\[ \text{Nu}_x = 0.332 \frac{Re_x^{1/2} Pr^{1/3}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}} \]

\[ \Rightarrow \theta \left( \frac{x}{a} \right) \]

To see why \( \theta = 0.332 \), let's look at the limit of \( x_0 = 0 \):

\[ \text{Nu}_x = 0.332 \frac{Re_x^{1/2} Pr^{1/3}}{\left[1 - \left(\frac{0}{x_0}\right)^{3/4}\right]^{1/3}} = 0.332 \text{Re}_x^{1/2} \text{Pr}^{1/3} \]

The solution collapses to our initial solution solved on page 89 of notes (go check). The only way it does this is if the integral on the previous page = 0.332.

END OF LECTURE II

So what if we had steps of heating instead of only 1 step?

\[ \theta (x, y, z) = \frac{T - T_\infty}{T_w - T_\infty} \]

\[ \Theta (z) = \frac{T - T_\infty}{T_w - T_\infty} \]

This will satisfy the differential equation

\[ \frac{T - T_\infty}{T_w - T_\infty} = 1 - \Theta = f(x, y, 3) \quad \Rightarrow \quad \Theta (z) = \frac{1}{2} \frac{y}{z} \]

\[ \Theta (z) = f(x, y) \Rightarrow \text{page 100 of notes} \]

\[ \frac{S}{S_T} = f(x_0, x) \Rightarrow \text{page 100 of notes} \]

\[ f(x, 0, 3) = 1 \Rightarrow \text{Wall temperature at } y = 0 \Rightarrow T_w \]

\[ f(x, \infty, 3) = 0 \Rightarrow \text{at } y \to \infty, T = T_\infty \]

So now if we have temperature jumps, we can use superposition to solve.
\[ (T - T_\infty) = \Delta T_0 \cdot f(x, y, z) \]

1. (equals to)

\[ (T - T_\infty) = \Delta T_0 \cdot f(x, y, z) \]

2. (equals to)

\[ (T - T_\infty) = \Delta T_1 \cdot f(x, y, z) \]

So we can use superposition to solve, making our lives a lot easier:

\[ (T - T_\infty) = \sum_{j=0}^{n} \Delta T_j \cdot f(x, y, z_j), \quad 3_n < x < 3_{n+1} \]

Similarly, we can show:

\[ q_{y=0} = 0.332 \left( \frac{k}{x} \right) Re_x^{1/2} Pr^{1/3} \sum_{j=0}^{n} \frac{\Delta T_j}{1 - \left( \frac{3_i}{x} \right)^{3/4}}^{1/3}, \quad 3_n < x < 3_{n+1} \]
What if our wall temperature difference was continuously changing:

\[ T_w - T_\infty \]

\[ \Rightarrow \text{Our summation becomes integration.} \]

\[ q''_{l,y=0} = C \int_0^x \frac{dT}{1 - \left(\frac{3}{x}\right)^{2/3}} \]

\[ q''_{l,y=0} = 0.332 \left(\frac{h}{x}\right) Re_x^{1/2} Pr^{1/3} \int_0^x \frac{d(T_w - T_\infty)}{\left[1 - \left(\frac{3}{x}\right)^{3/4}\right]^{1/3}} \, d3 \]

Internal Flow - Fully Developed Flow in Tubes

\[ U_\infty \]

\[ X_{el} = \text{entrance length or developing length. Velocity profile varies with radial position, } r, \text{ and axial location, } x. \]

We can estimate the magnitude of the entrance length, \( X_{el} \). We know from previous solution that the b.l. thickness in a laminar flow on a flat plate is:

\[ \frac{S}{x} = \frac{5.0}{\sqrt{Re_x}} \Rightarrow \text{Blasius solution} \]
Extra Derivation
Solve for the wall heat flux \( q'' \bigg|_{y=0} \) if \( (T_w - T_\infty) = \beta \sqrt{x} \).

We just figured out how to deal with this problem. Since the temperature change is continuous, let:

\[
3 = x
\]

\[
(T_w - T_\infty) = \beta \sqrt{3}
\]

\[
\frac{d(T_w - T_\infty)}{dx} = \frac{1}{2} \beta \frac{1}{\sqrt{3}}
\]

Let \( s = \frac{3}{x} \) \( \Rightarrow \) \( q'' \bigg|_{y=0} = 0.332 \left( \frac{k}{x} \right) \text{Re}_x^{1/2} \text{Pr}^{1/3} \beta \frac{1}{2} \frac{1}{\sqrt{3}} \int_0^1 \frac{ds}{s^{1/2} \left[ 1 - s^{3/4} \right]^{1/3}} \)

Multiply by \( \left( \frac{x}{s} \right) \), we will obtain

\[
q'' \bigg|_{y=0} = 0.332 \left( \frac{k}{x} \right) \text{Re}_x^{1/2} \text{Pr}^{1/3} \frac{x}{s^{1/2}} \beta \frac{1}{2} \frac{1}{\sqrt{3}} \int_0^1 \frac{ds}{s^{1/2} \left[ 1 - s^{3/4} \right]^{1/3}}
\]

\[
\int_0^1 \frac{ds}{s^{1/2} \left[ 1 - s^{3/4} \right]^{1/3}}, \quad \text{let } \lambda = (1 - s^{3/4}) \quad \text{We need to solve this}
\]

\[
S = (1 - \lambda)^{4/3}, \quad s^{1/2} = (1 - \lambda)^{2/3}, \quad \left[ 1 - s^{3/4} \right]^{1/3} = \lambda^{1/3}
\]

\[
= \frac{4}{3} \int_0^1 \frac{d\lambda}{(1 - \lambda)^{2/3} \lambda^{1/3}} = \frac{4}{3} \int_0^1 \frac{d\lambda}{\lambda^{1/3} (1 - \lambda)^{2/3}} = \frac{4}{3} \int_0^1 \frac{\lambda^{2/3 - 1} (1 - \lambda)^{2/3 - 1}}{d\lambda}
\]

From integral tables: \( \int_0^1 x^{-1} (1-x)^{2/3} \, dx = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)} = \frac{1.355^2}{0.893} = 2.06 \)

\[
q'' \bigg|_{y=0} = 0.332 \left( \frac{k}{x} \right) \text{Re}_x^{1/2} \text{Pr}^{1/3} \frac{x}{s^{1/2}} \beta \frac{1}{2} \frac{4}{3} \cdot (2.06)
\]

\[
Nu_x = \left( \frac{q'' \bigg|_{y=0} \cdot x}{\Delta T(x) \cdot k_f} \right) = 0.455 \text{Re}_x^{1/2} \text{Pr}^{1/3}
\]

\( \rightarrow \) Note we only get constant heat flux if \( T(x) = \beta(x) \)
We can estimate that \( S \sim \frac{D}{2} \) when the two b.l.'s merge

\[
\frac{D}{2x_e} \sim \frac{5.0}{\sqrt{Re_{x_e}}} = \frac{10}{\sqrt{\frac{\rho U_0 x_e}{\mu}}} = \frac{10}{\sqrt{\frac{\rho U_0 x_e D}{\mu}}} = \frac{10}{\sqrt{\frac{\rho U_0 D}{\mu}}} \sqrt{x_e}
\]

\[
\sqrt{\frac{D}{x_e}} \sim \frac{10}{\sqrt{Re_0}}
\]

\[
x_e \sim \frac{Re_0}{100} \approx 0.01 Re_0 \quad , \quad Re_0 = \frac{\rho U_0 D}{\mu}
\]

Note, the actual solution, experimentally verified is

\[
x_e \approx 0.05 Re_0 \quad \Rightarrow \text{We were fairly close given our assumptions. If } x > x_e, \text{ the flow is fully developed.}
\]

Now looking at the fully developed region, with Navier-Stokes: (x-momentum)

\[
p \left( \frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_x}{\partial \theta} + u_x \frac{\partial u_x}{\partial x} \right) = - \frac{\partial p}{\partial x} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \theta^2} + \frac{\partial^2 u_x}{\partial x^2} \right] + \rho g_x = 0
\]

We know for fully developed flow that \( u_\theta = u_r = 0 \)
\[
\frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial \theta} = 0
\]

So most of our terms drop out and we are left with

\[
- \frac{\partial p}{\partial x} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u_x}{\partial r} \right) = 0
\]

\[\text{B.C.'s: } \frac{\partial u_x}{\partial r} \bigg|_{r=0} = 0 \quad (1)\]

\[u_x(r=r_0) = 0 \quad (2)\]

From here, I will use \( u_x(r) = U(r) \). Dropping the x subscript for simplicity.
Integrating twice and applying our boundary conditions:

\[ u(r) = \frac{r_o^2}{4u} \left( -\frac{\partial p}{\partial x} \right) \left( 1 - \frac{r^2}{r_o^2} \right) \Rightarrow \text{Velocity profile in a pipe} \]

Now if we solve for our average velocity:

\[ \bar{u} = \frac{1}{\pi r_o^2} \int_0^{r_o} u \cdot 2\pi r dr = \frac{r_o^2}{4u} \left( -\frac{\partial p}{\partial x} \right) \left\{ \frac{1}{(1 - \lambda)} \right\} \frac{1}{2}, \quad \lambda = \frac{r^2}{r_o^2} \]

\[ \overline{u} = \frac{r_o^2}{4u} \left( -\frac{\partial p}{\partial x} \right) \]

\[ u(r) = 2\bar{u} \left( 1 - \frac{r^2}{r_o^2} \right) \]

Typically, we want to solve for our friction coefficient and pressure loss

\[ f = \frac{\frac{\Delta p}{\rho}}{(\frac{1}{2}) \frac{1}{2} \bar{u} V^2} \Rightarrow \text{Tube friction factor. Easy way to calculate pressure loss in tubes.} \]

\[ C_f = \frac{\frac{\Delta p}{\rho}}{(\frac{1}{2}) \frac{1}{2} \bar{u} V^2} \Rightarrow \text{Tube friction coefficient. We've determined before. It's related to the friction factor.} \]

Looking at a finite differential element in our flow and using a force balance.

We can say: \[ 4C_f = f \]

So what is our friction factor in a pipe
\[ f = \frac{\Delta P}{\left( \frac{L}{2} \right) \frac{1}{2} \rho V^2} \quad (1) \]

We've solved before that \( \bar{U} = \bar{V} = \frac{r_o^2}{8\mu} \left( \frac{\partial P}{\partial x} \right) \).

\[ \frac{8 \mu V}{r_o^2} = -\frac{2 \rho}{\partial x} = \frac{\Delta P}{L} \quad (2) \]

From (1):

\[ \frac{\Delta P}{L} \cdot \frac{20}{\rho V^2} = f = \frac{8 \mu V}{r_o^2} \cdot \frac{20}{\rho V^2} \]

\[ \frac{16 \mu D}{r_o^2 \rho V} = f = \frac{16 \mu D}{(\frac{D}{2}) (\frac{D}{2}) \rho V} = \frac{64 \mu}{\rho V D} = \frac{64}{Re_0} \]

\[ f = \frac{64}{Re_0} \Rightarrow \text{Pipe friction factor for laminar flow. Also known as the Darcy-Weisbach eqn.} \]

**Note,** \( f \cdot Re = \text{constant} \Rightarrow \text{For any cross section pipe.} \)

See Table 4.5, page 307 of Mills. \( \frac{O_H}{\rho} = \frac{4A}{\rho} \); \( A = \text{area} \)

Heat Transfer in the Pipe

Here we have a similar situation as the hydrodynamic developing (or entrance) length, but with temperature.

\[ \frac{X_{e,T}}{D} = 0.017 Re_o Pr \Rightarrow \text{Thermal developing length.} \]
To estimate the heat transfer, let's try a simple analysis

\[ h = \frac{q''_{\text{wall}}}{\Delta T} \]

\[ h \propto \frac{k_f}{\theta_T} \quad \text{since} \quad h \Delta T = k_f \frac{\Delta T}{\theta_T} \Rightarrow h \sim \frac{k_f}{\theta_T} \]

Assuming \( \theta_T \approx \frac{V_0}{2} \) (since pipe flow) \( \Rightarrow \)

\[ \frac{h}{V_0} \sim \frac{2k}{V_0} = 4 \frac{k}{D} \]

We know that \( \overline{Nu_0} = \frac{hD}{k_f} \Rightarrow \boxed{\overline{Nu_0} \approx 4} \Rightarrow \text{Just from a very simple analysis} \]

We'll see how accurate we are in a little bit.

It's important to note here that heat transfer for internal flow problems is calculated using the bulk fluid temperature.

\[ \overline{h} = \frac{q''_{\text{wall}}}{T_w - T_b} \quad , \quad T_b = \text{bulk fluid temperature} \]

Think of \( T_b \) as the uniform temperature of the pipe fluid if it was allowed to mix and come to an equilibrium temp. in an adiabatic way.

\[ T_b = \frac{1}{A/V} \int_A u(r) T \, dA \quad , \quad \text{where} \quad A = \text{cross sectional area} \quad V = \text{average velocity} \]

**Constant Wall Heat Flux** (\( q''_w \) = constant, Fully developed flow)

Writing out our energy equation: (we will derive this a little later)

\[ \rho C_p u \frac{\partial T}{\partial x} = k \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \]

\[ \boxed{\text{convection} \quad \text{conduction}} \]

We know already that: \( U = 2U \left( 1 - \frac{r^2}{R_o^2} \right) \)
Our B.C.'s are: \( \frac{\partial T}{\partial r} \bigg|_{r=r_0} = 0 \) (no heat flux across centerline due to symmetry) 
\[ T(r=r_0) = T_w(x) \]

Let: \[ \frac{T-T_b}{T_w-T_b} = f(r), \quad \frac{\partial}{\partial r} \left( \frac{1}{T_w-T_b} \right) \]

\[ q'' \bigg|_{r=r_0} = -k \frac{\partial T}{\partial r} \bigg|_{r=r_0} = -k \frac{\partial f}{\partial r} \bigg|_{r=r_0} \cdot (T_w-T_b) = \text{constant} \]

For fully developed flow, the temperature profile shape does not change: \( \frac{\partial f}{\partial r} \bigg|_{r=r_0} = \text{constant} \). So we have:

\[ -k \frac{\partial f}{\partial r} \bigg|_{r=r_0} (T_w-T_b) = q'' = \text{constant} \]

Constant \( \Rightarrow \) This means \( \frac{\partial T_b}{\partial x} = \frac{\partial T_w}{\partial x} \)

Note, if \( \frac{\partial T_b}{\partial x} \neq \frac{\partial T_w}{\partial x} \), then the slopes would cross one another and this is a clear violation of conservation of energy.

END OF LECTURE 12

Now we can solve our energy equation on a fluid element:

\[ \rho u_c T_2 \pi r dr \Rightarrow \frac{\partial}{\partial x} \left( \rho u_c T_2 \pi r \frac{dr}{dx} \right) \]