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**Example** | Calculate the boundary layer thickness for a jetliner.

$V_\infty = 400$ miles/hour (177 m/s) (Plane speed)
$\nu_{air} = 1.5 \times 10^{-5} \, m^2/s$
$L = 5m$ (Length of wing in the fuselage direction)

$$Re_L \approx \frac{V_\infty L}{\nu} = \frac{(177 m/s)(5m)}{1.5 \times 10^{-5} m^2/s} = 5.9 \times 10^7$$

$$\delta \approx \frac{L}{\sqrt{Re_L}} = \frac{5m}{\sqrt{5.9 \times 10^7}}$$

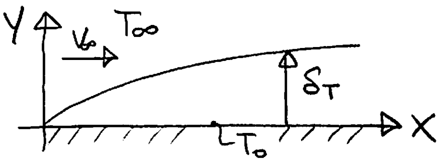
$\boxed{\delta = 0.65 mm}$ ⇒ Less than 1 mm thick! Very hard to see.

END OF LECTURE 9

## Heat Transfer

Note we are looking for heat transfer on the external surface

$$h = \frac{q''_{y=0}}{\Delta T} = \frac{q''_{y=0}}{T_0 - T_\infty} = ?$$




For a flat plate, laminar flow: $k, \rho, \mu, C_p$ = constant
Writing out our energy equation for a fluid element, which is the heat equation but extended to allow for fluid motion

$$\underbrace{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}}_{\text{Convection of heat}} = \underbrace{\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{\text{Conduction of heat}}$$

⇒ Usually we have $\nabla^2 T = 0$ or $\nabla^2 T = \underbrace{\frac{1}{\alpha}\frac{\partial T}{\partial t}}_{0 \text{ since steady.}}$

For a boundary layer at $T_0 = T_{wall}$ = constant

$$\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$$

Boundary conditions: $T(y=0) = T_0$
$T(y \to \infty) = T_\infty$

Non-dimensionalizing: Let $\Theta = \frac{T - T_0}{T_\infty - T_0}$

$$u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial y^2}$$

⇒ Note the similarity to the boundary layer eqns.

Previously we had:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \Rightarrow \text{Hydrodynamic boundary layer}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2} \Rightarrow \text{Thermal boundary layer}$$

B.C.'s :

$$\theta(y=0) = 0 \qquad\qquad \bar{u}(y=0) = 0$$

$$\theta(y\to\infty) = 1 \qquad \text{vs.} \qquad \bar{u}(y\to\infty) = 1$$

$$\left.\frac{\partial \theta}{\partial y}\right|_{y\to\infty} = 0 \qquad\qquad \left.\frac{\partial \bar{u}}{\partial y}\right|_{y\to\infty} = 0$$

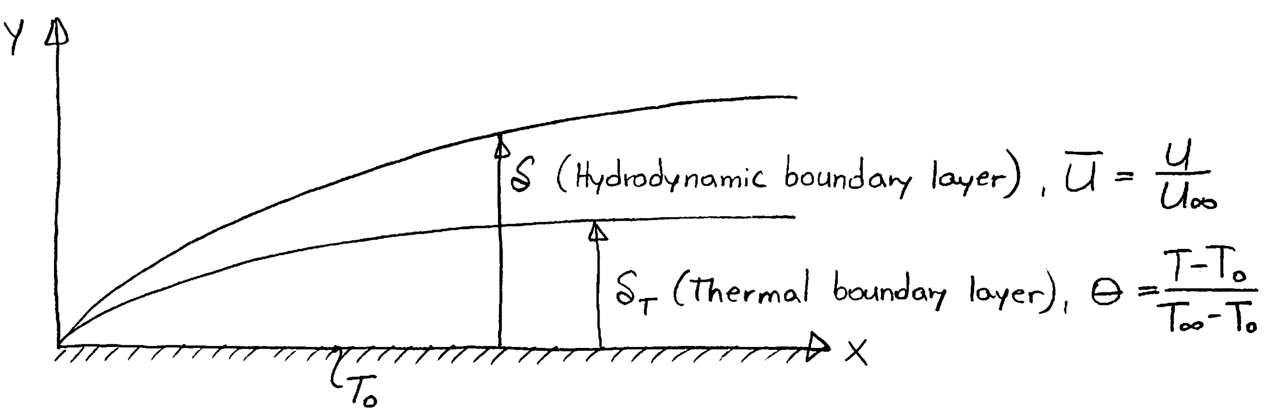
So our solution is already solved. Just use the b.l. soln's.

Here, we can define a usefull quantity called the Prandtl number

$$\boxed{Pr = \frac{\nu}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}}$$

If $Pr = 1$, $\theta = \bar{u}$, $\delta = \delta_T$ $\Rightarrow$ Hydrodynamic & thermal boundary layers are same.

But usually $\nu \neq \alpha$. If $\nu > \alpha$, the hydrodynamic boundary layer is thicker since you transfer momentum more efficiently than thermal energy.




Now lets solve the thermal boundary layer equation.

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If $Pr \neq 1$; $\theta(\eta)$, $\eta = y\sqrt{\frac{V_\infty}{x\nu}}$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} \qquad \partial\eta = \partial y\underbrace{\sqrt{\frac{V_\infty}{x\nu}}}_{\frac{\partial\eta}{\partial y}} = \partial y\left(\frac{\partial\eta}{\partial y}\right)$$

$$\frac{\partial\theta}{\partial y} = \frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial^2\theta}{\partial\eta^2}\cdot\left(\frac{\partial\eta}{\partial y}\right)^2$$

Now substituting back into our energy equation PDE,

$$\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2}F\,Pr\frac{\partial\theta}{\partial\eta} = 0 \quad , \text{ note } F = \int_0^\eta \phi\, d\eta = F(\eta)$$

$$\bar{u} = \phi(\eta) = \frac{u}{V_\infty} = F'$$

To solve we usually integrate but here we can use a trick.
Let:

$$Pr^{2/3}\cdot F(\eta) = F(\eta^*) \quad ; \quad \eta^* = \eta\, Pr^{1/3}$$
$$\partial\eta^* = Pr^{1/3}\partial\eta$$

Now our PDE becomes:

$$\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2}F(\eta^*)Pr^{1/3}\frac{\partial\theta}{\partial\eta} = 0$$

$$\frac{\partial^2\theta}{\partial\eta^2} + \frac{1}{2}F(\eta^*)Pr^{2/3}\frac{\partial\theta}{\underbrace{\partial\eta\, Pr^{1/3}}_{\partial\eta^*}} = 0$$

Multiplying through by $Pr^{-2/3}$

$$\underbrace{\frac{1}{Pr^{2/3}}\left(\frac{\partial^2\theta}{\partial\eta^2}\right.}_{\partial\eta^{*2}} + \frac{1}{2}F(\eta^*)\cancel{Pr^{2/3}}\frac{\partial\theta}{\partial\eta^*} \left. = 0\right)$$

Now our PDE becomes identical to before

$$\frac{\partial^2\theta}{\partial\eta^{*2}} + \frac{1}{2}F(\eta^*)\frac{\partial\theta}{\partial\eta^*} = 0 \quad \Rightarrow \quad \theta(\eta^*) = \bar{u}(\eta)$$

B.C.'s: $\theta(\eta^* = 0) = 0$ , $\theta = \frac{T - T_0}{T_\infty - T_0}$
$\theta(\eta^* \to \infty) = 1$

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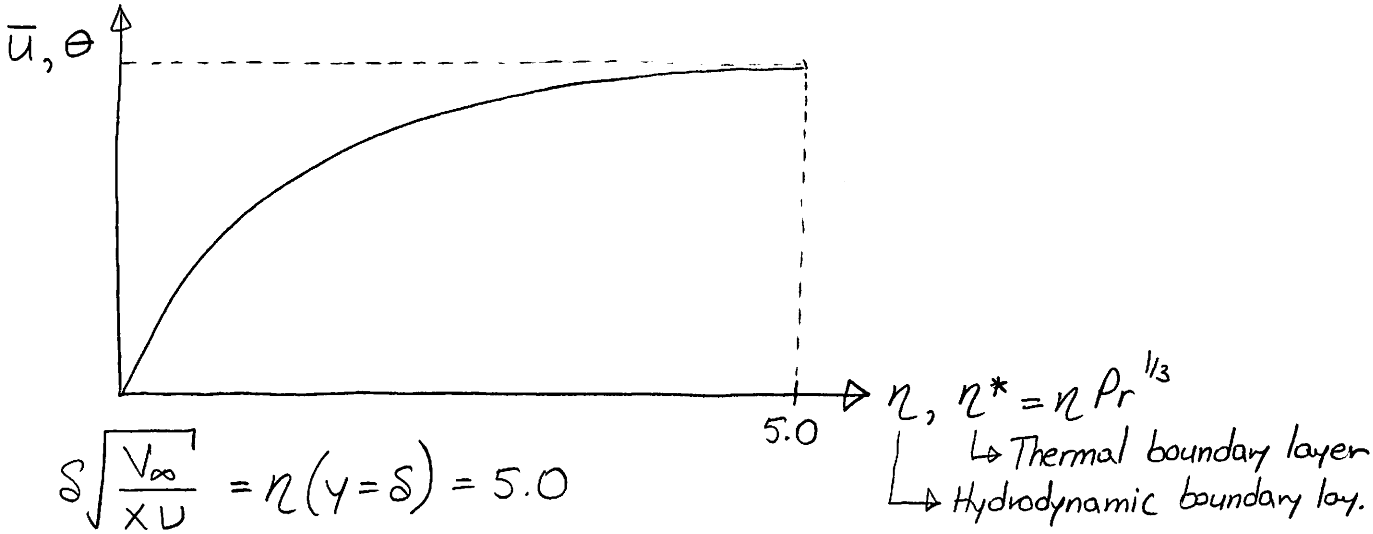


$$\delta\sqrt{\frac{V_\infty}{x\nu}} = \eta(y=\delta) = 5.0$$

$$Pr^{1/3}\,\delta_T\sqrt{\frac{V_\infty}{x\nu}} = \eta^*(y=\delta_T) = 5.0$$

Taking the ratio of our two boundary layer thicknesses:

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}}$$ $\Rightarrow$ Makes sense, the only difference is $\nu$ & $\alpha$
$Pr = \frac{\nu}{\alpha}$

Now we can solve for heat transfer

$$q''_{x=0} = -k\left.\frac{\partial T}{\partial y}\right|_{y=0} \Rightarrow \text{we know } \theta = \frac{T-T_0}{T_\infty - T_0},\ \partial\theta = \frac{\partial T}{T_\infty - T_0}$$

$$\eta^* = \eta Pr^{1/3} = y\left(\frac{V_\infty}{x\nu}\right)^{1/2} Pr^{1/3}$$

$$\partial\eta^* = \partial y\left(\frac{V_\infty}{x\nu}\right)^{1/2} Pr^{1/3}$$

$$q''_{x=0} = -k(T_\infty - T_0)\left(\frac{V_\infty}{x\nu}\right)^{1/2} Pr^{1/3}\left.\frac{\partial\theta}{\partial\eta^*}\right|_{\eta^*=0}$$

$$= \frac{k(T_0 - T_\infty)}{x}\underbrace{\left(\frac{V_\infty x}{\nu}\right)^{1/2}}_{Re_x^{1/2}\ \Rightarrow\ \text{Reynolds number}} Pr^{1/3}\underbrace{F''(0)}_{a_2 = 0.332}$$

$$q''_{x=0} = \frac{k\Delta T}{x} Re_x^{1/2} Pr^{1/3} a_2$$

$$Nu_x = \frac{hx}{k} = \underbrace{\frac{q''}{\Delta T}}_{h}\cdot\frac{x}{k} = a_2 Re^{1/2} Pr^{1/3} \Rightarrow \boxed{Nu_x = 0.332\,Re_x^{1/2} Pr^{1/3}}$$

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Looking at shear stress, we showed before that:

$$C_{f,x} = \frac{\tau}{\frac{1}{2}\rho V_\infty^2} = \frac{2a_2}{\sqrt{Re_x}} \quad \text{, where } C_{f,x} = \text{skin friction coefficient}$$

$$\underbrace{\left(\frac{Nu_x}{Re \cdot Pr}\right)} = \frac{1}{2}\overbrace{\left(\frac{2a_2}{\sqrt{Re_x}}\right)}^{C_{f,x}}\frac{1}{Pr^{2/3}} = \frac{1}{2}C_{f,x}\frac{1}{Pr^{2/3}}$$

$$\boxed{St = \frac{h}{\rho C_p V_\infty} \quad \Rightarrow \text{Stanton Number}}$$

Aside:

$$St = \frac{\frac{h\cancel{L}}{\cancel{k}}}{\left(\frac{\rho V_\infty \cancel{L}}{\cancel{\mu}}\right)(\cancel{\nu})\left(\frac{c_p \rho}{\cancel{k}}\right)}$$

$$St = \frac{h}{V_\infty \rho c_p}$$

$$\boxed{St = \frac{\text{Heat transfered to a fluid}}{\text{Thermal capacity of the fluid}}} \Rightarrow \text{Characterizes heat transfer in forced convection flows.}$$

We can write a general analogy that:

$$\boxed{St \cdot Pr^{2/3} = \frac{C_{f,x}}{2}} \Rightarrow \text{Colburn Analogy}$$

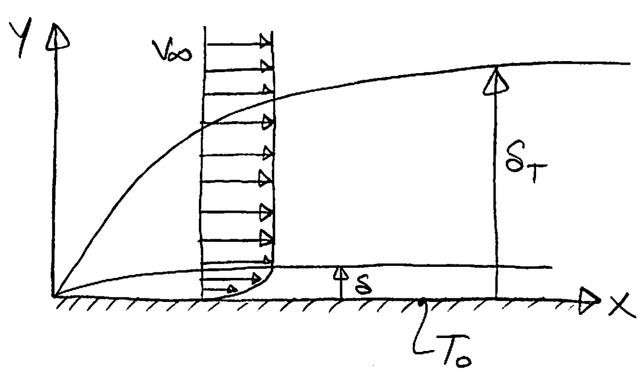
Aside: Another analogy Ratio of $C_{f,x}$, and $Nu_x$

Colburn j-factor $\Rightarrow$ $\boxed{\frac{C_{f,x}}{2} = \frac{h_x}{\rho C_p V_\infty} Pr^{2/3} = j_H}$

This is a very powerfull analogy relating heat, momentum, and mass transfer. Can relate heat transfer and temperature to shear & velocity, i.e. we can solve for shear & $V_\infty$ by measuring $q''$ & $T_o, T_\infty$.

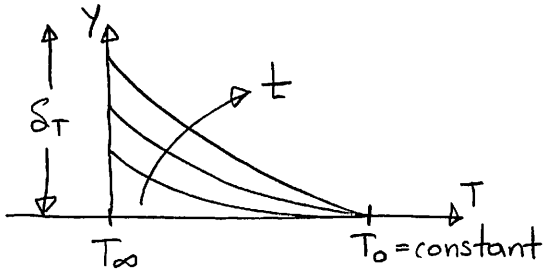
__Some limits__ $(Pr \ll 1)$

If a fluid has a very low Prandtl number $\Rightarrow Pr = \frac{\nu}{\alpha} \ll 1$




$\Rightarrow$ Looks a lot like a transient conduction problem.

We can rethink this as:




$$q''_{x=0} = \frac{k\Delta T}{\sqrt{\pi \alpha t}}$$ ⇒ Solved before for semi-infinite conduction. Check Lecture 8, Page 66 of my notes.

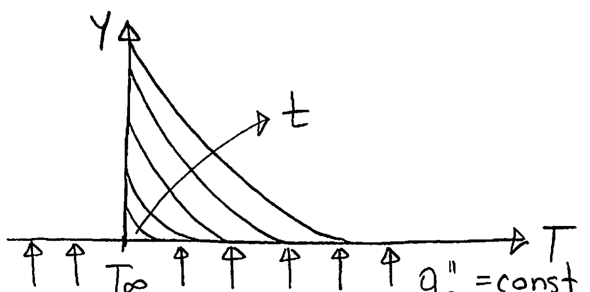
$$Nu_x = \left(\frac{q''}{\Delta T} \cdot \frac{x}{k}\right) = \frac{x}{\sqrt{\pi \alpha t}}$$

we know $t = \frac{x}{V_\infty}$

$$Nu_x = \frac{1}{\sqrt{\pi}} \cdot \left(\frac{V_\infty x}{\alpha}\right)^{1/2} \Rightarrow \boxed{Nu_x = \frac{1}{\sqrt{\pi}} \cdot Re_x^{1/2} Pr^{1/2}}$$

↳ For constant wall temperature, $T_o$, $Pr \ll 1$

Now if we have a constant heat flux: $(Pr \ll 1)$
$q''_o$ = constant




We know from our previous solution that: (Lecture 8, Page 67)

$$T - T_\infty = \frac{q''}{k}\left(\frac{4\alpha t}{\pi}\right)^{1/2} e^{-\frac{x^2}{4\alpha t}} - x \, \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$ ⇒ Evaluate at x=0

$$\underbrace{T_o - T_\infty}_{\Delta T} = \frac{q''_o}{k}\left(\frac{4\alpha t}{\pi}\right)^{1/2}$$

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$$Nu_x = \left(\frac{q_o''}{\Delta T}\cdot\frac{x}{k}\right) = \frac{4}{\sqrt{\pi}}\cdot\frac{x}{\sqrt{\alpha t}} = \frac{4}{\sqrt{\pi}}\cdot\left(\frac{V_\infty x}{\alpha}\right)^{1/2}, \quad t = \frac{x}{V_\infty}$$

$$\boxed{Nu_x = \frac{4}{\sqrt{\pi}}\cdot Re^{1/2} Pr^{1/2}} \Rightarrow \text{Constant heat flux, } q_o'',\ Pr \ll 1$$

Note that for these solutions, we've already defined $Pe = Re\,Pr$

$$Pe_x = \frac{V_\infty x}{\alpha} = \frac{\rho C_p V_\infty \Delta T}{k \Delta T} = \frac{\text{heat storage rate in the b.l.}}{\text{heat conductance through the b.l.}}$$

$$\boxed{Nu_x \propto Pe^{1/2}}$$

## Average Heat Transfer Coefficient $(\bar{h})$

$$\underbrace{\bar{h} = \frac{\overline{q''}}{\Delta T}}_{\text{Const. wall T.}} \quad \text{or} \quad \underbrace{\bar{h} = \frac{q''}{\overline{\Delta T}}}_{\text{Const. heat flux problems}}$$

## Uniform Wall Temperature:

$$\bar{h} = \frac{\overline{q''}}{\Delta T} = \frac{1}{\Delta T}\left[\frac{1}{L}\cdot\int_0^L q''\,dx\right] = \frac{1}{L}\int_0^L h(x)\,dx$$

## Uniform Heat Flux

$$\bar{h} = \frac{q''}{\overline{\Delta T}} = \frac{q''}{\frac{1}{L}\int_0^L \Delta T(x)\,dx}$$

The Nusselt number based on $\bar{h}$ and $L$ is $\overline{Nu_L}$

This is not the average of $Nu_x$.

For a flat plate case: $(x_o = x(0) = 0)$

$$\bar{h} = \frac{1}{L}\int_0^L \underbrace{h(x)}_{\frac{k}{x}Nu_x}\,dx = \frac{0.332\,k\,Pr^{1/3}}{L}\sqrt{\frac{V_\infty}{\nu}}\cdot\int_0^L \frac{\sqrt{x}}{x}\,dx = 0.664\,Re_L^{1/2}\,Pr^{1/3}\left(\frac{k}{L}\right)$$

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So now we see: $\bar{h} = 2h(x=L)$ in laminar flow

$$\boxed{\overline{Nu_L} = \frac{\bar{h}L}{k} = 0.664\, Re_L^{1/2} Pr^{1/3}}$$

For $Pr \ll 1$:

$$\boxed{\overline{Nu_L} = 1.13\, Pe_L^{1/2}}\ , \quad \boxed{Pe_L = Re_L\, Pr}$$

Some Observations and Notes

Previous results are valid under the following conditions:

1) $Re_x$ or $Re_L < 5.0 \times 10^5$ (Laminar flow)

2) $Ma = \frac{V_\infty}{\text{sound speed}} < 0.3$ (Incompressible flow)

3) $Ec = \text{Eckert number} = \frac{V_\infty^2}{c_p (T_0 - T_\infty)} \ll 1$ (Viscous dissipation heating is negligible)

$$= \frac{\text{Kinetic Energy}}{\text{Enthalpy}}$$

The higher the kinetic energy, the larger the effect of viscous dissipation.

4) We have always assumed that properties are constant. Need to evaluate properties at the average temperature of the boundary layer, or the film temperature:

$$T_f = \frac{T_0 + T_\infty}{2}$$

5) $h$ or $\bar{h} \propto \frac{1}{\sqrt{x}}$ or $\frac{1}{\sqrt{L}}$, $Nu_x \propto \sqrt{x}$

Thus $h \to \infty$ and $Nu_x \to 0$ at $x \to 0$. Of course, $h \to \infty$ will not occur at $x \to 0$ since the b.l. model breaks down at $x=0$

END OF LECTURE 10.