Nenad Miljkovic

Back substituting ② into ①

$$\frac{g\beta\Delta T}{\nu\left[-\phi''(0)\right]}\cdot\frac{\partial \delta_T^{\,3}}{\partial x} = \frac{\alpha}{\delta_T}\,\frac{\theta'(0)}{\int_0^1 \phi(1-\theta)\,d\eta}$$

Rewriting:

$$\boxed{\delta_T\,\frac{\partial \delta_T^{\,3}}{\partial x} = \frac{\alpha\nu}{g\beta\Delta T}\cdot[\gamma]} \quad ; \quad \boxed{[\gamma] = \frac{\theta'(0)\left[-\phi''(0)\right]}{\int_0^1 \phi(1-\theta)\,d\eta}}$$

Note, γ only depends on the temperature and velocity profiles that we assume, so it's usefull to take it out and call it γ separately by itself.

$$\delta_T\,\frac{\partial \delta_T^{\,3}}{\partial x} = \delta_T\, 3\,\delta_T^{\,2}\,\frac{\partial \delta_T}{\partial x} = 3\,\delta_T^{\,3}\,\frac{\partial \delta_T}{\partial x}$$

Aside:
(Think of $\delta_T = f(x)$, $\frac{d}{dx}\left[f(x)\right]^3 = 3f(x)^2\frac{df}{dx}$

$$3\delta_T^{\,3}\frac{\partial \delta_T}{\partial x} = \frac{\alpha\nu}{g\beta\Delta T}\gamma$$

$$\int_0^{\delta_T} 3\,\delta_T^{\,3}\,\partial\delta_T = \frac{\alpha\nu}{g\beta\Delta T}\gamma\int_0^x \partial x$$

$$\frac{3}{4}\,\delta_T^{\,4} = \frac{\alpha\nu x\gamma}{g\beta\Delta T}\cdot\left(\frac{x^4}{x^4}\right)$$

$$\left(\frac{\delta_T}{x}\right)^4 = \left[\frac{\nu\alpha}{g\beta\Delta T x^3}\right]\cdot\left(\frac{4}{3}\gamma\right) \quad \Rightarrow Ra = \frac{g\beta\Delta T x^3}{\nu\alpha}$$

$$\boxed{\frac{\delta_T}{x} = \frac{1}{Ra_x^{1/4}}\left(\frac{4}{3}\gamma\right)^{1/4}}$$

$\Rightarrow$ Note, before we had $\frac{\delta_T}{x} \sim \frac{1}{Ra^{1/4}}$
So we are very close!
See page 133 of notes.

We're almost there, we need to assume velocity and temperature profiles ($\phi(\eta)$, $\theta(\eta)$, $\eta = \frac{y}{\delta_T}$) and solve for γ.

Nenad Miljkovic

Our assumed profiles must satisfy our boundary conditions

Let:

$$\left.\begin{aligned}\theta(\eta) &= \frac{3}{2}\eta - \frac{1}{2}\eta^3 \\ \phi(\eta) &= \eta(1-\eta^2)\end{aligned}\right\} \quad \frac{\delta_T}{x} = \left(\frac{210}{Ra_x}\right)^{1/4}$$

$\Rightarrow$ After solving for γ and back substituting.

Now we can solve for heat transfer

$$\theta = \frac{T-T_w}{T_\infty - T_w} \qquad 1-\theta = \frac{T-T_\infty}{T_w-T_\infty} \qquad dT = d(1-\theta)\underbrace{(T_w-T_\infty)}_{\Delta T}$$

$$\left.q''\right|_{y=0} = -k_f\left.\frac{\partial T}{\partial y}\right|_{y=0} = -\frac{k_f}{\delta_T}\cdot\frac{d}{d\eta}(1-\theta)\Delta T$$

$$= \Delta T\,\frac{k_f}{\delta_T}\,\theta'(0)$$

$$= \frac{3}{2}\frac{k_f}{\delta_T}\Delta T$$

Now we can formulate our Nusselt number for natural convection on a heated vertical wall with T_w = constant.

$$\frac{\left(\frac{q''|_{y=0}}{\Delta T}\right)\cdot x}{k_f} = Nu_x = \frac{3}{2}\left(\frac{x}{\delta_T}\right) = \frac{1.5}{(210)^{1/4}}\cdot Ra_x^{1/4}$$

$$\boxed{Nu_x = 0.394\,Ra_x^{1/4}} \quad , \quad \boxed{Ra_x = \frac{g\beta\Delta T x^3}{\alpha\nu}}$$

We know that $h_x = C\cdot\frac{1}{x^{1/4}}$ since $Nu_x = \frac{h_x\cdot x}{k}$

$$\bar{h} = \frac{1}{L}\int_0^L h_x\,dx = \frac{C}{L}\int_0^L \frac{dx}{x^{1/4}} = \frac{4}{3}C\frac{L^{3/4}}{L} = \frac{4}{3}\frac{C}{L^{1/4}}$$

$$\bar{h} = \frac{4}{3}h_L$$

$$\boxed{Nu_L = \frac{\bar{h}L}{k_f} = 0.525\,Ra_L^{1/4}}$$

$\Rightarrow$ Note experiments show a very accurate result of $\boxed{Nu_{L,exp} = 0.52\,Ra_L^{1/4}}$

Nenad Miljkovic

Now if we wanted to solve for the constant heat flux case ($q''|_{y=0}$ = constant). Our energy integral would change a bit.

$$\frac{d}{dx}\int_0^{\delta} u(T-T_\infty)\,dy = \frac{1}{\rho c_p} q_0'' \quad \Big\} \text{ before was } -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}\Big|_{y=0}$$

Non-dimensionalizing just like before, we obtain:

$$\frac{d}{dx}\left(\delta_T \Delta T\, \Gamma(x)\right)\int_0^1 \phi(1-\theta)\,d\eta = \alpha\left(\frac{q_0''}{k}\right)$$

Substituting in our $\Gamma(x)$

$$\frac{g\beta}{\nu[-\phi''(0)]}\cdot\frac{d}{dx}\left(\delta_T^3 \Delta T^2\right)\int_0^1 \phi(1-\theta)\,d\eta = \alpha\left(\frac{q_0''}{k}\right) \quad (1)$$

$$q''_{y=0} = -k\frac{\partial T}{\partial y}\Big|_{y=0} = -\frac{k}{\delta_T}\left[\frac{\partial\left(\frac{T-T_\infty}{\Delta T}\right)}{\partial(y/\delta)}\right]\Bigg|_{y=0}\cdot \Delta T$$

$$= \frac{k}{\delta_T}\,\theta'(0)\cdot\Delta T$$

$$\boxed{\Delta T = \frac{\left(\frac{q_0''}{k}\right)\cdot \delta_T}{\theta'(0)}}$$

$\Rightarrow$ Back substituting into (1)

$$\frac{g\beta\left(\frac{q_0''}{k}\right)^{\cancel{2}}}{\nu[-\phi''(0)][\theta'(0)]^2}\cdot\frac{d\delta_T^5}{dx} = \alpha\,\cancel{\left(\frac{q_0''}{k}\right)}\cdot\frac{1}{\int_0^1\phi(1-\theta)\,d\eta}$$

$$\frac{d\delta_T^5}{dx} = \frac{\nu\alpha}{g\beta(q_0''/k)}\cdot\underbrace{\left[\frac{[-\phi''(0)]\cdot[\theta'(0)]^2}{\int_0^1\phi(1-\theta)\,d\eta}\right]}_{\text{Depends only on our assumed profiles for } \phi \text{ and } \theta.\ \text{Call this } [\gamma']}$$

$$\left(\frac{\delta_T}{x}\right)^5 = \left[\frac{\nu\alpha}{g\beta(q_0''/k)x^4}\right]\cdot[\gamma']$$

Nenad Miljkovic

$$\frac{x}{\delta_T} = \frac{(Ra_x^*)^{1/5}}{[\gamma']} \,, \qquad \boxed{Ra_x^* = \frac{g\beta(q_0''/k)x^4}{\nu\alpha}}$$

$$\left(\frac{q''|_{y=0}}{\Delta T}\right)\cdot\frac{x}{k_f} = Nu_x = \frac{\cancel{q''|_{y=0}}}{\frac{\cancel{q''|_{y=0}}}{\cancel{k_f}}\cdot\frac{\delta_T}{\theta'(0)}}\cdot\frac{x}{\cancel{k_f}}$$

$$Nu_x = \frac{3}{2}\frac{x}{\delta_T}$$

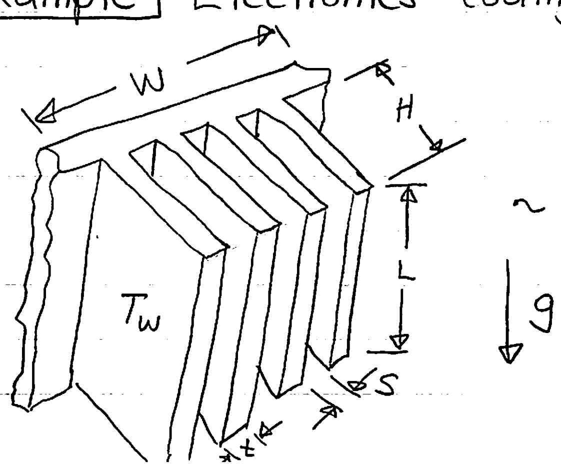
Assuming the same temperature and velocity profiles as before (ϕ, θ) and solving for $[\gamma']$, we get:

$$\boxed{Nu_x = 0.503\, Ra_x^{*\,1/5}}$$

<u>In general:</u> $Ra_L > 10^9 \Rightarrow$ Turbulent free convection
$Ra_L \leq 10^9 \Rightarrow$ Eq. 4.85 on pg. 326 in Mills
$10^9 \leq Ra_L \leq 10^{12} \Rightarrow$ Eq. 4.86
$10^6 \leq Ra_L \leq 10^9 \Rightarrow$ Eq. 4.87

<u>Note</u>: 1) All properties for natural convection problems are evaluated at $\bar{T} = (T_\infty + T_w)/2$
2) β is evaluated at T_∞. $\beta = \frac{1}{T}$ for ideal gases where T is in Kelvin.

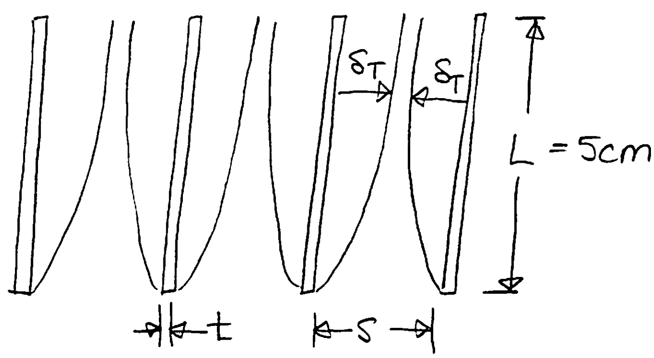
<u>Example</u> Electronics cooling heat sink



Nenad Miljkovic

How much heat can we dissipate if we assume $T_w = 80°C$, $t \ll S$, $S = 3\,mm$, $L = 5\,cm$, $n = 20$ (number of fins), and $H = 3\,cm$, $T_\infty = 25°C$.

Here, since $t \ll S$, we can assume isothermal vertical flat plates. We also assume that $\delta_T < \frac{S}{2}$ (non interacting b.l.'s). This makes the isolated vertical flat plate approxim. correct. Also, $\delta < \frac{S}{2}$



$n = 20$ fins
$H = 3\,cm$

$\sim T_\infty = 25°C$

Using our developed correlation (constant wall temperature)

$$T_\infty = 25°C = 298.15\,K, \quad \beta = \frac{1}{T_\infty} = \frac{1}{298.15K} = 0.00335\,K^{-1}$$

$$\bar{T} = \frac{T_\infty + T_w}{2} = 325.65\,K, \Rightarrow \underset{(at\ \bar{T})}{\text{Air Properties}}: \begin{cases} k_f = 0.0277\ W/m\cdot K \\ \nu = 17.9 \times 10^{-6}\,m^2/s \\ Pr = 0.71 = \frac{\nu}{\alpha} \end{cases}$$

$$\Delta T = 80°C - 25°C = 55°C$$

We first calculate $Ra_L = \frac{g\beta\Delta T L^3}{\alpha\nu} = \frac{(9.81)(0.00335)(55)(0.05)^3(0.71)}{(17.9\times10^{-6})^2}$

$$Ra_L = 5.0 \times 10^5 \ (\text{Laminar Flow} < 10^9)$$

Now we can calculate $\bar{h}_L$ from $Nu_L = \frac{\bar{h}_L L}{k_f} = 0.525\,Ra_L^{1/4}$

$$\bar{h}_L = \frac{Nu_L k_f}{L} = \frac{0.525\,Ra_L^{1/4} k_f}{L} = \frac{0.525\,(5.0\times10^5)^{1/4}(0.0277)}{0.05}$$

$$\bar{h}_L = 7.737\ W/m^2\cdot K$$

$$A_T = n \cdot 2 \cdot LH = (20\text{ fins})(2\text{sides/fin})(0.03m)(0.05m) = 0.06\,m^2$$

Nenad Miljkovic

$$Q_{TOT} = \bar{h}_L \cdot A_T \cdot \Delta T = (7.737\,W/m^2\cdot K)(0.06m^2)(55°C)$$

$$\boxed{Q_{TOT} = 25.53W}$$

Note, we can check our $\delta_T < \frac{S}{2}$ and $\delta < \frac{S}{2}$ approximation
We calculated previously that $\frac{\delta_T}{L} = \frac{(210)^{1/4}}{(Ra_L)^{1/4}}$

$$\delta_T = L \cdot \left(\frac{210}{Ra_L}\right)^{1/4} \text{ (see page 138 of notes)}$$

$$= (0.05m)\left(\frac{210}{5.0\times10^5}\right)^{1/4} = 0.0072m = 7.2mm > \frac{S}{2}$$

So our previous approximation was incorrect since the thermal boundary layers will overlap. However, the calculation is a good approximation or starting point. Luckily, many researchers have looked at this problem and have developed more accurate correlations ⇒ Let's re-calculate and see how off we were:

$$Nu_s = \frac{\bar{h}_L S}{k_f} = \left[\frac{576}{\left(Ra_s \cdot \frac{S}{L}\right)^2} + \frac{2.873}{\left(Ra_s \cdot \frac{S}{L}\right)^{0.5}}\right]^{-0.5}$$

$$Ra_s = \frac{g\beta\Delta T S^3 Pr}{\nu^2} = Gr_s \cdot Pr \quad ; \quad \frac{S}{L} = \frac{0.003}{0.05} = 0.06$$

$$= \frac{(9.81)(0.00335)(55)(0.003)^3(0.71)}{(17.9\times10^{-6})^2} = 108.14$$

$$Nu_s = \frac{\bar{h}_L S}{k_f} = \left[\frac{576}{(108.14\cdot0.06)^2} + \frac{2.873}{(108.14\cdot0.06)^{0.5}}\right]^{-0.5} = 0.259$$

$$\bar{h}_L = \frac{(0.259)(0.0277)}{0.003} = 2.4\,W/m^2\cdot K$$

$$Q_{TOT} = \bar{h}_L \cdot A_T \cdot \Delta T = (2.4W/m^2\cdot K)(0.06m^2)(55°C)$$

$\boxed{Q_{TOT} = 7.92\,W} < Q_{TOT}$ from our first try.

Makes sense since overlapping b.l's act to diminish heat transfer.

Nenad Miljkovic

In general:

Closely Packed Fins	Widely Spaced Fins
- Greater surfase area	- Higher heat transfer coefficient ($\bar{h}_L$)
- Smaller heat transfer coefficient ($\bar{h}_L$)	- Smaller surface area

For those of you who are interested, for T_w = constant

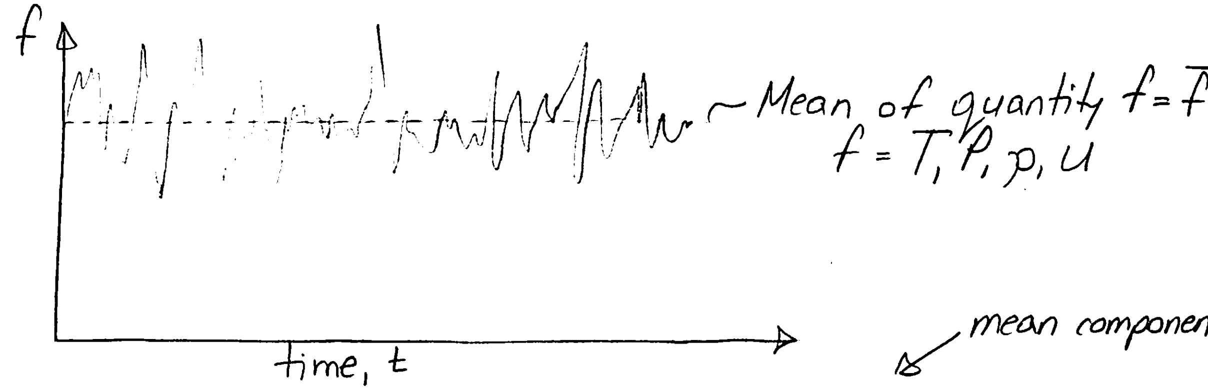
$$\boxed{S_{opt} = 2.714 \cdot \frac{L}{Ra_L^{1/4}}}$$

⇒ See paper on Dropbox by Bar-Cohen and Rohsenow (1984)

Turbulent Boundary Layers

Turbulence in a fluid can be seen as a spectrum of coexisting vortices (eddies) in which kinetic energy from larger ones is dissipated to succesively smaller ones until the very smallest of these vortices are damped out by viscous shear stress.

Turbulent flow causes fluctuations of the velocity components, pressure, temperature, and in compressible flows, density.



We can define our quantity f as: $f = \bar{f} + f'$

mean component ($\bar{f}$); fluctuating component (f')

We can prove that $\overline{f'} = 0$

$$\bar{f} = \overline{\bar{f} + f'} = \underbrace{\bar{\bar{f}} + \overline{f'}}_{\text{Additive law of expectation}} = \bar{f} + \overline{f'} \Rightarrow \overline{f'} = 0$$