Nenad Miljkovic

D'alembert's Paradox (Back to it)

Note in our derivation, we mentioned D'alembert's paradox. The drag force goes to zero on a body moving through a fluid at high Reynolds number.

What is the source of the problem & how was it solved? Well, let's take a look at what we did.

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{u} = \frac{u}{V_\infty}, \quad \bar{v} = \frac{v}{V_\infty}, \quad \bar{P} = \frac{P}{\rho V_\infty^2}$$

X-momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x}\cdot\frac{1}{\rho} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

If we non-dimensionalize like before

$$\bar{u}V_\infty \frac{\partial \bar{u}\cdot V_\infty}{\partial \bar{x}\cdot L} + \bar{v}V_\infty \frac{\partial \bar{u}\cdot V_\infty}{\partial \bar{y}\, L} = -\frac{\cancel{\rho} V_\infty^2}{\cancel{\rho} L}\frac{\partial \bar{P}}{\partial \bar{x}} + \nu\frac{V_\infty}{L}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}\cdot\frac{V_\infty^2}{L} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\frac{V_\infty^2}{L} = -\frac{V_\infty^2}{L}\frac{\partial \bar{P}}{\partial \bar{x}} + \nu\frac{V_\infty}{L^2}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

Divide through by $\frac{V_\infty^2}{L}$ on both sides

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \underbrace{\frac{\nu \cancel{V_\infty} \cancel{L}}{V_\infty^{\cancel{2}} L^{\cancel{2}}}}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

$$\frac{\nu}{V_\infty L} = \frac{\mu}{\rho V_\infty L} = \frac{1}{Re_L}$$

$$\boxed{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \underbrace{\frac{1}{Re_L}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]}}$$

Herein lies the problem. As $Re_L \to \infty$, our viscous terms completely drop out, so viscosity is negligible.

Nenad Miljkovic

If the viscous terms are negligible, the potential flow solution is valid and appropriate, which says there is no drag on the body. We all know this is wrong since drag increases with Re_L.

So where did we go wrong?

Turns out hydrodynamicists back then were not aware of the boundary layer, and scaled the N-S equation incorrectly just like we did.

What we should have used is: $V_\delta = \frac{\delta}{L} V_\infty$, δ is critical y-dim.

$$\bar{u} = \frac{u}{V_\infty}, \quad \underbrace{\bar{V} = \frac{V}{V_\delta} = \left(\frac{L}{\delta}\right)\frac{V}{V_\infty}}_{\text{This is the correct scaling}}, \quad \bar{X} = \frac{X}{L}, \quad \underbrace{\bar{Y} = \frac{Y}{\delta}}_{\text{This is the correct scaling.}}, \quad \bar{P} = \frac{P}{\rho V_\infty^2}$$

So let's try this again: $\partial u = V_\infty \partial \bar{u}$; $\partial v = \frac{\delta}{L} V_\infty \partial \bar{v}$, $\partial x = L \partial \bar{x}$, $\partial y = \delta \partial \bar{y}$ ...

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}\left(V_\infty^2 \frac{1}{L}\right) + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\left(V_\infty\left(\frac{\delta}{L}\right)V_\infty \cdot \frac{1}{\delta}\right) = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial \bar{x}} \cdot \frac{\rho V_\infty^2}{L}$$

$$+ \nu\left[\frac{V_\infty}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{V_\infty}{\delta^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

Dividing through by $\left(\frac{V_\infty^2}{L}\right)$ on both sides

$$\boxed{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{Re_L}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \left(\frac{L^2}{\delta^2}\right)\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]}$$

↳ This is the correct scaled equation.

Nenad Miljkovic

We can see that if we assume $\frac{\delta}{L} \ll 1, \Rightarrow \frac{L}{\delta} \gg 1$

As $Re_L \to \infty$, we can't say much about the last viscous term.

Now if we wanted an order of magnitude, we can use our correct scaled equation to say some things:

$$\bar{u} = \frac{u}{V_\infty} \sim \frac{V_\infty}{V_\infty} \sim O(1) \quad , \quad \bar{v} = \left(\frac{L}{\delta}\right)\frac{v}{V_\infty} \sim \left(\frac{L}{\delta}\right)\left(\frac{\delta}{L}\right)\frac{V_\infty}{V_\infty} \sim O(1)$$

$$\bar{x} = \frac{x}{L} \sim \frac{L}{L} \sim O(1) \quad , \quad \bar{y} = \frac{y}{\delta} \sim \frac{\delta}{\delta} \sim O(1)$$

$$\bar{p} = \frac{p}{\rho V_\infty^2} \sim \frac{\rho V_\infty^2}{\rho V_\infty^2} \sim O(1)$$

$$\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}}_{\sim O(1)} + \underbrace{\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}}_{\sim O(1)} = -\underbrace{\frac{\partial \bar{p}}{\partial \bar{x}}}_{\sim O(1)} + \frac{1}{Re_L}\left[\underbrace{\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}}_{\sim O(1)} + \left(\frac{L}{\delta}\right)^2 \underbrace{\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}_{\sim O(1)}\right]$$

For this equation to balance, which it needs to for the boundary layer approximation (where viscosity $\sim$ inertia)

Then: $$\frac{1}{Re_L}\left(\frac{L}{\delta}\right)^2 \sim O(1)$$

$$\frac{L^2}{\delta^2} = Re_L$$

$$\frac{\delta}{L} = \frac{1}{Re_L} \Rightarrow \boxed{\delta \sim \frac{L}{\sqrt{Re_L}}} \Rightarrow \text{Boundary layer thickness}$$

So we see how Prandtl resolved D'alembert's paradox by elucidating the boundary layer concept.

Nenad Miljkovic

So now we can write down the more generalized boundary layer equations (without the flat plate assumptions)
We know $\frac{\delta}{L} \ll 1$, this helps us simplify our N-S eqn.

$$\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}}_{\sim O(1)} + \underbrace{\bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}}_{\sim O(1)} = \underbrace{-\frac{\partial \bar{P}}{\partial \bar{x}}}_{\sim O(1)} + \underbrace{\underbrace{\frac{1}{Re_L}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}}_{\sim O\left(\frac{1}{Re_L}\right)}}_{Re_L \sim \frac{L^2}{\delta^2}} + \underbrace{\frac{L^2}{\delta^2}\frac{1}{Re_L}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}_{\sim O(1)}$$

So our x-viscosity term becomes: $\frac{\delta^2}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \ll 1$, since $\frac{\delta}{L} \ll 1$

Therefore, our equation simplifies to:

$$\boxed{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{L^2}{\delta^2}\frac{1}{Re_L}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}} \Rightarrow \text{x-momentum}$$

Now we can look at y-momentum:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y}\cdot\frac{1}{\rho} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

Non-dimensionalizing:

$$V_\infty^2 \bar{u}\left(\frac{\delta}{L}\right)\left(\frac{1}{L}\right)\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\left(\frac{\delta}{L}\right)^{\cancel{2}} V_\infty^2\left(\frac{1}{\cancel{\delta}}\right)\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\cancel{\rho} V_\infty^2}{\cancel{\rho}\delta}\frac{\partial \bar{P}}{\partial \bar{y}} + V_\infty \nu \frac{\delta}{L}\left[\frac{1}{L^2}\frac{\partial^2 \bar{v}}{\partial \bar{x}}\right.$$

$$\left(\frac{\delta}{L}\right)\frac{V_\infty^2}{L}\left[\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y}\right] = -\frac{V_\infty^2}{\delta}\frac{\partial \bar{P}}{\partial \bar{y}} + V_\infty \nu\left(\frac{\delta}{L}\right)\left[\frac{1}{L^2}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} \cdots + \frac{1}{\delta^2}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right]$$

We know that $\frac{\delta}{L} \ll 1$, so both the inertia & viscosity ~ 0

$$\cancel{-\frac{V_\infty^2}{\delta}}\frac{\partial \bar{P}}{\partial \bar{y}} = 0 \Rightarrow \frac{\partial \bar{P}}{\partial \bar{y}} = 0, \quad \boxed{\bar{P} \neq f(\bar{y}) = \text{constant}}$$

↳ y-momentum

And now for continuity: $\bar{u} = \dfrac{u}{V_\infty}$, $\bar{v} = \dfrac{v}{V_\delta} = \dfrac{L}{\delta}\dfrac{v}{V_\infty}$, $\bar{x} = \dfrac{x}{L}$

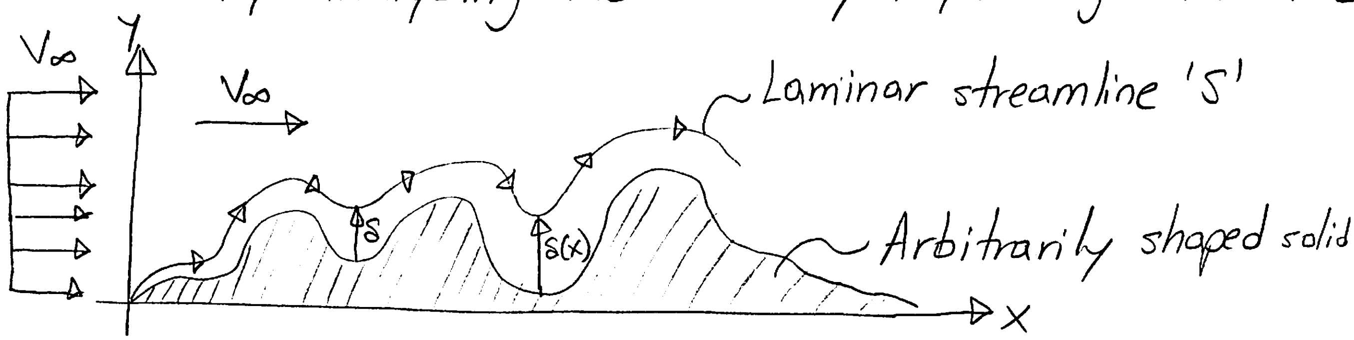
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \partial\bar{u} = \frac{\partial u}{V_\infty} , \quad \partial\bar{v} = \frac{\partial v}{V_\delta} = \partial v \frac{L}{\delta V_\infty} , \quad \partial\bar{x} = \frac{\partial x}{L}$$

Non-dimensionalizing

$$\frac{V_\infty \partial\bar{u}}{L\,\partial\bar{x}} + \left(\frac{\cancel{\delta}}{L}\right)\frac{V_\infty \partial\bar{v}}{\cancel{\delta}\,\partial\bar{y}} = 0$$

$$\cancel{\frac{V_\infty}{L}}\left(\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}}\right) = 0 \Rightarrow \boxed{\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}} = 0} \Rightarrow \text{Continuity}$$

Now finally analyzing the boundary layer edge streamline



At 'S' $\Rightarrow \dfrac{\partial u}{\partial y} = 0$, $u = V_\infty$, viscosity ~ 0 (potential flow)

So our N-S equation on the streamline is:

$$u\frac{\partial u}{\partial x} + v\cancelto{0}{\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \cancelto{0}{\frac{\partial^2 u}{\partial y^2}}\right]$$

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \Rightarrow \boxed{\bar{V}_\infty \frac{\partial \bar{V}_\infty}{\partial \bar{x}} = -\frac{\partial \bar{P}}{\partial \bar{x}}} \Rightarrow \text{Non-dimensional form. } \bar{V}_\infty = \left.\frac{u}{V_\infty}\right|_{y=\delta}$$

$$\boxed{V_\infty \frac{\partial V_\infty}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}} \Rightarrow \text{Solves for } \frac{\partial P}{\partial x} \text{ in x-momentum.}$$

So we have 4 generalized boundary layer equations for laminar flow over any solid body: (in dimensional form)

Nenad Miljkovic

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$P(y) = \text{constant} \quad (2)$$

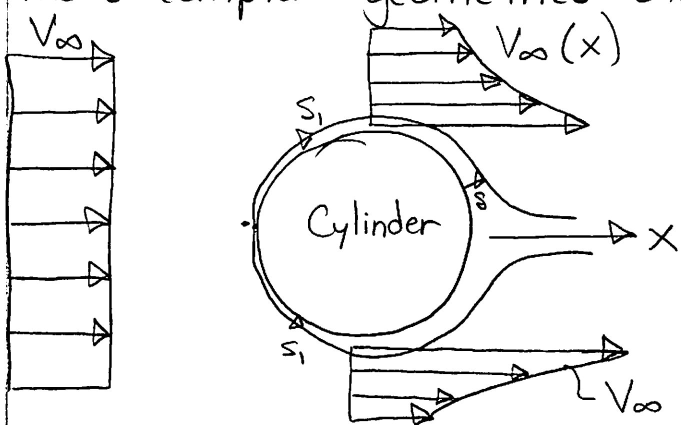
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (3)$$

$$V_\infty\frac{\partial V_\infty}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0 \quad (4)$$

Sometimes (1) is written with (4) as:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = V_\infty\frac{\partial V_\infty}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2} \quad (5)$$

You can see that the above equations are very cumbersome and difficult to solve. No analytical solutions exist for more complex geometries other than a flat plate. For ex:



This is a case where equation (4) has to be solved and we cannot assume $\frac{\partial V_\infty}{\partial x} = 0$.

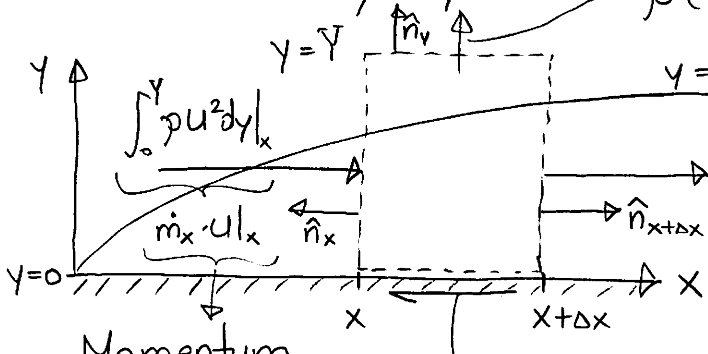
V_∞ changes with x-coordinate.

Thankfully, a dude by the name of Theodore von Kármán found a much simple way to solve these problems. He just integrated across the boundary layer to solve the conservation of momentum and energy. His technique is called the momentum integral technique.

Nenad Miljkovic

<u>Integral Technique</u> (Very usefull for arbitrary shapes where N-S not)
The integral form of the momentum equation can be derived across the boundary layer.

$\rho(vu)\big|_{y=Y}\cdot\Delta x = \underbrace{\rho v\big|_Y \Delta x}_{\dot m_y}\cdot V_\infty$



$\int_0^Y \rho u^2 dy\big|_{x+\Delta x}$ $\underbrace{\left(\dot m_{x+\Delta x}\cdot u\big|_{x+\Delta x}\right)}_{\text{momentum Leaving our CV}}$

Momentum Entering our CV

$\mu \frac{\partial u}{\partial y}\Big|_{y=0}\cdot\Delta x$ } Net force on our CV

Our momentum eqn. becomes: (From Control Volume analysis)

$$\underbrace{\int_0^Y \rho u^2 dy\Big|_{x+\Delta x} + \rho v u\Big|_Y \Delta x - \int_0^Y \rho u^2 dy\Big|_x}_{\text{net momentum outflow } (m\vec a)} = \underbrace{-\mu \frac{\partial u}{\partial y}\Big|_0 \Delta x}_{\Sigma F}$$

Divide through by $\rho\cdot\Delta x$ and let $\Delta x \to 0$ (Do a Taylor series expansion as well)

$$\frac{\partial}{\partial x}\int_0^Y u^2 dy + vu\Big|_Y = -\nu \frac{\partial u}{\partial y}\Big|_0$$

We know $u|_Y = V_\infty$, and from continuity: $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$v(Y) = v|_0 - \int_0^Y \frac{\partial u}{\partial x} dy = -\int_0^Y \frac{\partial u}{\partial x} dy \quad \Rightarrow \text{Substitute back}$$

$$\frac{d}{dx}\int_0^Y u^2 dy - \int_0^Y V_\infty \frac{\partial u}{\partial x} dy = -\nu \frac{\partial u}{\partial y}\Big|_0 \quad \Rightarrow \text{We can re-write this as:}$$

let $\delta = Y$ since as $y > \delta$, our integral $= 0$ so not impor.

$$\boxed{\frac{d}{dx}\int_0^\delta u(V_\infty - u)\, dy = \nu\left(\frac{\partial u}{\partial y}\right)_0} \Rightarrow \text{Momentum integral}$$

Nenad Miljkovic

If we non-dimensionalize our momentum integral

$$\bar{u} = \frac{u}{V_\infty} = \phi(\eta) \, , \quad \eta = \frac{y}{\delta}$$

$$\frac{d}{dx}\int_0^\delta \bar{u}(1-\bar{u})\,dy = \frac{\nu}{V_\infty}\cdot\frac{\partial \bar{u}}{\partial y} \Rightarrow d\eta = \frac{dy}{\delta} \, , \ \bar{u} = \phi \ \text{(Back substitute)}$$

$$\frac{\partial \delta}{\partial x}\int_0^1 \phi(1-\phi)\,d\eta = \frac{\nu}{V_\infty \delta}\left(\frac{\partial \phi}{\partial \eta}\right)\Big|_0$$

$$\delta\frac{\partial \delta}{\partial x} = \frac{\nu}{V_\infty}\beta \, , \quad \boxed{\beta = \frac{\phi'(0)}{\int_0^1 \phi(1-\phi)\,dy}}$$

Integrating, we obtain:

$$\frac{\delta^2}{2} = \frac{\nu x}{V_\infty}\cdot\beta$$

$$\delta^2 = \frac{\nu x}{V_\infty}2\beta$$

$$\left(\frac{\delta}{x}\right)^2 = \frac{2\beta}{Re_x} \Rightarrow \boxed{\frac{\delta}{x} = \frac{\sqrt{2\beta}}{\sqrt{Re_x}}}$$

$\Rightarrow$ Note same form as before but $5.0 = \sqrt{2\beta}$

The way to solve the momentum integral equation is to assume the shape of a velocity profile. Developed by von Karman (another of Prandtl's students). We don't care what happens in the b.l.
Looking at our B.C.'s

$$\phi(0) = \bar{u}(0) = \frac{u}{V_\infty}\Big|_0 = 0$$

$$\phi(1) = 1$$

$\phi = \eta$

$$\phi'(1) = 0 \Rightarrow \frac{\partial \phi}{\partial \eta} = 1 \neq 0 \ \times$$

$\phi = 2\eta - \eta^2$

$$\phi''(0) = 0 \Rightarrow \frac{\partial^2 \phi}{\partial \eta^2} = -2 \ \times$$

Since $\phi'(0)$ = constant (constant wall shear stress)

$\phi = \sin\left(\frac{\pi}{2}\eta\right)$

or

$\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$

Satisfies all B.C.'s

Now we can solve for $\sqrt{2\beta}$, we obtain

ϕ	$\sqrt{2\beta}$
η	3.464
$2\eta-\eta^2$	5.477
$\sin\left(\frac{\pi}{2}\eta\right)$	4.795
$\frac{3}{2}\eta-\frac{1}{2}\eta^3$	4.641

Note all of our solutions for $\sqrt{2\beta}$ are close to 5.0 that we got before. So we chose our ϕ based on our B.C.'s. Let's say we wanted to solve for shear:

$$\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

$$\tau(x) = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\rho\nu V_\infty}{\delta}\cdot\frac{\partial \bar{u}}{\partial \eta} = \frac{3}{2}\cdot\frac{\nu V_\infty}{\left(\frac{\delta}{x}\right)x} \quad ; \quad \bar{u}=\frac{u}{V_\infty},\ \eta=\frac{y}{\delta} \quad \partial u = V_\infty \partial\bar{u},\ \partial y = \delta\partial\eta$$

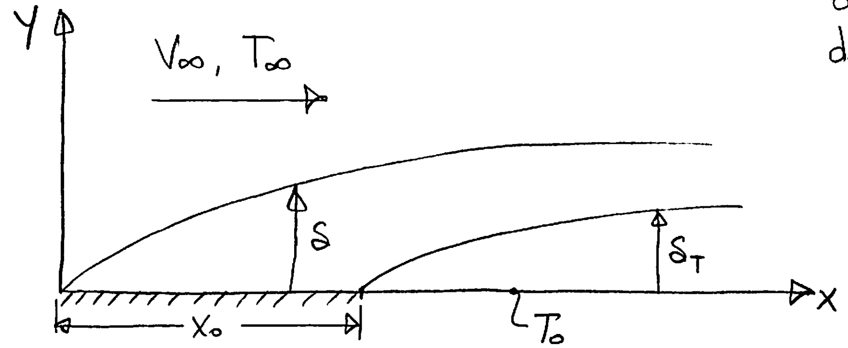
$$\frac{\tau(x)}{\frac{1}{2}\rho V_\infty^2} = \frac{3\nu}{V_\infty\left(\frac{\delta}{x}\right)x} = \frac{3}{Re_x\left(\frac{\delta}{x}\right)} = \frac{3}{\sqrt{2\beta}}\cdot\frac{1}{\sqrt{Re_x}}$$

$$\boxed{C_{f,x} = \frac{\tau(x)}{\frac{1}{2}\rho V_\infty^2} = \frac{0.646}{Re_x^{1/2}}} \Rightarrow \text{Before we had } \frac{0.664}{Re_x^{1/2}}$$

Very close to exact soln!

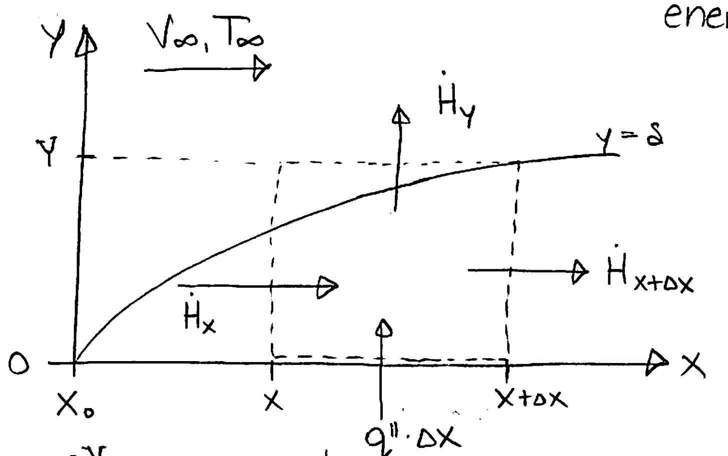
<u>Energy Integral:</u>

What if we had the following problem: (Difficult to solve using analytical technique developed before)



Nenad Miljkovic

Looking at our control volume in the thermal b.l., and considering energy flow in and out:



$$\dot{H}_x = \int_0^Y \rho c_p u T \, dy \Big|_{x=x_0} \quad ; \quad \dot{H}_y = \rho v c_p T \Big|_Y \cdot \Delta x$$

$$\dot{H}_{x+\Delta x} + \dot{H}_y - \dot{H}_x = q'' \Delta x$$

$$\int_0^Y \rho c_p u T \, dy \Big|_{x+\Delta x} + \rho v c_p T \Big|_Y \Delta x - \int_0^Y \rho u c_p T \, dy \Big|_x = -k \frac{\partial T}{\partial y}\Big|_0 \Delta x$$

Divide through by $\rho c_p \Delta x$ and let $\Delta x \to 0$ (Do Taylor series exp)

$$\frac{d}{dx}\int_0^Y u T \, dy + v T \Big|_Y = -\alpha \frac{\partial T}{\partial y}\Big|_0$$

We know $T\big|_Y = T_\infty$, and from continuity: $v(Y) = -\int_0^Y \left(\frac{\partial u}{\partial x}\right) dy$

So we have:

$$\frac{d}{dx}\int_0^Y u T \, dy - \int_0^Y T_\infty \frac{\partial u}{\partial x} \, dy = -\alpha \frac{\partial T}{\partial y}\Big|_0$$

Rearanging, we obtain:

$$\boxed{\frac{d}{dx}\int_0^Y u (T - T_\infty) \, dy = -\alpha \frac{\partial T}{\partial y}\Big|_0} \Rightarrow \text{Energy integral}$$

Integrating from 0 to δ_T (the integral above δ_T is zero, so below δ_T is the only usefull solution)

Nenad Miljkovic

$$\frac{d}{dx}\int_0^{\delta_T} \bar{u}\left(\frac{T-T_\infty}{T_0-T_\infty}\right)dy = -\frac{\alpha}{V_\infty}\left.\frac{\partial\left(\frac{T-T_\infty}{T_0-T_\infty}\right)}{dy}\right|_0 ;$$

Let:

$$\theta = \frac{T-T_0}{T_\infty-T_0} = \theta(\eta_T)$$

$$\eta_T = \eta Pr^{1/3} = \frac{y}{\delta_T}$$

we see that:

$$\frac{T-T_\infty}{T_0-T_\infty} = 1-\frac{T-T_0}{T_\infty-T_0} = 1-\theta$$

$$\frac{d}{dx}\int_0^{\delta_T}\bar{u}(1-\theta)\,dy = \frac{\alpha}{V_\infty}\left.\frac{\partial\theta}{\partial y}\right|_0 \quad (1)$$

So our boundary conditions are:

$$\theta(0) = \frac{T_0-T_0}{T_\infty-T_0} = 0$$

$$\theta(1) = \theta(\eta_T=1) = 1 = \frac{T_\infty-T_0}{T_\infty-T_0}$$

$\theta'(1) = 0 \Rightarrow$ No heat transfer at or above δ_T

$\theta''(0) = 0 \Rightarrow$ Linear temperature profile at $y=0 \Rightarrow \left.\frac{\partial\theta}{\partial y}\right|_{y=0} = \text{const.}$

We know $\eta_T = \frac{y}{\delta_T} \Rightarrow \partial\eta = \frac{1}{\delta}\partial y \Rightarrow$ Back substituting into (1)

$$\frac{d}{dx}(\delta_T)\int_0^1 \bar{u}(1-\theta)\,d\eta_T = \frac{\alpha}{V_\infty}\theta'(0)$$

Using a clever mathematical trick: (Assume the following solution)

$$\bar{u}\frac{\delta}{\delta_T} = \theta(\eta^*), \quad \eta^* = \eta Pr^{1/3} \text{ (From before)}$$

This assumption is OK since: (check)

$$\bar{u} = \phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3 = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

Multiply trough by $\frac{\delta}{\delta_T}$ on both sides:

$$\theta = \frac{3}{2}\left(\frac{y}{\delta_T}\right) \underbrace{-\frac{1}{2}\left(\frac{y^3}{\delta^2\delta_T}\right)}_{\text{We can neglect this term since}}$$

at $y \Rightarrow$ small, $y \ll \delta_T$
$y^3 \to 0$, integral goes to 0.
at $y \Rightarrow$ large, $y \to \delta_T$
$(1-\theta) \to 0$, so our integral goes to 0.

Nenad Miljkovic

Now our solution becomes: $\theta(\eta^*) = \frac{3}{2}\left(\frac{y}{\delta_T}\right)$

$$\frac{d}{dx}\left(\frac{\delta_T^2}{\delta}\right)\int_0^1 \theta(1-\theta)\,d\eta_T = \frac{\alpha}{U\delta_T}\left(\frac{U}{V_\infty}\theta'(0)\right)$$

$\Rightarrow \theta(\eta^*) = \bar{u}\frac{\delta}{\delta_T}$

$\bar{u} = \frac{\delta_T}{\delta}\theta$

$$\boxed{\frac{d}{dx}\left(\frac{\delta_T^2}{\delta}\right) = \frac{1}{Pr\cdot\delta_T}\left[\frac{U}{V_\infty}\cdot\frac{\theta'(0)}{\int_0^1 \theta(1-\theta)\,d\eta}\right]}$$

But note that this is nothing but:

$$\delta_T\frac{d}{dx}\left(\frac{\delta_T^2}{\delta}\right) = \frac{1}{Pr}\left(\frac{U}{V_\infty}\cdot\beta\right)$$

$\Rightarrow$ We defined β before (pg. 101)

$$\delta_T\frac{d}{dx}\left(\frac{\delta_T^2}{\delta}\right) = \frac{1}{Pr}\cdot\underbrace{\delta\frac{\partial\delta}{\partial x}}_{=\beta\left(\frac{U}{V_\infty}\right)} \quad (1)$$

$\Rightarrow$ Derived this previously

Doing another math trick: (Divide both sides of (1) by $\delta^{1/2}$)

$$\frac{\delta_T}{\delta^{1/2}}\frac{d}{dx}\left(\frac{\delta_T^2}{\delta}\right) = \frac{1}{Pr}\delta^{1/2}\frac{\partial\delta}{\partial x}$$

$$\int_{x_0}^{x}\frac{\delta_T}{\delta^{1/2}}\frac{d}{dx}\left(\frac{\delta_T^2}{\delta}\right) = \int_{x_0}^{x}\frac{1}{Pr}\delta^{1/2}\frac{\partial\delta}{\partial x}$$

$\Rightarrow$ Integrate both sides:

Aside:

$$\int_{x_0}^{x}\frac{\delta_T}{\delta^{3/2}}\,d\left(\delta_T^2\right) = \int_{x_0}^{x}\frac{\delta_T}{\delta^{3/2}}\,2\delta_T\,d\delta_T = \int_{x_0}^{x}\frac{2\delta_T^2}{\delta^{3/2}}\,d\delta_T = \frac{2}{3}\frac{\delta_T^3}{\delta^{3/2}}\Big|_{x_0}^{x}$$

$$\frac{\cancel{2}}{\cancel{3}}\left(\frac{\delta_T^2}{\delta}\right)^{3/2}\Bigg|_{x_0}^{x} = \frac{1}{Pr}\cdot\frac{\cancel{2}}{\cancel{3}}\delta^{3/2}\Big|_{x_0}^{x}$$

$$\left(\frac{\delta_T^2}{\delta}\right)^{3/2} = \frac{1}{Pr}\cdot\delta^{3/2}\left[1-\left(\frac{\delta_0}{\delta}\right)^{3/2}\right]$$

But we know in our problem that at $x = x_0$, $\delta_T = 0$, and $\delta_0 = \underbrace{C\sqrt{x}}$

since $\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \Rightarrow \delta \sim C\sqrt{x}$

Nenad Miljkovic

$$\left(\frac{\delta_T}{\delta}\right)^3 = \frac{1}{Pr}\left(1-\left(\frac{X_o}{X}\right)^{3/4}\right)$$

$$\boxed{\frac{\delta}{\delta_T} = \frac{Pr^{1/3}}{\left[1-\left(\frac{X_o}{X}\right)^{3/4}\right]^{1/3}}}$$

$\Rightarrow$ Note, we obtain this analytical profile without assuming anything about the δ_T profile.

So now we can calculate the heat transfer:

$\theta = \theta(\eta_T)$

$\eta_T = \frac{y}{\delta_T}$

$d\eta_T = \frac{dy}{\delta_T}$

$$q''_{y=0} = -k\left.\frac{\partial T}{\partial y}\right|_0 = -\frac{k}{\delta_T}\cdot\frac{\partial\left(\frac{T-T_o}{T_\infty-T_o}\right)}{\partial\eta_T}\cdot\underbrace{(T_\infty-T_o)}_{\Delta T}$$

For $X > X_o$

$$q''_{y=0} = \frac{k\Delta T}{\delta_T}\theta'(0) = \overbrace{\frac{k\Delta T}{X}\cdot\frac{\theta'(0)}{\left(\frac{\delta_T}{\delta}\right)\left(\frac{\delta}{X}\right)}}^{\text{Trick}}$$

$\Rightarrow$ We know $\frac{\delta_T}{\delta}$

$\frac{\delta}{X} = \frac{\sqrt{2\beta}}{Re_x^{1/2}}$

$$q''_{y=0} = \left(\frac{k\Delta T}{X}\right)\frac{Pr^{1/3}Re_x^{1/2}\theta'(0)}{\left[1-\left(\frac{X_o}{X}\right)^{3/4}\right]^{1/3}\sqrt{2\beta}}$$

So our complete solution becomes:

$$\boxed{q''_{y=0} = \left(\frac{k\Delta T}{X}\right)\underbrace{\left[\frac{\theta'(0)}{2}\int_0^1\theta(1-\theta)d\eta_T\right]^{1/2}}\cdot\frac{Re_x^{1/2}\cdot Pr^{1/3}}{\left[1-\left(\frac{X_o}{X}\right)^{3/4}\right]^{1/3}}}$$

$q''_{y=0} = h\Delta T = k_f\frac{\Delta T}{X}$

$\left(\frac{X q''|_{y=0}}{k\Delta T}\right) = \frac{hX}{k_f} = Nu_x$

To solve this integral, we need to assume a velocity profile and solve.

However, there is a trick, since the integral is not a function of X_o at all, it has to equal to 0.332. Let's see what I mean.