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Now we can write our conduction terms:

$$-k\frac{\partial T}{\partial r}2\pi r\Delta x\Big|_{r+\Delta r} \Rightarrow \text{Heat conduction at } r+\Delta r$$

$$-k\frac{\partial T}{\partial r}2\pi r\Delta x\Big|_{r} \Rightarrow \text{Heat conduction at } r$$

Putting everything together (energy balance on our control volume)

$$\rho u c_p T 2\pi r\Delta r\Big|_{x+\Delta x} - \rho u c_p T 2\pi r\Delta r\Big|_{x} = -k\frac{\partial T}{\partial r}2\pi r\Delta x\Big|_{r} + k\frac{\partial T}{\partial r}2\pi r\Delta x\Big|_{r+\Delta r}$$

Rearanging and dividing by $2\pi\Delta x\Delta r$

$$\rho u c_p r\frac{\partial T}{\partial x} = k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

or

$$\boxed{u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)}$$ $\Rightarrow$ Governing energy equation we must solve.

We know that $u = 2\bar{u}\left[1-\frac{r^2}{r_o^2}\right]$

$$2\bar{u}\left[1-\frac{r^2}{r_o^2}\right]\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \Rightarrow \text{PDE since } T=T(x,r)$$

But note that we've already proven that $\frac{\partial T}{\partial x} = \frac{\partial T_b}{\partial x} = \text{constant}$
By simple energy balance:

$$q_o'' \cancel{\pi}\cancel{D}\cancel{\Delta x} = \rho\bar{u}c_p\frac{\cancel{\pi}D^{\cancel{2}}}{4}\frac{\partial T}{\partial x}\cdot\cancel{\Delta x}$$, note, $\bar{u}$ here, not $u(r)$, since we are doing the whole cross section of flow.

$$\boxed{\frac{\partial T}{\partial x} = \frac{2q_o''}{\rho\bar{u}c_p r_o}} = \text{constant}$$

So now we have to solve for the radial temperature profile:

$$2\bar{u}\left(1-\frac{r^2}{r_o^2}\right)\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Rearanging and subbing in α:

$$\left(1-\frac{r^2}{r_o}\right)\frac{4q_o''}{kr_o} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \Rightarrow \text{ODE, we can solve now}$$

B.C.'s: $\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$ (Symmetry)

$T(r=r_o) = T_w$,

Integrating once:

$$r\frac{\partial T}{\partial r} = \frac{4q_o''}{kr_o}\left[\frac{r^2}{2} - \frac{r^4}{4r_o^2}\right] + C_1$$

$$\frac{dT}{dr} = \frac{4q_o''}{kr_o}\left[\frac{r}{2} - \frac{r^3}{4r_o^2}\right] + \underbrace{\frac{C_1}{r}}$$

$C_1 = 0$ since at $r \to 0$, this term explodes

$$T(r) = \frac{4q_o''}{kr_o}\left[\frac{r^2}{4} - \frac{r^4}{16r_o^2}\right] + C_2$$

Using our second boundary condition $(T(r=r_o) = T_w)$

$$T(r_o) = \frac{4q_o''}{kr_o}\left[\frac{r_o^2}{4} - \frac{r_o^{\cancel{4}\,2}}{16\cancel{r_o^2}}\right] + C_2 = T_w$$

$$C_2 = T_w - \frac{4q_o''}{kr_o}\left[\frac{3}{16}r_o^2\right]$$

So our solution becomes:

$$T(r) = T_w - \frac{4q_o''}{kr_o}\left(\frac{3r_o^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_o^2}\right) \Rightarrow \text{Quartic temperature profile}$$

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Now we can solve for $T_b = \dfrac{\int_0^{r_o} uT\,2\pi r\,dr}{\pi r_o^2 \bar{u}}$

Solving (I'll leave the steps for you), we obtain:

$$T_b = T_w - \frac{11}{24}\frac{q_o'' r_o}{k}$$

or

$$\boxed{T_w - T_b = \frac{11}{24}\frac{q_o'' r_o}{k}}$$

$\Rightarrow$ Not a function of x, makes sense, since $\frac{\partial T_b}{\partial x} = \frac{\partial T_w}{\partial x} = \text{constant}$.

Now we can solve for our Nusselt number

$$\bar{h} = \frac{q_o''}{T_w - T_b} = \frac{24k}{11 r_o} = \frac{48k}{11D}$$

$$\overline{Nu}_D = \frac{\bar{h}D}{k} = \frac{48}{11} = 4.364$$

$$\boxed{\overline{Nu}_D = 4.364}$$

$\Rightarrow$ Laminar flow in a tube with constant heat flux conditions.

We can do a similar analysis to show that for a constant wall temperature boundary condition:

$$\boxed{\overline{Nu}_D = \frac{hD}{k} = 3.66}$$

$\Rightarrow T_w = \text{constant}$

We see that our previous solution of $Nu_D \approx 4$ was pretty close to the exact analytical solutions.

Note, both of these solutions are valid only if:

$$\boxed{Re_D = \frac{\rho \bar{u} D}{\mu} \lesssim 2300}$$

$\Rightarrow$ Laminar flow in a smooth pipe, fully developed.

We will discuss turbulence a little later.

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<u>Some physical insights:</u>

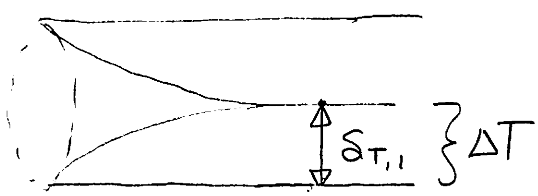
On page 79 of the notes, we defined the Nusselt number as the non-dimensional temperature gradient

$$\underbrace{-\frac{\partial \theta}{\partial n^*}}_{\text{Measure of the gradient}} = \underbrace{\frac{hL}{k_f}}_{\text{Nusselt \#}} = Nu, \qquad n^* = \frac{n}{L} \text{ or } n^* = \frac{n}{D} \text{ for a pipe}$$

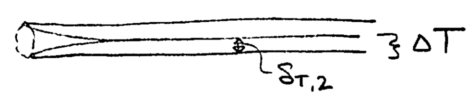
$$\theta = \frac{T - T_f}{T_s - T_f}$$

This tells us why it's so advantageous to go to mini or micro channel flows for cooling. By reducing the channel dimension (diameter), we reduce the effective boundary layer thickness (thermal, δ_T) and pump up our heat transfer.

<u>Large Channel:</u> <u>Micro Channel:</u>



$\Longleftrightarrow$



$$-\frac{\partial \theta}{\partial n^*} = \frac{h_1 D_1}{k_f} = Nu \approx 4 \qquad -\frac{\partial \theta}{\partial n^*} = \frac{h_2 D_2}{k_f} = Nu \approx 4$$

$$h_1 = \frac{4 k_f}{D_1} \qquad h_2 = \frac{4 k_f}{D_2}$$

Since $D_2 \ll D_1$, $h_2 \gg h_1$

This is the direction of research these days, pushing dimensions to the micron length scale to maximize h. One drawback to this is the excess pumping power required to drive the fluid.

$$\Delta P = f \cdot \left(\frac{L}{D}\right) \frac{1}{2} \rho \bar{V}^2, \quad \text{as } D\downarrow,\ \Delta P \uparrow \text{ non-linearly}$$

since $f = \frac{64}{Re_D} = \frac{64\mu}{\rho \bar{V} D}$ and $\bar{V} \propto D^2$ for a constant $\dot{m}$.

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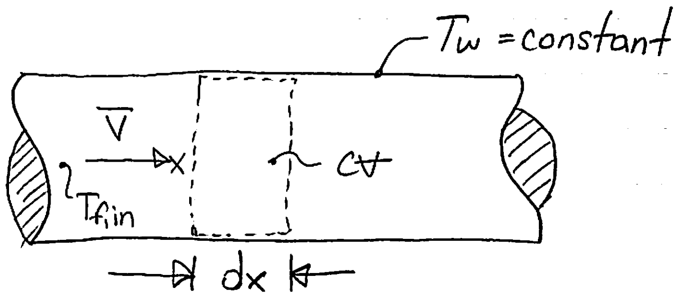
To get an order of magnitude feel for the enhancement

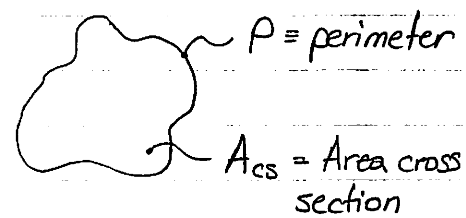
$k_f = 0.6 \ W/m \cdot K$
$D_1 = 1 \ cm, \quad D_2 = 100 \ \mu m$
$Nu = 4$

$$h_1 = \frac{Nu \, k_f}{D_1} = \frac{(4)(0.6 W/m \cdot K)}{0.01 m} = 240 \ W/m^2 \cdot K$$

$$h_2 = \frac{Nu \cdot k_f}{D_2} = \frac{(4)(0.6 W/m \cdot K)}{0.0001} = 24\,000 \ W/m^2 \cdot K = 24 \ kW/m^2 \cdot K$$

Example | Heat Exchange to a liquid flowing in a tube, with heat transfer coefficient h. Find T_{out} given that T_w = constant, and fully developed flow.





Apply an enegy balance on our control volume

$$\underbrace{\rho A_{cs} \bar{V}}_{\dot{m}} C_p \cdot dT_f = h \underbrace{P dx}_{dA_{surface}} (T_w - T_f)$$

Let $T^* = \dfrac{T_f - T_w}{T_{f,in} - T_w}$, $x^* = \dfrac{x}{L}$, L = length of pipe

$dT^* = \dfrac{dT_f}{T_{f,in} - T_w}$, $dx^* = \dfrac{dx}{L}$

$$\rho A_{cs} \bar{V} C_p \cancel{(T_{f,in} - T_w)} dT^* = h P L dx^* (-T^*) \cancel{(T_{f,in} - T_w)}$$

$$\rho A_{cs} \bar{V} C_p dT^* + h P L T^* dx^* = 0$$

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$$\int_{T_{f,in}}^{T_f} \frac{dT^*}{T^*} + \int_{x=0}^{x} \frac{hPL}{\rho A_{cs} \bar{V} c_p} dx^* = 0$$

$$\ln\left(\frac{T^*}{T_{f,in}^*}\right) + \frac{hPLx^*}{\rho A_{cs} \bar{V} c_p} = 0$$

$$T^* = T_{f,in}^* e^{-\frac{hPL}{\rho A_{cs} \bar{V} c_p} x^*} \Rightarrow T_{f,in}^* = \frac{T_{f,in} - T_w}{T_{f,in} - T_w} = 1$$

$$\boxed{T_f = T_w + (T_{f,in} - T_w) e^{-\frac{hP}{\rho A_{cs} \bar{V} c_p} x}}$$

So remember here, $T_f = T_b$ (bulk fluid temperature)

$$\boxed{T_b = \frac{1}{\rho A_{cs} \bar{V} c_p} \int_{A_{cs}} \rho c_p V(x,y) \cdot T(x,y)\, dA}$$

$\Rightarrow$ Note, already defined on page 113 of notes

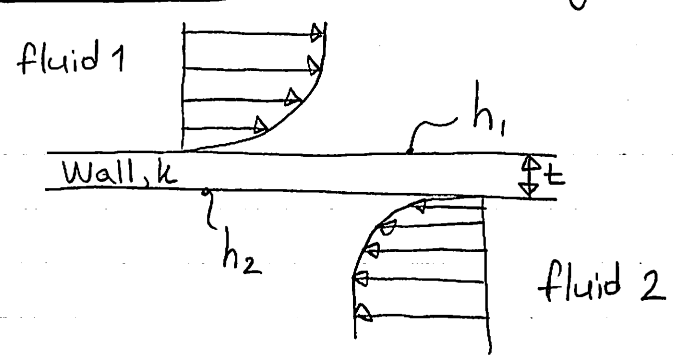
Note also $q = hPdx(T_w - T_b)$

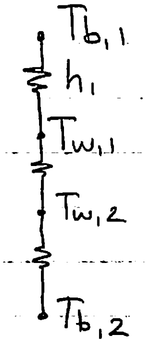
$\hookrightarrow$ But by definition, h (or Nu) $= f(T_b)$

For problems like this, always base property values on:

$$\boxed{\bar{T} = \frac{T_{b,in} + T_{b,out}}{2}}, \text{ where } T_{b,in} = T_{f,in}, \; T_{b,out} = T_{f,out}$$

Example #2 | Heat exchanger.





Using our thermal resistance concepts:

From our resistance network, we can write

$$U = \frac{1}{\left\{\frac{1}{h_1} + \frac{t}{k} + \frac{1}{h_2}\right\}} \Rightarrow \frac{1}{U\cancel{A}} = \frac{t}{k\cancel{A}} + \frac{1}{h_1\cancel{A}} + \frac{1}{h_2\cancel{A}}$$

So now we can analyze our heat exchanger: (Perimeter, P)

① $\longrightarrow$ $\dot{m}_1, T_{b1}, c_{p,1}$ (or: $\dot{M}, T, C$)

$\rightarrow| dx |\leftarrow$

② $\longrightarrow$ $\dot{m}_2, T_{b2}, c_{p,2}$ (or: $\dot{m}, t, c$)

Across our element dx: $dq = -\dot{M} C \, dT$

$$dq = +\dot{m} c \, dt$$

$$dT = -\frac{dq}{\dot{M}C} \quad \text{and} \quad dt = +\frac{dq}{\dot{m}c}$$

Aside: $d(T-t) = dT - dt$

$$d(T-t) = dq\left\{-\frac{1}{\dot{M}C} - \frac{1}{\dot{m}c}\right\}$$

$$d(T-t) = \frac{dq}{\dot{M}C}\left\{-1 - \frac{\dot{M}C}{\dot{m}c}\right\}$$

But we know that $dq = UP\,dx\,(T-t)$, where $P\,dx = dA$

$$\frac{d(T-t)}{(T-t)} = \frac{U}{\dot{M}C}\left\{-1 - \frac{\dot{M}C}{\dot{m}c}\right\} dA$$

Integrating both sides from inlet (a) to outlet (b), we obtain

$$\ln\left\{\frac{T_b - t_b}{T_a - t_a}\right\} = \frac{UA}{\dot{M}C}\left\{-1 - \frac{\dot{M}C}{\dot{m}c}\right\} \quad (1)$$

For the entire heat exchanger, we know:

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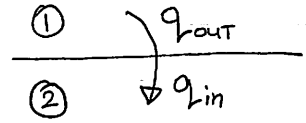
$$\overbrace{-\dot{M}C\left(T_b - T_a\right)}^{\text{Energy out of stream (1), } q_{OUT}=-q} = \overbrace{\dot{m}c\left(t_b - t_a\right)}^{\text{Energy in to stream (2), } q_{in}=q}$$



rearanging:

$$-\frac{\dot{M}C}{\dot{m}c} = \frac{t_b - t_a}{T_b - T_a} \qquad (2)$$

Back substituting (2) into (1)

$$\ln\left\{\frac{T_b - t_b}{T_a - t_a}\right\} = \frac{UA}{\dot{M}C}\left\{-1 + \frac{t_b - t_a}{T_b - T_a}\right\}$$

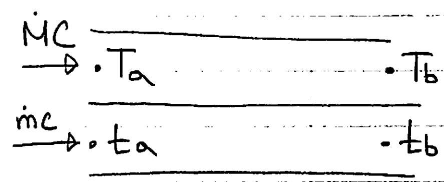
Rewriting this, we obtain:

$$\ln\left\{\frac{T_b - t_b}{T_a - t_a}\right\} = \frac{UA}{\underbrace{\dot{M}C\left(T_b - T_a\right)}_{-q \text{ (net heat transfer)}}}\overbrace{\left\{\left(T_a - T_b\right) + t_b - t_a\right\}}^{-\left\{\left(T_b - t_b\right) - \left(T_a - t_a\right)\right\}}$$

Since $q = UA\Delta T$, now we can write

$$\Delta T = LMTD$$

$$\boxed{\Delta T = \frac{\left(T_b - t_b\right) - \left(T_a - t_a\right)}{\ln\left\{\dfrac{T_b - t_b}{T_a - t_a}\right\}}}$$



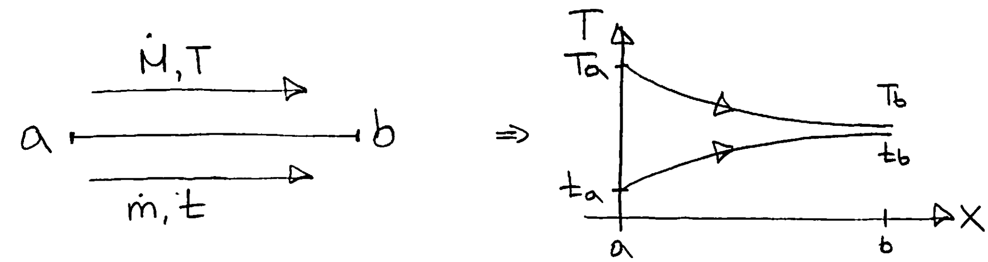
$$\boxed{q = UA\left(LMTD\right)}$$

$\Downarrow$
Log Mean Temperature Difference

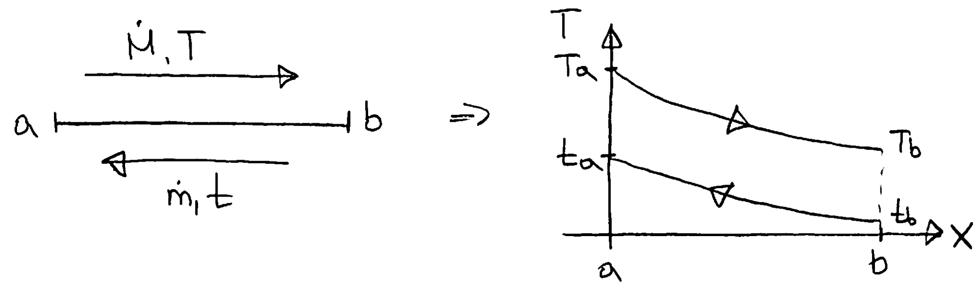
Note, it doesn't matter which side is a and b. You can reverse it and still get the same answer. (i.e. switch a & b)

It also desn't matter which direction the flows are traveling in, as long as "a" and "b" refer to some physical end of the heat exchanger.

Parallel Flow:



Counter Flow:



Think of the LMTD as a convenient way to define a ΔT between two streams whose temperature is continuously varying.

END OF LECTURE 13

Spetial Case ⇒ Balanced counter flow heat exchanger

Assuming $\dot{M} = \dot{m}$, and $C = c$, then the LMTD becomes undefined:

$$\mathrm{LMTD} = \frac{\Delta T_b - \Delta T_a}{\ln\left(\frac{\Delta T_b}{\Delta T_a}\right)} \quad ; \quad \Delta T = T - t$$

In this case, $\Delta T_b = \Delta T_a$, so $\mathrm{LMTD} = \frac{0}{0}$ ⇒ undefined

The way to resolve this is the following:

Suppose $\Delta T_b = \Delta T_a + \varepsilon$, where $\varepsilon \ll 1$

$$\mathrm{LMTD} = \frac{\Delta T_a + \varepsilon - \Delta T_a}{\ln\left\{\frac{\Delta T_a + \varepsilon}{\Delta T_a}\right\}} = \frac{\varepsilon}{\ln\left(1 + \frac{\varepsilon}{\Delta T_a}\right)}$$

But we know that $\ln(1+x) \approx x$ for $x \ll 1$

$$\ln\left(1 + \frac{\varepsilon}{\Delta T_a}\right) \approx \frac{\varepsilon}{\Delta T_a} \Rightarrow \mathrm{LMTD} = \frac{\cancel{\varepsilon}}{\cancel{\varepsilon}/\Delta T_a} = \Delta T_a = \Delta T_b = \Delta T \quad \text{for } \varepsilon \ll 1$$

So our undefined problem is resolved.