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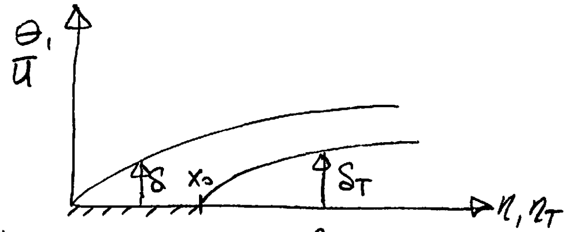
$$\boxed{Nu_x = 0.332 \frac{Re_x^{1/2} Pr^{1/3}}{\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}} \iff$$



To see why $=0.332$, lets look at the limit of $x_0 = 0$

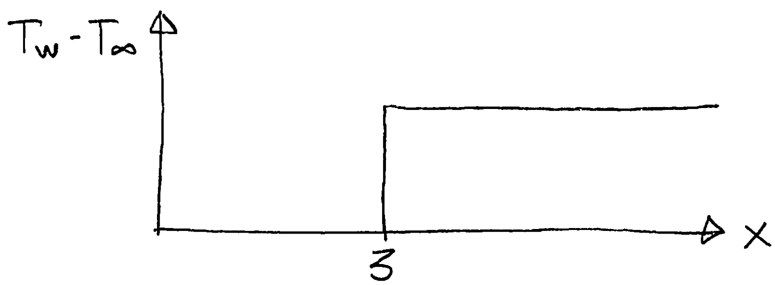
$$Nu_x = 0.332 \frac{Re_x^{1/2} Pr^{1/3}}{\underbrace{\left[1-\left(\frac{0}{x_0}\right)^{3/4}\right]^{1/3}}_{1}} = 0.332\, Re_x^{1/2} Pr^{1/3}$$

The solution collapses to our initial solution solved on page 89 of notes (go check). The only way it does this is if the integral on the previous page $= 0.332$.

Show b. l. NSF video

END OF LECTURE 11

So what if we had steps of heating instead of only 1 step?



$$\boxed{\theta(x, y, 3) = \frac{T - T_w}{T_\infty - T_w}}, \quad 3 < x$$

↳ This will satisfy the differential equation

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \theta = f(x, y, 3) \Rightarrow \quad \theta = \theta(\eta_T) = f\left(\frac{y}{\delta_T}\right) = \frac{3}{2}\frac{y}{\delta_T}$$

$$T - T_\infty = (T_w - T_\infty) \cdot f(x, y, 3) \qquad \frac{\delta}{\delta_T} = f(x_0, x) \Rightarrow \text{page 100 of notes}$$

$$f(x, 0, 3) = 1 \Rightarrow \text{Wall temperature at } y = 0 \Rightarrow T_w$$

$$f(x, \infty, 3) = 0 \Rightarrow \text{at } y \to \infty,\ T = T_\infty$$

So now if we have temperature jumps, we can use superposition to solve.

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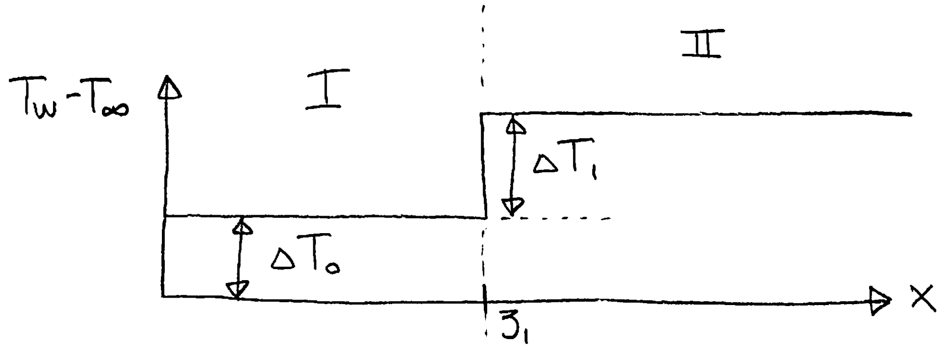
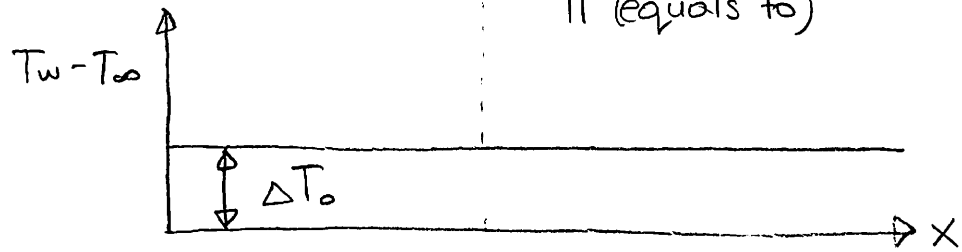


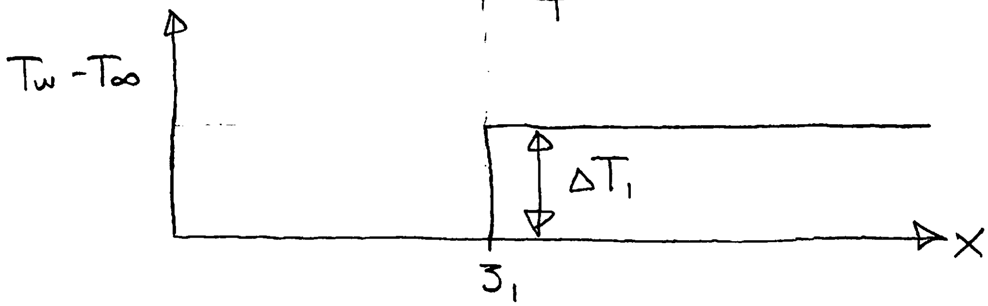
$(T - T_\infty)$
$= \Delta T_0 f(x, y, 0)$
$+ \Delta T_1 f(x, y, \xi)$

|| (equals to)



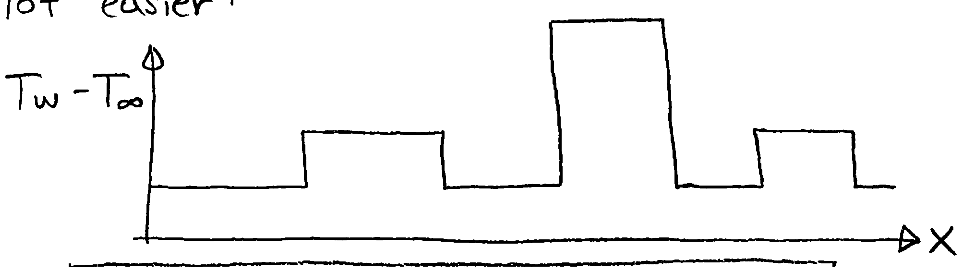
$(T - T_\infty)_{\mathrm{I}}$
$= \Delta T_0 f(x, y, 0)$

$+$



$(T - T_\infty)_{\mathrm{II}}$
$= \Delta T_1 f(x, y, \xi_1)$

So we can use superposition to solve, makes our lives a lot easier:



$$\boxed{(T - T_\infty) = \sum_{j=0}^{n} \Delta T_j \cdot f(x, y, \xi_j)}, \quad \xi_n < x < \xi_{n+1}$$

Similarly we can show:

$$\boxed{q''_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \sum_{j=0}^{n} \frac{\Delta T_j}{\left[1 - \left(\frac{\xi_i}{x}\right)^{3/4}\right]^{1/3}}}, \quad \xi_n < x < \xi_{n+1}$$

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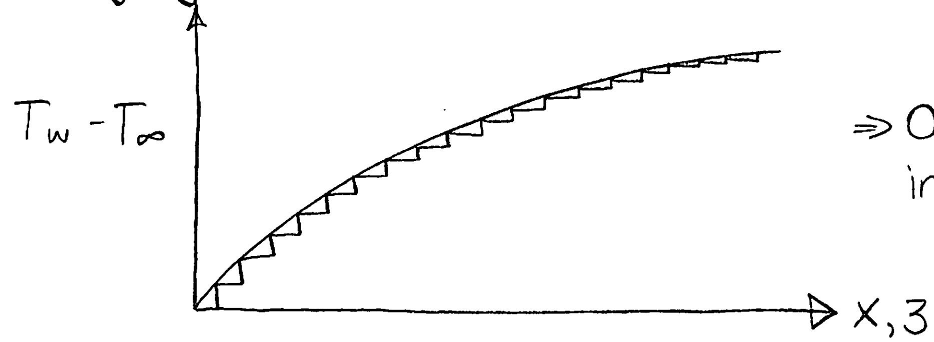
What if our wall temperature difference was continuously changing:

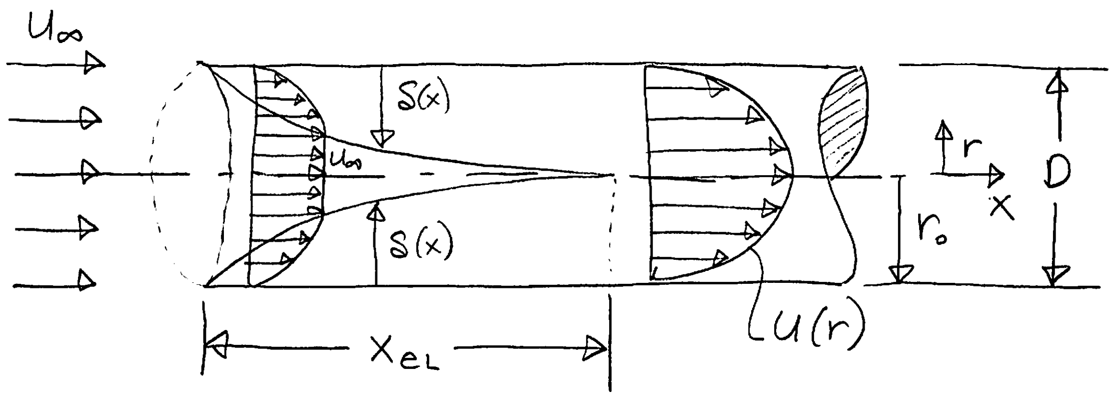


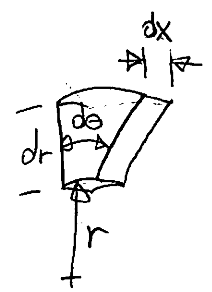
$\Rightarrow$ Our summation becomes integration.

$$q''_{y=0} = C \int_0^x \frac{dT}{\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{1/3}} \cdot \frac{\partial \xi}{\partial \xi}$$

$$\boxed{q''_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \int_0^x \frac{\frac{d(T_w - T_\infty)}{d\xi}}{\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{1/3}} d\xi}$$

<u>Internal Flow</u> - Fully Developed Flow in Tubes





x_{EL} = entrance length or developing length. Velocety profile varies with radial position, r, and axial location, x.

We can estimate the magnitude of the entrance length, x_{EL}. We know from previous solution that the b.l. thickness in a laminar flow on a flat plate is:

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \Rightarrow \text{Blasius solution}$$

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Extra Derrivation

Solve for the wall heat flux $\left(q''|_{y=0}\right)$ if $(T_w - T_\infty) = \beta\sqrt{x}$.

We just figured out how to deal with this problem. Since the temperature change is continuous, let:

$$\xi = x$$

$$(T_w - T_\infty) = \beta\sqrt{\xi}$$

$$\frac{d(T_w - T_\infty)}{d\xi} = \frac{1}{2}\beta\frac{1}{\sqrt{\xi}}$$

Let $s = \frac{\xi}{x} \Rightarrow q''|_{y=0} = 0.332\left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \beta \frac{1}{2}\int_0^x \frac{d\xi}{\xi^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{1/3}}$

Multiply by $\left(\frac{x}{x}\right)$, we will obtain

$$q''|_{y=0} = 0.332\left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} x^{1/2} \beta \frac{1}{2}\underbrace{\int_0^1 \frac{ds}{s^{1/2}\left[1-s^{3/4}\right]^{1/3}}}$$

$\int_0^1 \frac{ds}{s^{1/2}\left[1-s^{3/4}\right]^{1/3}}$, let $\lambda = (1-s^{3/4})$ We need to solve this

$s = (1-\lambda)^{4/3}$, $s^{1/2} = (1-\lambda)^{2/3}$, $\left[1-s^{3/4}\right]^{1/3} = \lambda^{1/3}$

$$= \frac{4}{3}\int_0^1 \frac{(1-\lambda)^{1/3} d\lambda}{(1-\lambda)^{2/3}\lambda^{1/3}} = \frac{4}{3}\int_0^1 \frac{d\lambda}{\lambda^{1/3}(1-\lambda)^{1/3}} = \frac{4}{3}\int_0^1 \lambda^{2/3-1}(1-\lambda)^{2/3-1} d\lambda$$

From integral tables: $\int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{1.355^2}{0.893} = 2.06$

$$\therefore \quad q''|_{y=0} = 0.332\left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \underbrace{x^{1/2}\beta}_{\Delta T(x)} \cdot \frac{1}{2}\cdot\frac{4}{3}\cdot(2.06)$$

$$\boxed{Nu_x = \left(\frac{q''|_{y=0}\cdot x}{\Delta T(x)\cdot k_f}\right) = 0.455 Re_x^{1/2} Pr^{1/3}}$$

↳ Note we only get constant heat flux if $T(x) = \beta(x)$

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We can estimate that $\delta \sim \frac{D}{2}$ when the two b.l.'s merge

$$\frac{D}{2x_{eL}} \sim \frac{5.0}{\sqrt{Re_{x_{eL}}}}$$ ⇒ Note, I don't use equals (=) since not a flat plate

$$\frac{D}{x_{eL}} \sim \frac{10}{\sqrt{Re_{x_{eL}}}} = \frac{10}{\sqrt{\frac{\rho U_\infty x_{eL}}{\mu}}} = \frac{10}{\sqrt{\frac{\rho U_\infty x_{eL}}{\mu}\cdot\frac{D}{D}}} = \frac{10}{\sqrt{\frac{\rho U_\infty D}{\mu}}\cdot\sqrt{\frac{x_{eL}}{D}}}$$

$$\sqrt{\frac{D}{x_{eL}}} \sim \frac{10}{\sqrt{Re_D}}$$

$$\boxed{\frac{x_{eL}}{D} \sim \frac{Re_D}{100} \sim 0.01 Re_D}\ ,\ \boxed{Re_D = \frac{\rho U_\infty D}{\mu}}$$

Note, the actual solution, experimentally verified is

$$\boxed{\frac{x_{eL}}{D} = 0.05 Re_D}$$ ⇒ We were fairly close given our assumptions. If $x > x_{eL}$, the flow is fully developed.

Now looking at the fully developed region, with Navier-Stokes: (x-momentum)

$$\rho\left(\frac{\partial u_x}{\partial t} + u_r\frac{\partial u_x}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_x}{\partial \phi} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_x}{\partial \phi^2} + \frac{\partial^2 u_x}{\partial x^2}\right] + \rho g_x = 0$$

We know for fully developed flow that $u_\phi = u_r = 0$

$$\frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial \phi} = 0$$

So most of our terms drop out and we are left with

$$-\frac{\partial p}{\partial x} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) = 0$$

B.C.'s: $\left.\frac{\partial u_x}{\partial r}\right|_{r=0} = 0$ (1)

$u_x(r=r_o) = 0$ (2)

From here, I will use $u_x(r) = u(r)$. Dropping the x subscript for simplicity.

Integrating twice and applying our boundary conditions:

$$\boxed{u(r) = \frac{r_o^2}{4\mu}\left(-\frac{\partial P}{\partial x}\right)\left(1 - \frac{r^2}{r_o^2}\right)} \Rightarrow \text{Velocity profile in a pipe}$$

Now if we solve for our average velocity:

$$\bar{u} = \frac{1}{\pi r_o^2}\int_0^{r_o} u \cdot 2\pi r\,dr = \frac{r_o^2}{4\mu}\left(-\frac{\partial P}{\partial x}\right)\underbrace{\int_0^1 (1-\lambda)\,\partial\lambda}_{\frac{1}{2}}, \quad \lambda = \frac{r^2}{r_o^2}$$

$$\boxed{\bar{u} = \frac{r_o^2}{8\mu}\left(-\frac{\partial P}{\partial x}\right)}$$

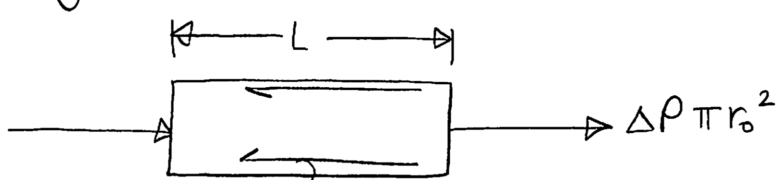
$$\boxed{u(r) = 2\bar{u}\left(1 - \frac{r^2}{r_o^2}\right)}$$

Typically, we want to solve for our friction coefficient and Pressure loss

$$f = \frac{\Delta P}{\left(\frac{L}{D}\right)\frac{1}{2}\rho V^2}$$ $\Rightarrow$ Tube friction factor. Easy way to calculate pressure loss in tubes.

$$C_f = \frac{\tau}{\frac{1}{2}\rho V^2}$$ $\Rightarrow$ Tube friction coefficient. We've determined before. It's related to the friction factor.

Looking at a finite differential element in our flow and using a force balance.



$$2\cancel{\pi}\tau_o \cancel{r_o} L = \Delta P \cancel{\pi} r_o^{\cancel{2}} = \frac{\Delta P D}{2}$$

$$4\tau_o = \frac{\Delta P}{\left(\frac{L}{D}\right)}$$

We can say: $\boxed{4C_f = f}$

So what is our friction factor in a pipe

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$$f = \frac{\Delta P}{\left(\frac{L}{D}\right)\frac{1}{2}\rho V^2} \quad (1)$$

We've solved before that $\bar{u} = V = \frac{r_o^2}{8\mu}\left(-\frac{\partial P}{\partial x}\right)$

$$\frac{8\mu V}{r_o^2} = -\frac{\partial P}{\partial x} = \frac{\Delta P}{L} \quad (2)$$

From (1): $\frac{\Delta P}{L} \cdot \frac{2D}{\rho V^2} = f = \frac{8\mu \cancel{V}}{r_o^2} \cdot \frac{2D}{\rho V^{\cancel{2}}}$

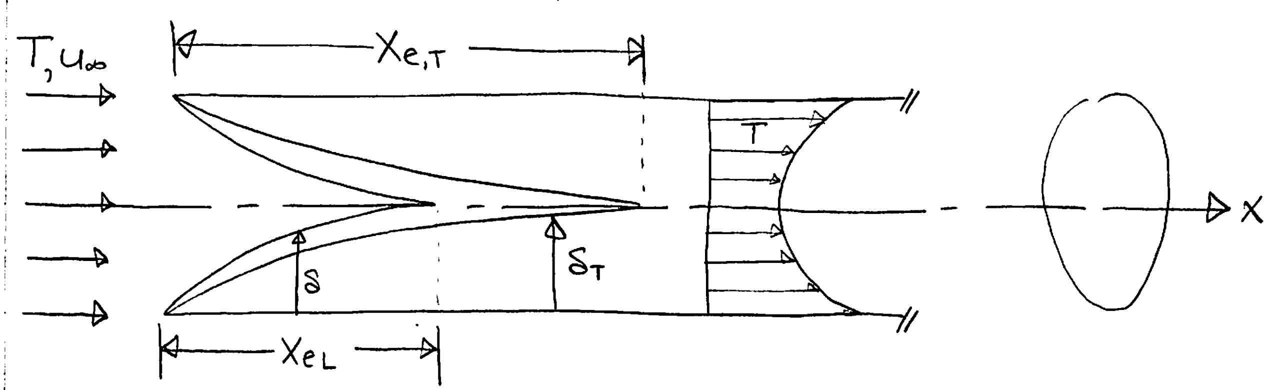
$$\frac{16\mu D}{r_o^2 \rho V} = f = \frac{16\mu \cancel{D}}{\left(\frac{\cancel{D}}{2}\right)\left(\frac{D}{2}\right)\rho V} = \frac{64\mu}{\rho V D} = \frac{64}{Re_D}$$

$\boxed{f = \frac{64}{Re_D}}$ ⇒ Pipe friction factor for laminar flow. Also known as the Darcy-Weisbach eqn.

Note, $\boxed{f \cdot Re = \text{constant}}$ ⇒ For any cross section pipe.

See Table 4.5, page 307 of Mills. $\boxed{D_H = \frac{4A}{P}}$; A = area ; P = perimeter

Heat Transfer in the Pipe



Here we have a similar situation as the hydrodynamic developing (or entrance) length, but with temperature.

$\boxed{\frac{X_{e,T}}{D} = 0.017\, Re_D\, Pr}$ ⇒ Thermal developing length.

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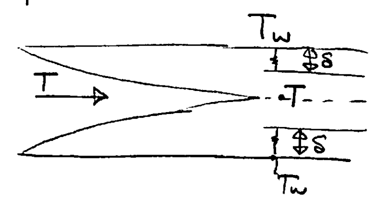
To estimate the heat transfer, let's try a simple analysis

$$h = \frac{q''_{wall}}{\Delta T}$$

$$h \propto \frac{k_f}{\delta_T} \quad \text{since} \quad h\cancel{\Delta T} = k_f \frac{\cancel{\Delta T}}{\delta_T} \Rightarrow h \sim \frac{k_f}{\delta_T}$$

Assuming $\delta_T \approx \frac{r_o}{2}$ (since pipe flow) $\Rightarrow$



$$\bar{h} \sim \frac{2k}{r_o} = 4\frac{k}{D}$$

We know that $\overline{Nu}_D = \frac{\bar{h}D}{k_f} \Rightarrow \boxed{Nu_D \approx 4} \Rightarrow$ Just from a very simple analysis

We'll see how accurate we are in a little bit.

It's important to note here that heat transfer for internal flow problems is calculated using the bulk fluid temperature.

$$\bar{h} = \frac{q''_{wall}}{T_w - T_b}, \quad T_b = \text{bulk fluid temperature}$$

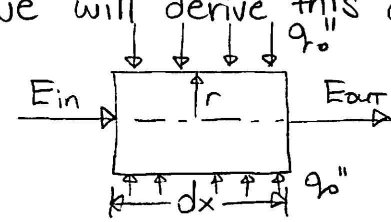
Think of T_b as the uniform temperature of the pipe fluid if it was allowed to mix and come to an equilibrium temp. in an adiabatic way.

$$\boxed{T_b = \frac{1}{A\bar{V}} \int_A u(r)\, T\, dA}$$, where A = cross sectional area, $\bar{V}$ = average velocity

Constant Wall Heat Flux (q_o'' = constant, Fully developed flow)

Writing out our energy equation: (we will derive this a little later)

$$\underbrace{\rho C_p u \frac{\partial T}{\partial x}}_{\text{convection}} = \underbrace{k \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)}_{\text{conduction}}$$



We know already that: $u = 2\bar{u}\left(1 - \frac{r^2}{r_o^2}\right)$

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Our B.C.'s are: $\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$ (no heat flux across centerline due to symmetry)

$$T(r=r_o) = T_w(x)$$

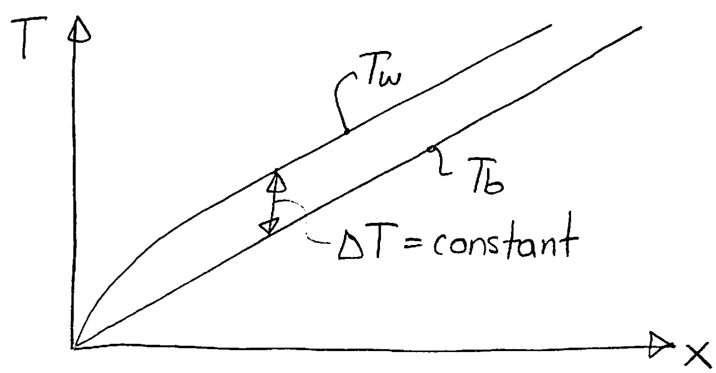
Let: $\frac{T - T_b}{T_w - T_b} = f(r)$, $\quad \partial f = \partial T \left(\frac{1}{T_w - T_b}\right)$

$$q''\Big|_{r=r_o} = -k \left.\frac{\partial T}{\partial r}\right|_{r_o} = -k \left.\frac{\partial f}{\partial r}\right|_{r_o} \cdot (T_w - T_b) = \text{constant}$$

For fully developed flow, the temperature profile shape does not change: $\left.\frac{\partial f}{\partial r}\right|_{r_o} = \text{constant}$. So we have:

$$-k \left.\frac{\partial f}{\partial r}\right|_{r_o} \underbrace{(T_w - T_b)}_{\text{constant}} = q_o'' = \text{constant}$$

$\Rightarrow$ This means $\frac{\partial T_b}{\partial x} = \frac{\partial T_w}{\partial x}$



Note, if $\frac{\partial T_b}{\partial x} \neq \frac{\partial T_w}{\partial x}$, then the slopes would cross one another and this is a clear violation of conservation of energy.

Note: Did informal feedback, & discussed quiz.

END OF LECTURE 12

Now we can solve our energy equation on a fluid element: