-\epsilon_2}{A_2\epsilon_2}} \Rightarrow \text{multiply by } \frac{A_1}{A_1}$$

$$q_{1 \rightarrow 2} = \frac{A_1 (e_{b1} - e_{b2})}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \frac{A_1}{A_2} \left(\frac{1}{\epsilon_2} - 1\right)}$$

Some special cases:

① $A_2 \gg A_1$

This means $\frac{A_1}{A_2} \rightarrow 0$ and $F_{12} = 1$

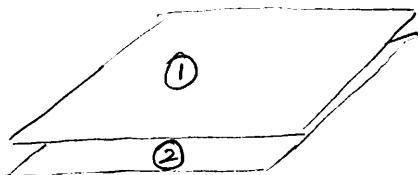
Back substituting into our solution

$$q_{1 \rightarrow 2} = \frac{A_1 (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} - 1 + 1 + 0 \left(\frac{1}{\epsilon_2} - 1 \right)}^0$$

$$q_{1 \rightarrow 2} = A_1 \epsilon_1 (e_{b1} - e_{b2})$$

\Rightarrow Smaller body behaves like a black body with finite emissivity.

② Parallel Plates (small gap between them)

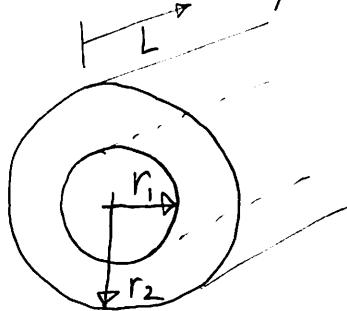


$$\frac{F_{12}}{A_1} = 1 = F_{21}$$

$$q_{1 \rightarrow 2} = \frac{A_1 (e_{b1} - e_{b2})}{\left(\frac{1}{\epsilon_1} - 1 \right) + \frac{1}{F_{12}} + \frac{A_1}{A_2} \left(\frac{1}{\epsilon_2} - 1 \right)} = \frac{A_1 (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} - 1 + 1 + \frac{1}{\epsilon_2} - 1}$$

$$q_{1 \rightarrow 2} = \frac{A_1 (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

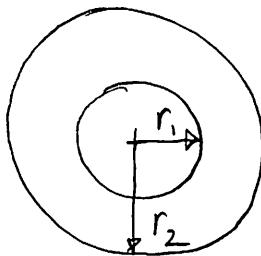
3. Concentric Cylinders (Length L)



$$F_{12} = 1 ; \quad A_1 = 2\pi r_1 L \quad A_2 = 2\pi r_2 L \quad \left\{ \frac{A_1}{A_2} = \frac{r_1}{r_2} \right.$$

$$q_{1 \rightarrow 2} = \frac{2\pi r_1 L (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{r_1}{r_2} \left(\frac{1}{\epsilon_2} - 1 \right)}$$

4. Concentric Spheres



$$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} ; \quad F_{12} = 1$$

$$q_{1 \rightarrow 2} = \frac{4\pi r_1^2 (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{r_1^2}{r_2^2} \left(\frac{1}{\epsilon_2} - 1 \right)}$$

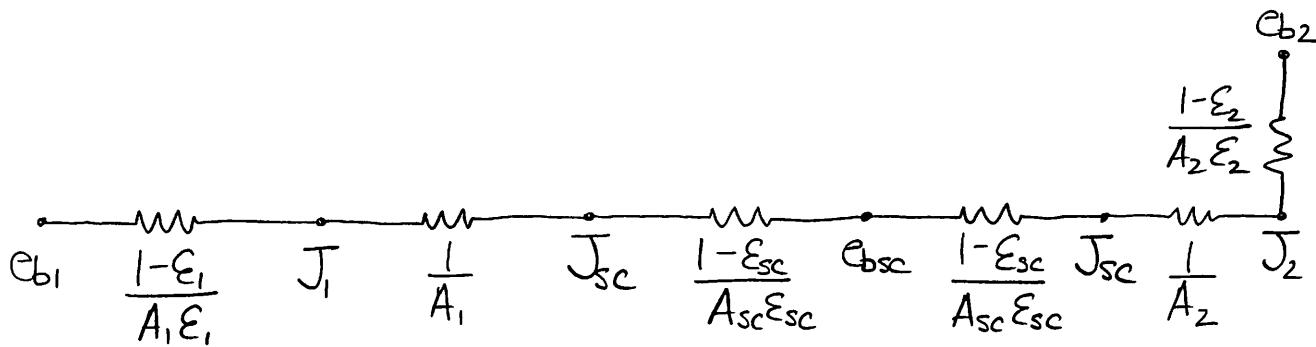
Radiation Shields

Used to reduce radiative heat loss
For a flat plate arrangement:

_____ ①, ϵ_1

- - - - - Sc (shield), ϵ_{sc}

_____ ②, ϵ_2



Since $A_1 = A_{sc} = A_2$ (parallel plates)
 $F_{12} = F_{sc,2} = F_{2,sc} = 1$

Our result simplifies to:

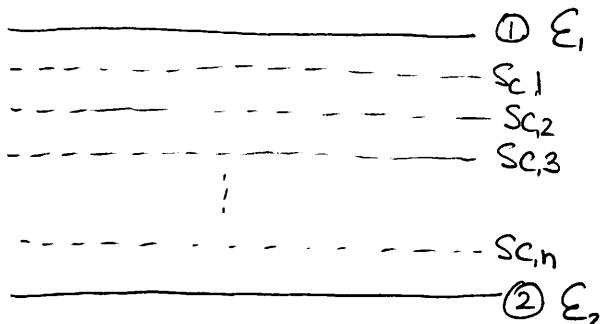
$$q_{12} = \frac{A (e_{b1} - e_{b2})}{\left(\frac{1}{\epsilon_1} - 1\right) + 1 + \left(\frac{1}{\epsilon_2} - 1\right) + 2 \left(\frac{1}{\epsilon_{sc}} - 1\right) + 1}$$

$$q_{12} = \frac{A (e_{b1} - e_{b2})}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) + \left(\frac{2}{\epsilon_{sc}} - 1\right)}$$

⇒ For 1 shield

Note the solution is identical to the parallel plate solution from before except now we have an additional term in the denominator that diminishes q_{12} .

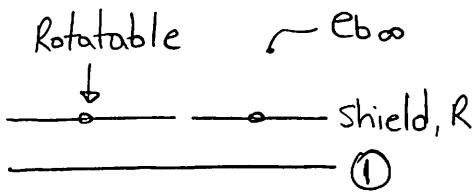
For n shields in series: (with identical radiative properties)



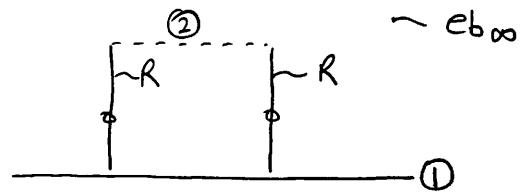
$$q_{12} = \frac{A (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{2n}{\epsilon_{sc}} - (n+1)}$$

Example Consider NASA's rotatable radiation shield shown below. Calculate its effectiveness in the on & off positions: $E_{sc} = 0.1$, $\epsilon_i = 0.5$

State I: ON



State II: OFF



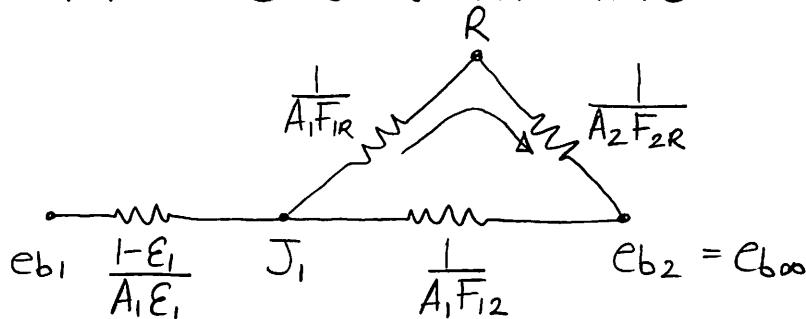
Let's solve the ON case first:

$$e_{b1} \frac{1-\epsilon_i}{A_i E_i} J_i \frac{1}{A_i F_{isc}} J_{sc} \frac{1-E_{sc}}{A_{sc} E_{sc}} e_{b_{sc}} \frac{1-E_{sc}}{A_{sc} E_{sc}} J_{sc} \frac{1}{A_{sc} F_{sc,\infty}} e_{b\infty}$$

We know $A_i = A_{sc} = A$ (parallel infinite plates)
 $F_{isc} = F_{sc2} = 1$

$$q_{100} = \frac{A_i (e_{b1} - e_{b\infty})}{\frac{1}{E_i} + \frac{2}{E_{sc}} - 1} \Rightarrow \text{State I, ON}$$

The OFF case is a bit more tricky to solve:



* The shields in the OFF state act as adiabatic surfaces due to symmetry, so $e_{bR} = J_R$

From our view factor tables:

$$F_{12} = \sqrt{2} - 1$$

$$F_{1R} = 2 - \sqrt{2}$$

$$q_{12} = \frac{A_1(e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} - 1 + \sqrt{2}} \Rightarrow \text{State II, OFF}$$

Now we can calculate the ratio of heat transfer from the thermal resistance ratios (denominator):

$$\frac{R_I}{R_{II}} = \frac{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_{sc}} - 1}{\frac{1}{\epsilon_1} - 1 + \sqrt{2}} = 13.2$$

So when the shield is ON, it is 13.2 times better at limiting radiative heat transfer.

Note here the importance of choosing ϵ_{sc} . If $\epsilon_{sc} \ll 1$, then radiation shield is much better performing. This is why scientists use highly reflective surfaces for radiation shielding like aluminum foil.

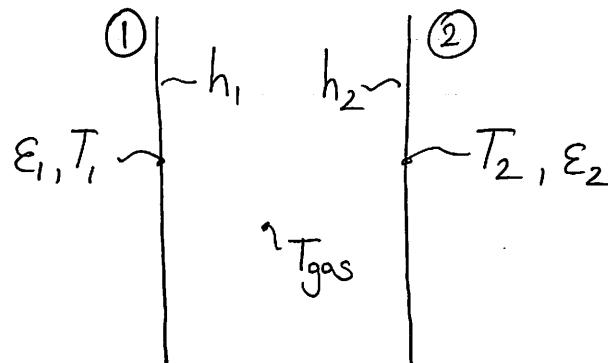
Multi-Mode Heat Transfer

What if we have radiation & convection. How do we handle this?

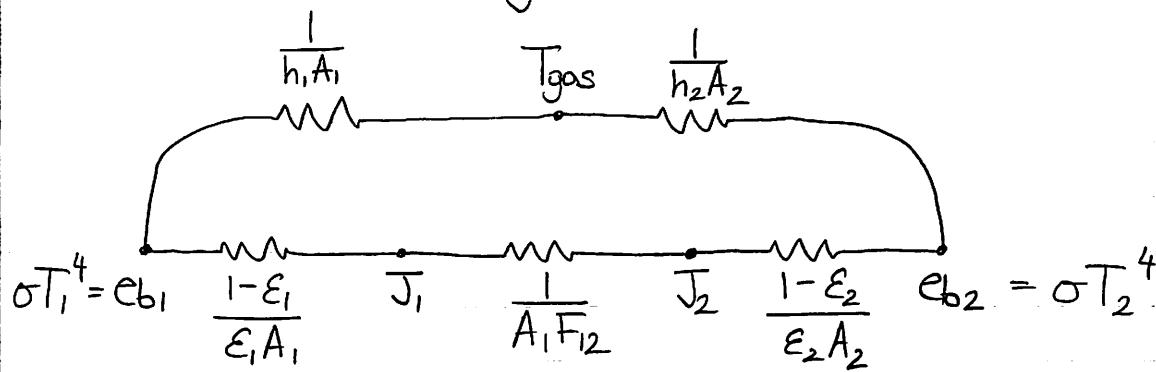
Well, since:

Radiation \Rightarrow independent of medium } Parallel paths.
 Convection \Rightarrow medium dependent }

For example:

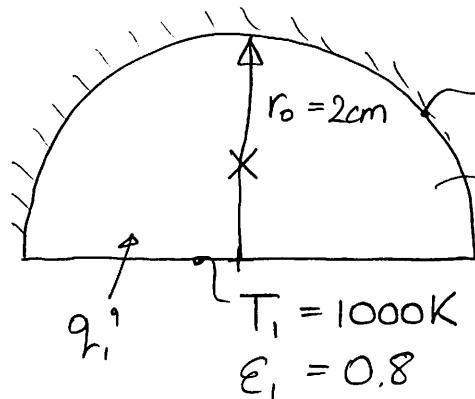


Our resistance diagram becomes:



We need to solve this complete resistance diagram and it usually involves iteration since T_{gas} is unknown.

Example Consider an air heater : (semi-circular)



Adiabatic, $\epsilon_2 = 0.8$

Air flows through here into the page. $T_m = 400 \text{ K}$, $m = 0.01 \text{ kg/s}$

Find the temperature of the adiabatic surface, and the heat required per unit length (q_h') to maintain $T_1 = 1000 \text{ K}$. Assuming fully developed internal flow:

Look up properties in Table A.4 or Incropera (Textbook)

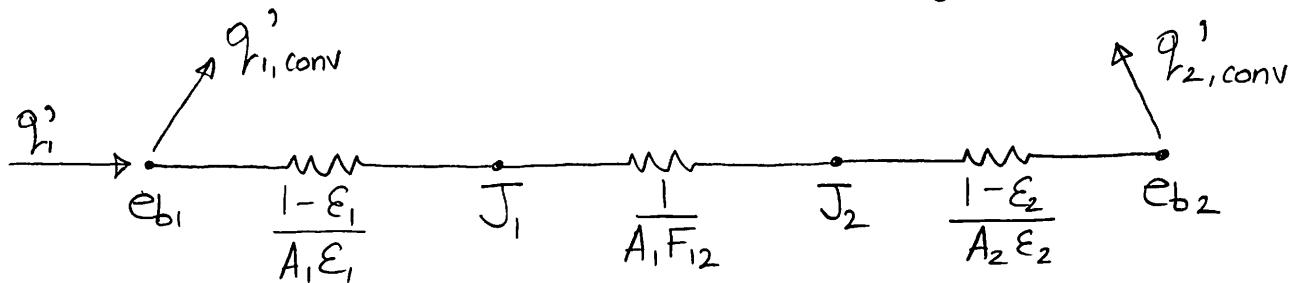
$$K_{\text{Air}} = 0.0338 \text{ W/m}\cdot\text{K}$$

$$\mu = 230 \times 10^{-7} \text{ kg/(s}\cdot\text{m)}$$

$$C_p = 1014 \text{ J/kg}\cdot\text{K}$$

$$\Pr = 0.69$$

Let's draw our thermal resistance diagram:



Since surface 2 is insulated : $\text{q}_{h2,\text{rad}}' = \text{q}_{h2,\text{conv}}' \quad ①$

Net radiative transfer into 2. (79)

We can evaluate the radiative and convective balance on surface 2 as:

$$\frac{\sigma(T_1^4 - T_2^4)}{\frac{1-\epsilon_1}{A_1\epsilon_1} + \frac{1}{A_1F_{12}} + \frac{1-\epsilon_2}{A_2\epsilon_2}} = h A_2 (T_2 - T_m) \quad \text{Air temperature}$$

We know: $F_{12} = 1$ (doesn't see itself, $F_{11} = 0$)

$$A_1 = 2r_0 \text{ (area per unit length)}$$

$$A_2 = \pi r_0 \text{ (area per unit length)}$$

Now for the fluids problem: (convection)

$$Re_0 = \frac{\rho U D_h}{\mu} = \frac{\dot{m} D_h}{A_c U} = \frac{\dot{m} D_h}{(\pi r_0^2/2) U}$$

\hookrightarrow Cross sectional area

$$D_h = \frac{4A_c}{\rho} = \frac{2\pi r_0^2}{\rho(\pi+2)} = \frac{0.04\pi}{\pi+2} = 0.0244 \text{ m}$$

$$Re_0 = \frac{(0.01 \text{ kg/s})(0.0244 \text{ m})}{(\pi/2)(0.02 \text{ m})^2 (230 \times 10^{-7} \text{ kg/m.s})} = 16900$$

So we have turbulent fully developed flow.

Flipping back to our correlation table for internal flow:
Table 8.4, pg 137 of notes.

We can use $(8.60)^d \Rightarrow$ Dittus-Boelter equation

$$Nu_0 = \frac{h D_h}{k_{air}} = 0.023 Re_0^{4/5} Pr^{0.4}$$

$$Nu_0 = 0.023 (16900)^{4/5} (0.69)^{0.4} = 47.8$$

$$h = \frac{k_{air} Nu_0}{D_h} = \frac{(0.0338 \text{ W/m}\cdot\text{K})(47.8)}{(0.0244 \text{ m})} = 66.2 \text{ W/m}^2\cdot\text{K}$$

Now we can work with our energy balance on surface 2.

$$\frac{\sigma (T_1^4 - T_2^4)}{\frac{1-\epsilon_1}{A_1 \epsilon_1} + \frac{1}{A_1 F_{12}} + \frac{1-\epsilon_2}{A_2 \epsilon_2}} = h A_2 (T_2 - T_m) \Rightarrow \begin{matrix} \text{Divide through} \\ \text{by } A_1 = 2r_0 \\ A_2 = \pi r_0 \end{matrix}$$

$$\frac{(5.67 \times 10^{-8})(1000^4 - T_2^4)}{\frac{1-0.8}{0.8} + 1 + \frac{1-0.8}{0.8} \left(\frac{2}{\pi}\right)} = (66.2) \left(\frac{\pi}{2}\right) (T_2 - 400)$$

Simplifying, we obtain:

$$(5.67 \times 10^{-8}) T_2^4 + 146.5 T_2 - 115313 = 0$$

Using iteration (guess T_2 & check solution), we obtain:

$$\boxed{T_2 = 696 \text{ K}}$$

Now we can solve for q'_1 at the heated surface 1.

An energy balance reveals:

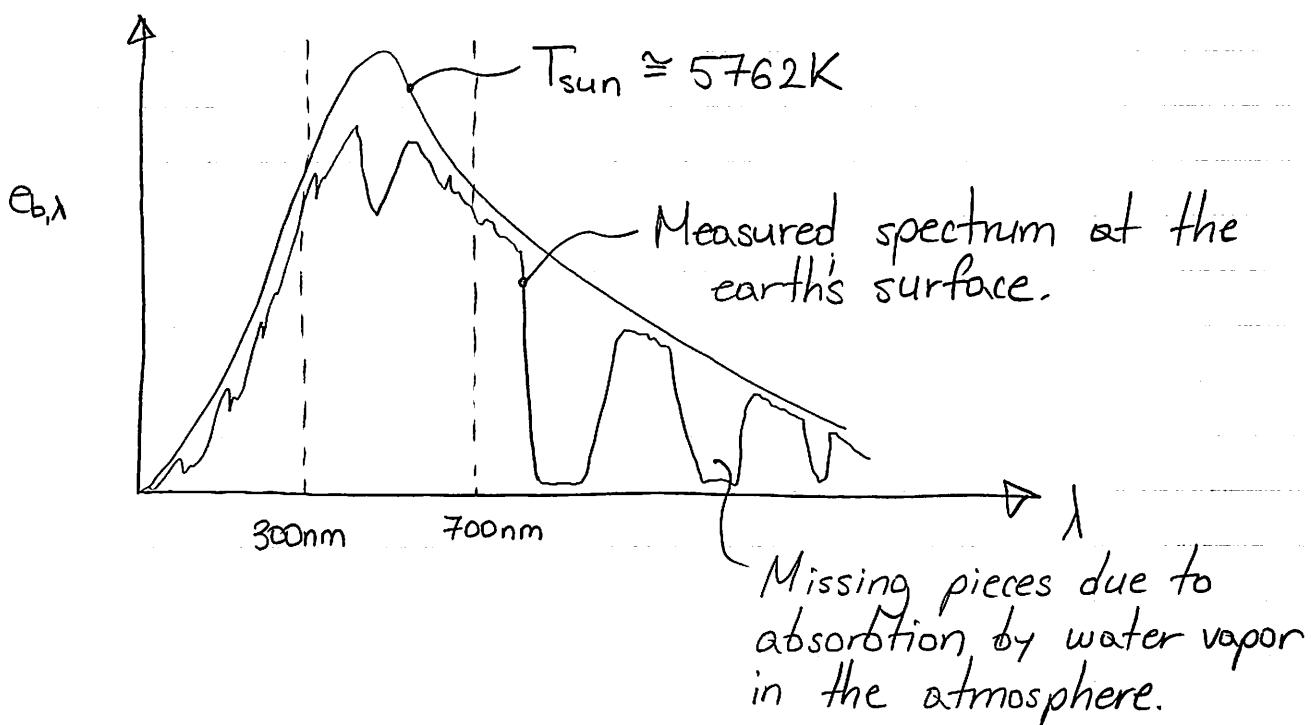
$$q'_1 = q'_{1,\text{rad}} + q'_{1,\text{conv}} = \underbrace{q'_{2,\text{conv}}}_{\text{All heat in } (q'_1) \text{ is taken away by convection.}} + q'_{1,\text{conv}}$$

(181)

So on a per unit length basis:

$$\begin{aligned} q'_h &= h(\pi r_o)(T_2 - T_m) + h(2r_o)(T_1 - T_m) \\ &= 66.2 \left[(\pi(0.02m))(696 - 400) + 2(0.02m)(1000 - 400) \right] \\ q'_h &= 2820 \text{ W/m} \end{aligned}$$

Solar Radiation



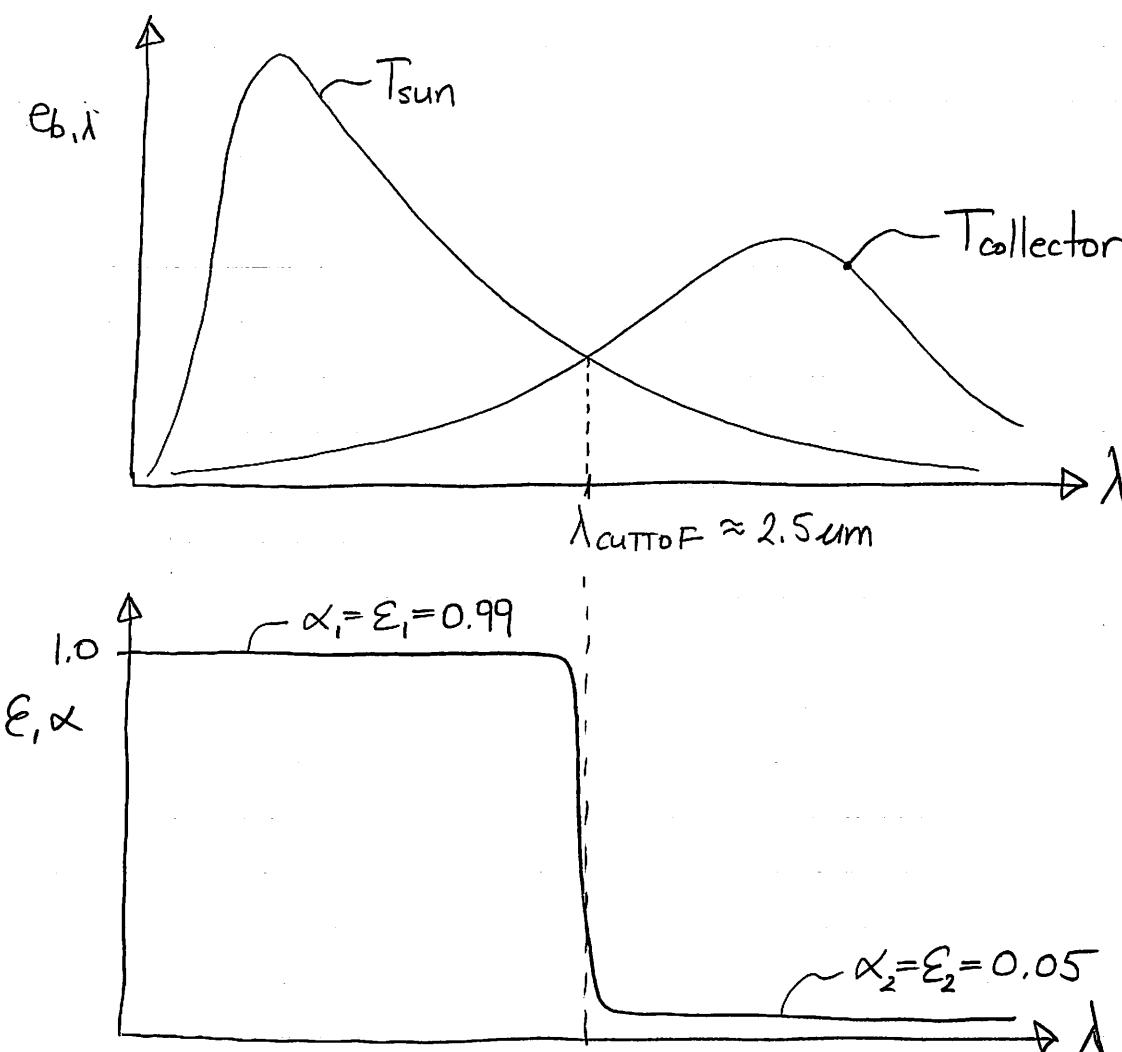
$$q''_{\text{sun}} = 1353 \text{ W/m}^2 \text{ (in space)}$$

$$q''_{\text{sun,earth}} = 636 \text{ W/m}^2 \text{ (on the earth's surface)}$$

Note, the total arriving energy from sun to earth is $1.7 \times 10^{14} \text{ kW}$! Peak demand in the US is $1 \times 10^9 \text{ kW}$!

Most solar applications try to absorb some of the heat from the sun.

They use fancy surfaces called selective surfaces:



Because the absorption and emission spectra don't overlap, we can simplify our calculations:

$$E_{\text{absorbed}} = q''_{\text{sun}} \cdot \alpha_1 \cdot A$$

; α_1 = absorbtivity in the solar spectrum

$$E_{\text{emitted}} = \epsilon_2 A \cdot \sigma T_{\text{surface}}^4$$

; ϵ_2 = emissivity in the IR spectrum