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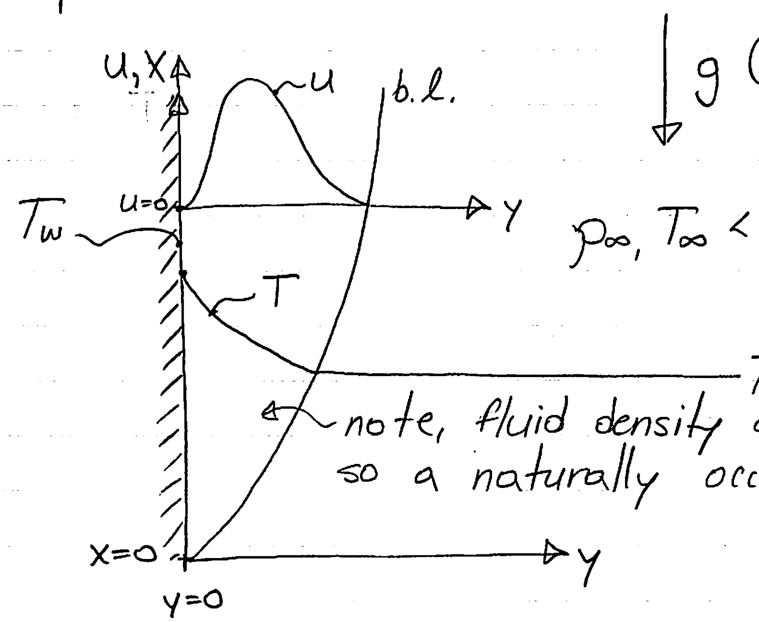
Natural Convection

Unlike forced convection, in which the fluid motion driving force is external to the fluid, natural convection processes are driven by body forces exerted directly within the fluid as a result of heating or cooling.

What is natural convection?
Simplest case:



g (gravity pointing down)

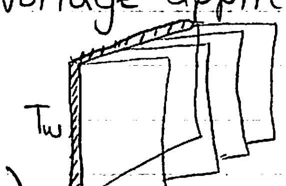
ρ_∞, $T_\infty < T_w$

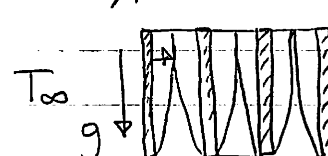
note, fluid density decreases with higher temperat. so a naturally occuring boyant force arises

This flow is similar to a flow past a flat plate with a boundary layer developing, however here $u=0$ at the wall and at $y \to \infty$.

We see natural convection all around us. When we are stationary, we are undergoing natural convection heat loss to the environment.

Many passive heat sinks (fins) are natural convection. I.e. transformer cooling (high voltage applications), home appliances, electronics cooling.
&
Cooling towers in nuclear power plants (Stack Effect, Natural Convect.)





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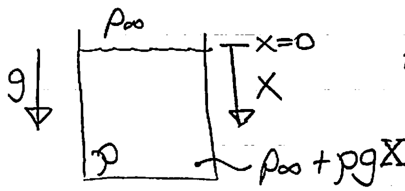
We can now solve for our governing equatins.

<u>x-momentum:</u> (steady state)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \underbrace{-\, g}_{\text{Body force (gravity in } -x)} \quad (1)$$

We know in the x-direction for a fluid that:

$$p(x) = P_\infty + \rho g x$$



Archimedes Principle.

Developed by Arichmedes of Syracuse.

$$\frac{\partial p}{\partial x} = -\rho_\infty g \quad \text{(negative since x pointig against gravity)}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{\rho_\infty g}{\rho} \quad (2)$$

Back substitute (2) into (1)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\left(\frac{\rho_\infty - \rho}{\rho}\right) + \nu\frac{\partial^2 u}{\partial y^2} \quad (3)$$

However, we need to know the relation between density, ρ and temperature, T. In general for a fluid we can define a quantity called the isothermal compressibility, β.

$$\beta = \frac{1}{v}\frac{\partial v}{\partial T}\bigg|_p \quad \text{(or coefficient of thermal expansion)}$$

specific volume.

$$= -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\bigg|_p \quad \underset{\text{proof}}{\Rightarrow}$$

Aside: $\rho = \frac{P}{R_{specific} T}$, $\frac{\partial \rho}{\partial T} = \frac{P}{R_s}\frac{-1}{T^2}$

$\frac{1}{\rho} = v$, $v = \frac{R_s T}{P}$, $\frac{\partial v}{\partial T} = \frac{R_s}{P}$

$$\frac{\cancel{P}}{\cancel{R_s} T}\cdot\frac{\cancel{R_s}}{\cancel{P}} = \frac{1}{T} = \frac{1}{v}\frac{\partial v}{\partial T}\bigg|_p$$

$$-\frac{\cancel{R_s}\cancel{T}}{\cancel{P}}\cdot\frac{\cancel{P}}{\cancel{R_s}}\left(-\frac{1}{T^{\cancel{2}}}\right) = \frac{1}{T} = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\bigg|_p$$

We can rewrite as:

$$\beta = -\frac{1}{\rho}\frac{(\rho_\infty - \rho)}{T_\infty - T}$$

$$\frac{\rho_\infty - \rho}{\rho} = \beta(T - T_\infty) \quad (4)$$

Back substitute ④ into ③, we obtain

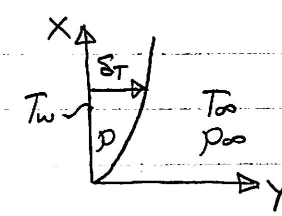
$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{Inertia Forces, I}} = \underbrace{g\beta(T-T_\infty)}_{\text{Boyancy Forces, }\beta} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\text{Viscous Forces, }\nu}$$

Now if we do our simple and tested scaling analysis

$$\delta_T \sim \sqrt{\alpha t} \sim \sqrt{\frac{\alpha x}{u}} \quad \text{(Speed of heat propagation via conduction)}$$

$$u \sim \frac{\alpha x}{\delta_T^2}\ , \quad \partial u \sim \frac{\alpha \partial x}{\delta_T^2}$$

$$y \sim \delta_T\ , \quad dy \sim \delta_T$$



Inertia, I	Viscosity, ν	Boyancy, β
$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \sim \frac{u^2}{x} \sim \frac{\alpha^2 x}{\delta_T^4}$	$\nu\frac{\partial^2 u}{\partial y^2} \sim \nu\frac{u}{y^2} \sim \frac{\nu\alpha x}{\delta_T^4}$	$g\beta(T-T_\infty) = g\beta\Delta T$

Now if we take a ratio of our forces with respect to our boyant force term:

I/β	ν/β	β/β
$\frac{\alpha^2 x}{\delta_T^4 g\beta\Delta T}$	$\frac{\nu\alpha x}{\delta_T^4 g\beta\Delta T}$	1

Assuming that $\nu/\beta \sim O(1)$ (Inside the natural convection b.l.)

$$\frac{\nu\alpha x}{\delta_T^4 g\beta\Delta T} \sim 1$$

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Multiplying both sides by $\frac{x^3}{x^3}$

> **Aside:**
> Lord Rayleigh was the same guy who developed Rayleigh scattering.

$$\frac{\nu\alpha}{g\beta\Delta T x^3}\left(\frac{x}{\delta_T}\right)^4 \sim 1$$

$$\boxed{\frac{x}{\delta_T} \sim \left(\frac{g\beta\Delta T x^3}{\alpha\nu}\right)^{1/4}} \Rightarrow (\text{Rayleigh Number})^{1/4}$$

$$\boxed{Ra = \frac{g\beta\Delta T x^3}{\alpha\nu}}, \quad \boxed{\frac{x}{\delta_T} \sim Ra^{1/4}}$$

$$Ra = \underbrace{\frac{\text{Boyancy Force}}{\text{Viscous Force}}}_{Gr} \cdot \underbrace{\frac{\text{Momentum Diffusivity}}{\text{Thermal Diffusivity}}}_{Pr} = Gr \cdot Pr$$

↳ Grashof number

If: $Ra < Ra_{crit}$, conduction dominates $(Gr \ll 1, Pr \ll 1)$
$Ra > Ra_{crit}$, convection dominates $(Gr > 1, Pr > 1)$

So what about heat transfer?

$$q''\big|_{y=0} = -k\frac{\partial T}{\partial y}\bigg|_{y=0} \sim k\frac{\Delta T}{\delta_T}$$

$$\left(\frac{q''|_{y=0}}{\Delta T}\right)\cdot\frac{x}{k_f} = Nu_x \sim \frac{x}{\delta_T} \sim Ra_x^{1/4}, \quad \boxed{Nu_x \sim Ra_x^{1/4}}$$

So we can tell that our Nusselt number will be proportional to the Rayleigh number to the 1/4 power just by looking at our Fourier conduction equation.

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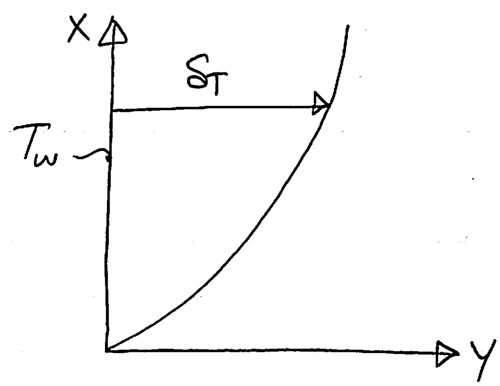
Looking back at our natural convection b.l.



$\delta_T \sim X^{1/4}$ $\left(\text{since } \dfrac{\nu \alpha X}{\delta_T^4 \beta \Delta T g} \sim 1\right)$

Our heat transfer will slowly decrease with increasing X.

Constant Heat Flux Case

Instead of T_w = constant, what if $q''|_{y=0}$ = constant

$$q_0''|_{y=0} \sim \frac{k \Delta T}{\delta_T}$$

$$\Delta T \sim \frac{q_0'' \delta_T}{k}$$

We can re-do our scaling analysis with our new ΔT definition

Inertia, I	Viscosity, ν	Boyancy, β
$\dfrac{\alpha^2 X}{\delta_T^4}$	$\dfrac{\nu \alpha X}{\delta_T^4}$	$g \beta \delta_T (q_0''/k)$

Again, assuming (Viscous Force)/(Boyancy Force) ~ 1 (inside b.l)

$$\frac{\nu \alpha X}{\delta_T^5 g \beta (q_0''/k)} \sim 1 \cdot \left(\frac{X^4}{X^4}\right)$$

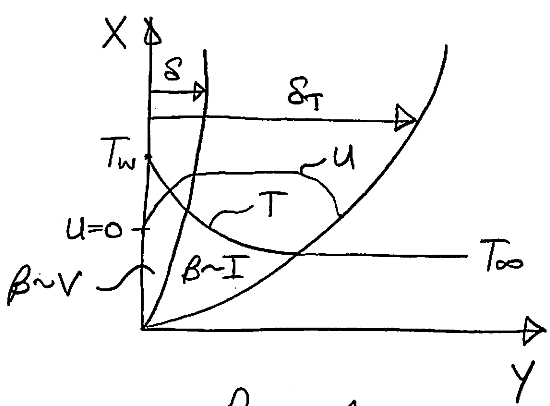
$$\boxed{\frac{X}{\delta_T} = \left[\frac{g \beta q_0'' X^4}{\nu \alpha k}\right]^{1/5}} \quad , \quad \boxed{Ra^* = \frac{g \beta q_0'' X^4}{\alpha \nu k}} \Rightarrow \text{Modified Rayleigh Number.}$$

↳ For constant heat flux

$$\boxed{Nu_x \sim Ra_x^{*\,1/5}}$$

Note our two previous solutions are valid if $\frac{\text{Viscosity}}{\text{Boyancy}} \sim 1$

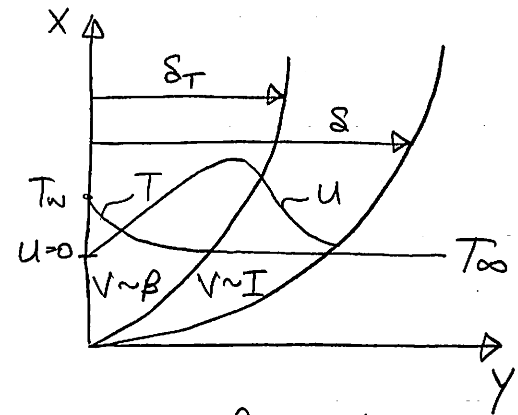
If we have a very viscous fluid, we must solve our boundary layer equations fully.



$Pr \ll 1$
(Liquid Metal)

$\Downarrow$

Temperature diffuses far and boyancy moves fluid up as wall drags a thin b.l. (hydrodynamic) where Boyancy ~ Viscosity. More like flow over a flat plate with U_∞ where in U_∞ region Boyancy ~ Inertia



$Pr \gg 1$
(Synthetic Oil)

$\Downarrow$

Temperature has small diffusion but momentum diffusion dominates and creates a thick hydrodynamic b.l. where Viscosity ~ Inertia, $B = 0$ since no temperature change.

Now we can go back to our analysis and solve for our x-momentum equation. Using the integral technique we solved for earlier, we can solve.

<u>Energy Integral Technique</u> (for natural convection)

Assume $\delta \sim \delta_T$ (Viscosity ~ Boyancy), T_w = constant

$$\delta \sim \delta_T \quad , \quad \theta = \frac{T - T_w}{T_\infty - T_w} \quad , \quad \eta = \frac{y}{\delta_T} \quad , \quad \theta = f(\eta)$$

$$u = \Gamma(x)\,\phi(\eta) \, , \quad \Gamma(x) = \text{function of } x \text{ only}$$

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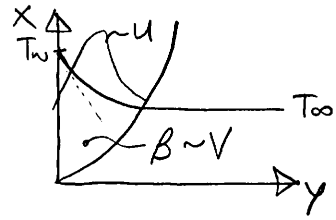

So our boundary conditions are:

$\theta(0) = 0 \quad (T = T_w \text{ at } y=0)$
$\theta(1) = 1 \quad (T = T_\infty \text{ at } y = \delta_T)$
$\theta'(1) = 0 \quad (\partial T/\partial y|_{y=\delta_T} = 0)$
$\theta''(0) = 0 \quad (\partial T/\partial y|_{y=0} = \text{constant at the wall})$

$\phi(0) = 0 \quad (u = 0 \text{ at } y = 0)$
$\phi(1) = 0 \quad (u = 0 \text{ at } y = \delta_T)$
$\phi'(1) = 0 \quad (\partial u/\partial y|_{y=\delta_T} = 0)$

Our energy integral equation for this b.l. is:

$$\frac{d}{dx}\int_0^{\delta_T} u(T - T_\infty)\,dy = -\alpha \left.\frac{\partial T}{\partial y}\right|_{y=0}$$ $\Rightarrow$ Check page 103 of notes for derivation.

Non-dimensionalizing

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \theta \quad , \quad u = \Gamma(x)\phi(\eta) \, , \quad \eta = \frac{y}{\delta_T} \Rightarrow dy = \delta_T\, d\eta$$

$$\theta = \frac{T - T_w}{T_\infty - T_w} \Rightarrow d\theta = \frac{dT}{T_\infty - T_w}$$

Back substituting into our energy integral equation:

$$\frac{d}{dx}(\delta_T \Gamma)\int_0^1 \phi(\eta)(1 - \theta(\eta))\,d\eta = \frac{\alpha}{\delta_T}\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} \quad (1)$$

Note, we can use a trick to solve for $\Gamma(x)$ since we know at the wall, Boyancy ~ Viscosity (one of our assumptions)

$$g\beta\Delta T + \nu\frac{\partial^2 u}{\partial y^2} = 0 \quad (\text{Inertia is negligible since } u|_{y=0} = 0)$$

$dy = \delta_T\, d\eta \, ; \; \partial^2 u \Rightarrow u = \Gamma(x)\phi(\eta) \Rightarrow \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\Gamma(x)}{\delta_T^2}\underbrace{\dfrac{\partial^2 \phi}{\partial \eta^2}}_{\phi''(0)}$

$$\Gamma(x) = \frac{g\beta\Delta T \delta_T^2}{\nu[-\phi''(0)]} \quad (2)$$

END OF LECTURE 15