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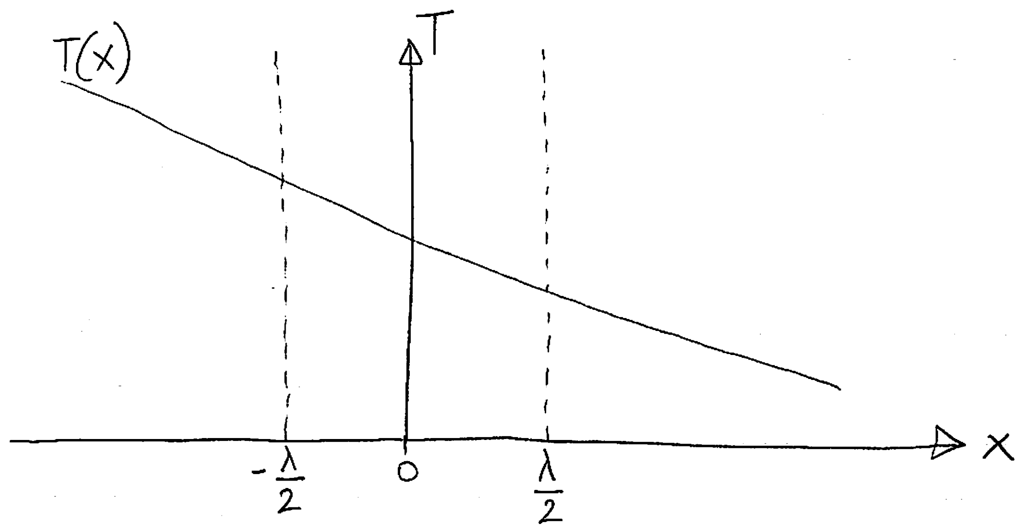
ME 420 - Intermediate Heat Transfer

Mechanisms of Heat Transfer
- Conduction = Convection $\Downarrow$ Convection with bulk fluid motion.
- Radiation

$$q_x'' = -k\frac{\partial T}{\partial x}, \quad \text{where } k = \text{thermal conductivity } [W/m\cdot K]$$

When we have gradients, we have diffusional transport



$$q_{x+}'' = n m c_p \bar{c}\, T\left(x - \frac{\lambda}{2}\right), \quad \text{where}$$

n = # of particles/m^3
m = mass per particle
c_p = specific heat capacity
$\bar{c}$ = average speed of a parti

$c_p \Rightarrow \left[\frac{J}{kg\cdot K}\right]$, $n \Rightarrow \frac{\#}{m^3}$, $m = \frac{kg}{\#}$

We see that: $n\cdot m = \rho$

$$q_{x+}'' = +\rho c_p \bar{c}\, T\left(x - \frac{\lambda}{2}\right)$$

$$q_{x-}'' = +\rho c_p \bar{c}\, T\left(x + \frac{\lambda}{2}\right)$$

$$q_x'' = q_{x+}'' - q_{x-}'' = \rho c_p \bar{c}\left[T\left(x - \frac{\lambda}{2}\right) - T\left(x + \frac{\lambda}{2}\right)\right]$$

$$= -\rho c_p \bar{c}\left[T\left(x + \frac{\lambda}{2}\right) - T\left(x - \frac{\lambda}{2}\right)\right]$$

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Now if we multiply our expression by $\frac{\lambda}{\lambda}$

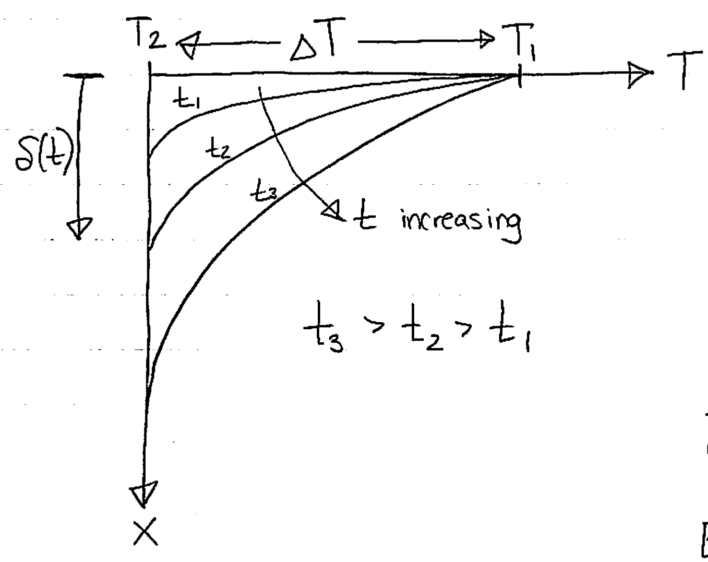
$$q'' = -\rho c_p \bar{c} \lambda \underbrace{\left[\frac{T(x+\frac{\lambda}{2}) - T(x-\frac{\lambda}{2})}{\lambda} \right]}_{\frac{\partial T}{\partial x}}$$

$$\boxed{q'' = -\rho c_p \bar{c} \lambda \cdot \frac{\partial T}{\partial x}} \Rightarrow \text{Gradient dependent}$$

We can also see that: $\boxed{k = \rho c_p \bar{c} \lambda} \Rightarrow$ Thermal conductivity

Remember, this derivation is valid for gases, however the kinetic approach is also often used analogously in solid state physics to study phonons.

Now lets look more closely at time for heat transfer



$$t_3 > t_2 > t_1$$

$\delta(t)$ = thermal penetration depth
ΔT = temperature difference
t = time

on the order of

$$E \sim \rho \delta c_p \Delta T \quad \left[\frac{J}{m^2}\right]$$

$$\dot{E} \sim \rho \delta c_p \frac{\partial T}{\partial t}$$

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$$\dot{E} = \dot{Q}_{in} = -k\left(\frac{\partial T}{\partial x}\right)_0 \sim k\,\frac{\Delta T}{\delta}$$

Equating our two expressions:

$$\rho c_p \delta \underbrace{\frac{\partial T}{\partial t}} \sim k\,\frac{\Delta T}{\delta}$$

$$\frac{\partial T}{\partial t} = \frac{\partial \delta}{\partial t}\cdot\frac{\partial T}{\partial \delta}$$

$$\rho c_p \delta \frac{\partial \delta}{\partial t}\cdot\frac{\partial T}{\partial \delta} \sim k\,\frac{\partial T}{\partial \delta}$$

$$\rho c_p \delta \frac{\partial \delta}{\partial t} \sim k$$

$$\rho c_p \frac{\delta}{t} \sim k$$

$$\delta^2 \sim \frac{k}{\rho c_p}\cdot t$$

$$\boxed{\delta \sim \sqrt{\frac{k}{\rho c_p}\cdot t} \sim \sqrt{\alpha t}} \Rightarrow \text{Diffusion distance} \sim \sqrt{t}$$

$$\boxed{\alpha = \frac{k}{\rho c_p}} \Rightarrow \text{Thermal Diffusivity}$$

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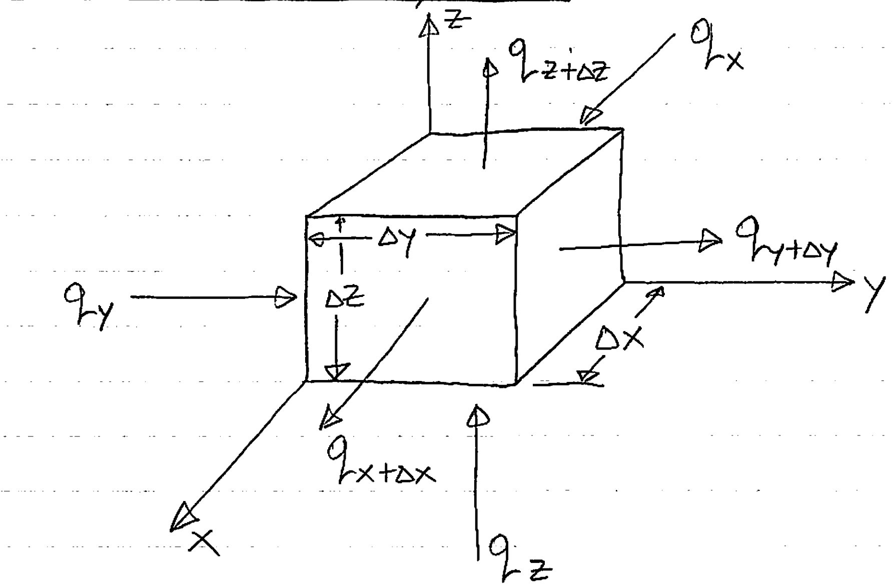
Heat Diffusion Equation



Note, q_x, q_y, q_z etc... are heat fluxes $\left[\frac{W}{m^2}\right] \Rightarrow q''$

From 1'st law of thermodynamics ($E_{in} = E_{OUT}$)

$$\dot{Q}_{in} - \dot{Q}_{OUT} + \dot{Q}_{gen} = \dot{Q}_{ST}$$

Entering & leaving through the surfaces are

$$\dot{Q}_{in} = q_x A_x + q_y A_y + q_z A_z$$

$$\dot{Q}_{OUT} = q_{x+\Delta x} A_{x+\Delta x} + q_{y+\Delta y} A_{y+\Delta y} + q_{z+\Delta z} A_{z+\Delta z}$$

For the cartesian coordinate system

$$A_x = A_{x+\Delta x} = \Delta y \Delta z$$
$$A_y = A_{y+\Delta y} = \Delta x \Delta z$$
$$A_z = A_{z+\Delta z} = \Delta x \Delta y$$

Within the control volume, we have energy generation $\dot{Q}'''$

$$\boxed{\dot{Q}_{gen} = \dot{Q}''' \forall = \dot{Q}''' \Delta x \Delta y \Delta z}$$

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Within the control volume, the change in stored energy is

$$\dot{Q}_{ST} = \frac{\partial U}{\partial t} = \frac{\partial (Mu)}{\partial t} = \frac{\partial (\rho \forall u)}{\partial t}$$

Assuming a constant density ρ

$$\dot{Q}_{ST} = \rho \frac{\partial u}{\partial t} \Delta x \Delta y \Delta z$$

Adopting an equation of state (simple incompressible substance with constant specific heat)

Aside:

$$U = M c_p (T - T_{ref})$$
$$u = c_p (T - T_{ref})$$

$$\boxed{\dot{Q}_{ST} = \rho c_p \frac{\partial T}{\partial t} \Delta x \Delta y \Delta z}$$

Now we plug everything together

$$\dot{Q}_{in} - \dot{Q}_{OUT} = \Delta y \Delta z (q_x - q_{x+\Delta x}) + \Delta x \Delta z (q_y - q_{y+\Delta y}) + \Delta x \Delta y (q_z - q_{z+\Delta z})$$

Using Taylor series expansion

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 + \cdots + \frac{1}{n!} \frac{\partial^n q_x}{\partial x^n} \Delta x^n$$

So,

$$q_x - q_{x+\Delta x} = -\frac{\partial q_x}{\partial x} \Delta x - \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 - \cdots - \frac{1}{n!} \frac{\partial^n q_x}{\partial x^n} \Delta x^n$$

As $\Delta x \to 0$, Δx^2 really $\to 0$, so higher order terms can be dropped. Maybe not very rigorous but will reconsile later

$$q_x - q_{x+\Delta x} = -\frac{\partial q_x}{\partial x} \Delta x$$

$$q_y - q_{y+\Delta y} = -\frac{\partial q_y}{\partial y} \Delta y$$

etc...

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Back substituting: Note, could have kept the H.O.T. terms here.

$$\dot{Q}_{in} - \dot{Q}_{OUT} = -\Delta y \Delta z\left(\frac{\partial q_x}{\partial x}\Delta x\right) - \Delta x \Delta z\left(\frac{\partial q_y}{\partial y}\Delta y\right) - \Delta x \Delta y\left(\frac{\partial q_z}{\partial z}\Delta z\right)$$

Rearranging:

$$\boxed{\dot{Q}_{in} - \dot{Q}_{OUT} = -\Delta x \Delta y \Delta z\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)}$$

$$\dot{Q}_{in} - \dot{Q}_{OUT} + \dot{Q}_{gen} = \dot{Q}_{ST}$$

$$-\cancel{\Delta x \Delta y \Delta z}\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + \dot{Q}'''\cancel{\Delta x \Delta y \Delta z} = \rho c_p \frac{\partial T}{\partial t}\cancel{\Delta x \Delta y \Delta z}$$

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}$$ (Let $\Delta x, \Delta y, \Delta z \to 0$, H.O.T. terms drop out)

But we've already proven before that $q_x = -k\frac{\partial T}{\partial x}$, etc...

Substituting the constitutive equations into our first law

$$\boxed{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}}$$

Second order PDE is the conservation of thermal energy for an isotropic, incompressible substance, with density and specific heat independent of time.

A well-posed model requires two boundary conditions in each coordinate (x, y, and z), and an initial condition.

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If thermal conductivity does not depend on location

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

or

$$\boxed{\nabla^2 T + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

So now we have a new tool.
For constant properties, and 1-D conduction

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

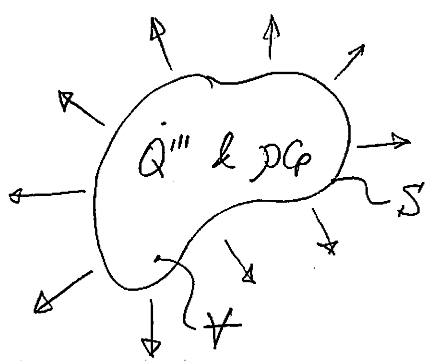
For 1-D conduction, and no generation

$$\boxed{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}}$$

For steady state (1-D, and no generation)

$$\boxed{\frac{\partial^2 T}{\partial x^2} = 0}$$

Note, there was an easier approach... but more abstract



$$\dot{Q}_{in} - \dot{Q}_{OUT} + \dot{Q}_{gen} = \dot{Q}_{ST}$$

$$\dot{Q}_{in} - \dot{Q}_{OUT} = -\oiint_S \vec{q} \cdot \hat{n}\, dS$$

$$\dot{Q}_{gen} = \iiint_{\forall} \dot{Q}'''\, d\forall$$

$$\dot{Q}_{ST} = \iiint_{\forall} \rho c \frac{\partial T}{\partial t}\, d\forall$$

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$$-\oint\!\!\!\oint_S \vec{q}\cdot\hat{n}\,dS + \iiint_{\forall} \dot{Q}'''\,d\forall = \iiint_{\forall} \rho c \frac{\partial T}{\partial t}\,d\forall$$

Applying Green's Theorem (Divergence Theorem)

$$\oint\!\!\!\oint_S \vec{V}\cdot d\vec{s} = \oint\!\!\!\oint_S \vec{V}\cdot\hat{n}\,dS = \iiint_{\forall} \nabla\cdot\vec{V}\,d\forall$$

So we obtain:

$$-\iiint_{\forall} \nabla\cdot\vec{q}\,d\forall + \iiint_{\forall} \dot{Q}'''\,d\forall = \iiint_{\forall} \rho c_p \frac{\partial T}{\partial t}\,d\forall$$

Rearanging:

$$-\iiint_{\forall} \left[-(\nabla\cdot\vec{q}) + \dot{Q}''' - \rho c_p \frac{\partial T}{\partial t}\right] d\forall = 0$$

Thus:

$$-(\nabla\cdot\vec{q}) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}$$

Using Fourier's Law in vector form $\vec{q} = -k\nabla T$

$$(\nabla\cdot k\nabla T) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}$$

Note, under the same assumptions used earlier, we get the following and did not assume a coordinate system

$$\boxed{\nabla^2 T + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}} \quad , \text{ Note } \boxed{(\nabla\cdot\nabla T) = \nabla^2 T}$$

↳ Divergence of a gradient is the Laplacian

For radial & spherical co-ordinates, we can use a more general formulation:

$d\forall = ds_1\,ds_2\,ds_3$, s_1, s_2, s_3 are coordinates in consideration

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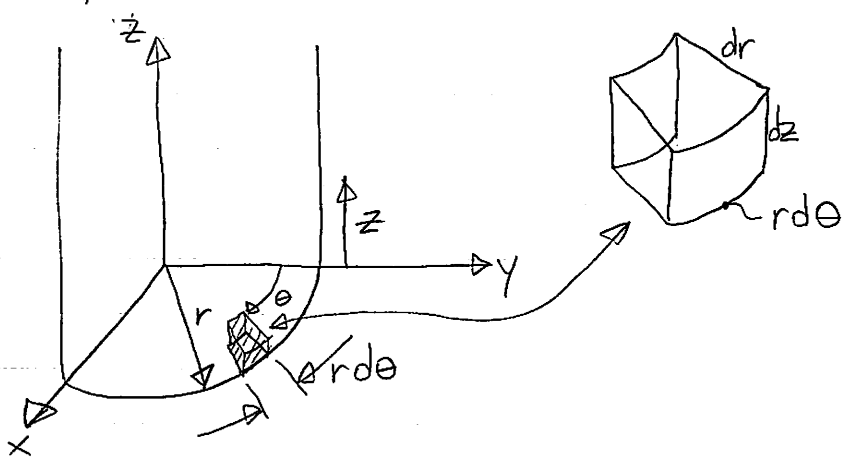
Recasting our heat equation

$$\boxed{\frac{1}{ds_1\,ds_2\,ds_3}\left\{\frac{\partial}{\partial s_1}\left(k\, ds_2 ds_3 \frac{\partial T}{\partial s_1}\right)ds_1 + \frac{\partial}{\partial s_2}\left(k\, ds_1 ds_3 \frac{\partial T}{\partial s_2}\right)ds_2 + \frac{\partial}{\partial s_3}\left(k\, ds_1 ds_2 \frac{\partial T}{\partial s_3}\right)ds_3\right\} + \dot{Q}''' = \frac{\partial}{\partial t}(\rho c_p T)}$$

Specific Cases:

① Cartesian, $ds_1 = dx$, $ds_2 = dy$, $ds_3 = dz$ } We'll get what we already solved

② Cylindrical



$ds_1 = dr$
$ds_2 = r d\theta$
$ds_3 = dz$

So our heat equation becomes

$$\frac{1}{r\,dr\,d\theta\,dz}\left\{\frac{\partial}{\partial r}\left(k r\, d\theta\, dz \frac{\partial T}{\partial r}\right)dr + \frac{1}{r}\frac{\partial}{\partial \theta}\left(k\, dr\, dz \frac{1}{r}\frac{\partial T}{\partial \theta}\right)r\,d\theta + \frac{\partial}{\partial z}\left(k r\, dr\, d\theta \frac{\partial T}{\partial z}\right)dz\right\} + \dot{Q}''' = \frac{\partial}{\partial t}(\rho c_p T)$$

Ask them to do spherical by themselves

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(k r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}}$$

END of LECTURE 1

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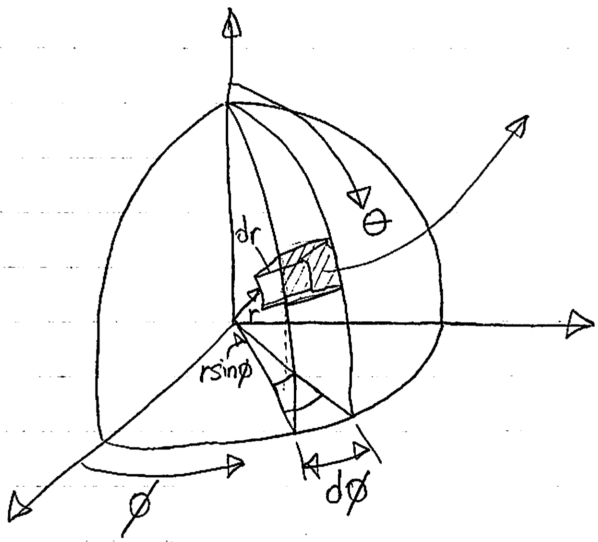
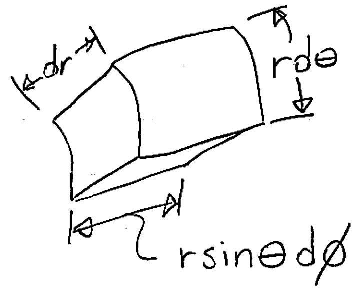
<u>Bonus Solution</u> - if we have time (Did not cover this in class)

③ Spherical





$$dS_1 = dr$$
$$dS_2 = r\,d\theta$$
$$dS_3 = r\sin\theta\,d\phi$$

Back substituting into our general formulation

$$\frac{1}{r^2\cancel{dr}\cancel{d\theta}\sin\theta\cancel{d\phi}}\Bigg\{\frac{\partial}{\partial r}\left(k r\cancel{d\theta} r\sin\theta\cancel{d\phi}\frac{\partial T}{\partial r}\right)\cancel{dr} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{k\cancel{dr}\cdot r\cdot\sin\theta\cancel{d\phi}}{r}\cdot\frac{\partial T}{\partial\theta}\right)r\cancel{d\theta} + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\frac{k\cancel{dr}\cdot r\cdot\cancel{d\theta}}{r\sin\theta}\cdot\frac{\partial T}{\partial\phi}\right)r\sin\theta\cancel{d\phi}\Bigg\} + \dot{Q}''' = \frac{\partial}{\partial t}(\rho c_p T)$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(k r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \dot{Q}''' = \rho c_p\frac{\partial T}{\partial t}}$$