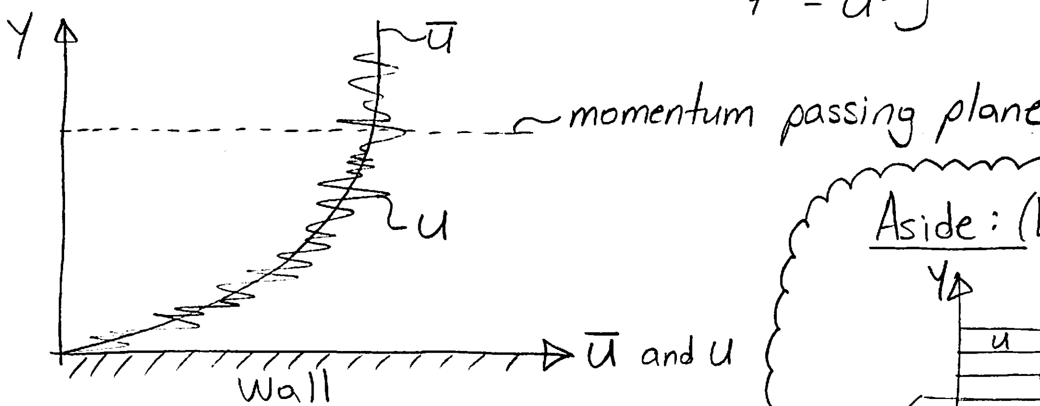
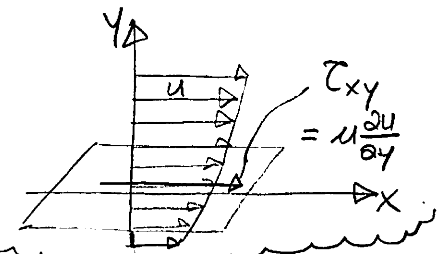
$$\overline{f}(t_0) = \frac{1}{\tau}\int_{t_0-\tau/2}^{t_0+\tau/2} f\,dt \quad \text{(Definition of } \overline{f}\text{)}$$

Note also that: $\overline{\overline{f}f'} = \underbrace{\overline{f}}_{\text{Constant}}\underbrace{\overline{f'}}_{0} = 0 \Rightarrow \overline{\overline{f}f'} = 0$

Now if we focus on velocity, i.e. $\left.\begin{aligned} f &= u \\ \overline{f} &= \overline{u} \\ f' &= u' \end{aligned}\right\} u = \overline{u} + u'$



Aside: (Laminar flow)



The average shear stress is written as:

$$\overline{\tau} = \underbrace{\overline{\mu\frac{\partial u}{\partial y}}}_{\substack{\text{Laminar} \\ \text{Component}}} - \underbrace{\overline{\rho uv}}_{\substack{\text{Turbulent} \\ \text{Component} \\ \text{(momentum leaving CV)}}}$$

$\Rightarrow$ Control volume on a fluid element inside the boundary layer, momentum balance. See page 100 of notes.

Note we typically write $\tau = \mu\frac{\partial u}{\partial y}$ for laminar fully developed flow over a flat plate since we evaluate $\partial u/\partial y$ at the wall, where $v = 0$. Also, in the fluid, $v' = 0$, and $\overline{v} \ll \overline{u}$, $\overline{v} \approx 0$.

$$\overline{\tau} = \mu\frac{\partial \overline{u}}{\partial y} - \rho\overline{uv}$$

$$u = \overline{u} + u'$$
$$v = \overline{v} + v'$$

For fully developed flow, $\overline{v} = 0$, $\overline{v'} = 0$

$$\overline{uv} = \overline{(\overline{u}+u')(\overline{v}+v')} = \overline{\overline{u}\,\overline{v}} + \overline{\overline{u}v'} + \overline{u'\overline{v}} + \overline{u'v'}$$
$$= \overline{u}\,\overline{v} + \overline{\overline{u}}\,\overline{v'} + \overline{u'}\,\overline{\overline{v}} + \overline{u'v'} = \overline{u}\,\overline{v} + \overline{u'v'}$$

$\overline{uv} = \overline{u}\,\overline{v} + \overline{u'v'} \Rightarrow$ Not trivial to calculate, but we can model

Note, $\overline{u'} = 0$ and $\overline{v'} = 0$, however $\overline{u'v'} \neq 0$.

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Note also, $\bar{v} \ll \bar{u}$, and $\bar{v} < v'$ (for boundary layer flow)

$\frac{\overline{u'v'}}{\bar{u}\bar{v}} \gg 1$, So we can assume $\bar{u}\bar{v} \approx 0$ since $\bar{v} \approx 0$

We see from before that: (Using some intuition)

$$\overline{u'v'} \to 0 \quad \text{as} \quad \frac{\partial \bar{u}}{\partial y} \to 0 \quad \text{(Outside the b.l.)}$$

$$\overline{u'v'} \uparrow \quad \text{as} \quad \frac{\partial \bar{u}}{\partial y} \uparrow \quad \text{(Shear at the wall is highest)}$$

To build a useful model by assuming $\overline{u'v'}$ is proportional to $\frac{\partial \bar{u}}{\partial y}$

Aside: $\overline{u'v'}$ = negative hence the turbulent stress increases overall shear $\bar{\tau}$.

$$\bar{\tau} = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{v'u'}$$

$$= \mu \frac{\partial \bar{u}}{\partial y} - \underbrace{\left(\text{factor which reflects turbulent mixing}\right)}_{= -\rho \varepsilon} \cdot \frac{\partial \bar{u}}{\partial y}$$

or

$$\boxed{\bar{\tau} = \rho(\nu + \varepsilon) \frac{\partial \bar{u}}{\partial y}} \quad \Rightarrow \quad \varepsilon = \text{eddy diffusivity} \left[\frac{m^2}{s}\right]$$

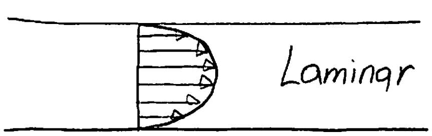
This model is often used in computational fluid dynamics (CFD) software packages to deal with turbulent flows.

We can also re-write this as:

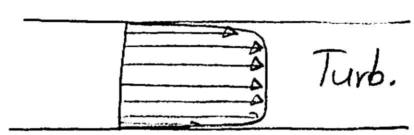
$$\bar{\tau} = (\underbrace{\rho\nu}_{\mu} + \underbrace{\rho\varepsilon}_{\mu_t}) \frac{\partial \bar{u}}{\partial y}$$

$\mu_t \Rightarrow$ Turbulent viscosity

Looking at our experimental velocity profiles:



Laminar $\left.\frac{\partial \bar{u}}{\partial y}\right|_{Lam}$

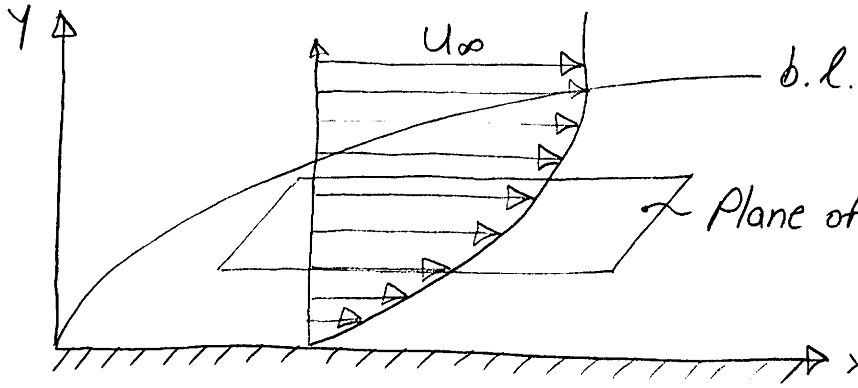


Turb. $\left.\frac{\partial \bar{u}}{\partial y}\right|_{Turb} > \left.\frac{\partial \bar{u}}{\partial y}\right|_{Lam}$

Since $\tau_o = \rho \overline{u'v'}$

$$u', v' \sim \sqrt{\frac{\tau_o}{\rho}} \Rightarrow \tau_o \cong \text{Turbulent shear stress or Reynolds stress}$$

We can now relate our viscosity to mixing length, ℓ, by creating an analogy with molecular viscosity.



We know that shear stress is the time rate of change of momentum. Looking at our plane of interest, and considering the rate of molecular motion across the plane:

Assuming:
1) n molecules per unit volume
2) $1/3$ of molecules have velocities along the y-dir.
3) Half of these, or $\frac{1}{6}n$ molecules per unit volume have mean velocity $\bar{V}$ in the $+y$-direction; the other half have a mean velocity $\bar{V}$ in the $-y$ direction.

From this, we can say at any given time there are $\frac{1}{6}n\bar{V}$ molecules that cross our plane from below, and $\frac{1}{6}n\bar{V}$ from above.

Also, molecules that cross have, on average, experienced their last collisions at a distance λ (λ = mean free path) from the plane.

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Since the mean velocity $u_x = u_x(y)$ is a function of y, we can write a balance equation for momentum.

The mean x-component of momentum transported per unit time per unit area across the plane upwards $= \frac{1}{6} n\bar{v}[m u_x(y-\lambda)]$ ①

The mean x-component of momentum transported per unit time per unit area across the plane downwards $= \frac{1}{6} n\bar{v}[m u_x(y+\lambda)]$ ②

$$\tau = \frac{1}{6} n\bar{v}m\left[u_x(y-\lambda) - u_x(y+\lambda)\right] \Rightarrow \text{Taylor series exp. H.O.T. terms drop}$$

$$\tau = \frac{1}{6} n\bar{v}m\left(-2\frac{\partial u_x}{\partial y}\cdot\lambda\right) = -\mu\frac{\partial u_x}{\partial y} \Rightarrow nm = \rho \text{ (density)}$$

$$\boxed{\mu = \frac{1}{3}\rho\bar{v}\lambda} \Rightarrow \text{Note if you substitute for } \bar{v}, \& \lambda, \text{ you will see that } \mu \neq f(\rho, \text{or } P)!$$

Now back to our turbulence derivation:

$\mu_t \sim \rho v_t \ell$, where v_t = turbulent or friction velocity

$$\frac{\mu_t}{\rho} = \varepsilon = v_t \cdot \ell \sim \sqrt{\frac{\tau_o}{\rho}}\cdot\ell, \quad v_t \sim v' \text{ or } u'$$

$$\frac{\tau_o}{\rho} = \varepsilon\frac{\partial\bar{u}}{\partial y}$$

$$\frac{\tau_o}{\rho} = \sqrt{\frac{\tau_o}{\rho}}\cdot\ell\frac{\partial\bar{u}}{\partial y} = \sqrt{\frac{\tau_o}{\rho}}\cdot Ky\frac{\partial u}{\partial y} \quad (1)$$

$\boxed{\ell = Ky}$, $K = 0.41 \Rightarrow$ Von Karman constant (from experiment)
ℓ = mixing length or eddy size

This says that turbulent eddies at a location y must be no bigger than the distance to the wall. Makes sense since eddies cannot cross into the wall.

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If we further non-dimensionalize:

$$u^+ = \frac{u}{\sqrt{\tau_o/\rho}} \quad , \quad y^+ = \frac{y\sqrt{\tau_o/\rho}}{\nu}$$

$$\partial u = \sqrt{\frac{\tau_o}{\rho}}\,\partial u^+ \quad , \quad \partial y = \frac{\nu\, dy^+}{\sqrt{\tau_o/\rho}}$$

Back substitute into ①

$$\frac{\cancel{\tau_o}}{\cancel{\rho}} = K y \sqrt{\frac{\tau_o}{\rho}} \cdot \frac{\partial u^+}{\partial y^+} \cdot \sqrt{\frac{\cancel{\tau_o}}{\cancel{\rho}}} \cdot \sqrt{\frac{\cancel{\tau_o}}{\cancel{\rho}}} \cdot \frac{1}{\nu}$$

$$1 = \underbrace{\frac{y}{\nu} \cdot \sqrt{\frac{\tau_o}{\rho}}}_{y^+} \cdot K \frac{\partial u^+}{\partial y^+}$$

$$K y^+ \frac{\partial u^+}{\partial y^+} = 1$$

$$\int \partial u^+ = \frac{1}{K} \int \frac{dy^+}{y^+}$$

$$\boxed{u^+ = \frac{1}{K} \ln y^+ + C}$$ $\Rightarrow$ The log layer (away from the wall)

Note, very near the wall, we can say that the turbulent eddies are very small since $\ell = Ky$. So:

$\ell \to 0$, and $\varepsilon = v'\ell \to 0$ $\Rightarrow$ Viscosity dominates
Turbulent viscosity $\to 0$

$$\tau_o = \mu \frac{\partial u}{\partial y} \approx \mu \frac{u}{y}$$

$$\approx \rho \nu \frac{u}{y} \Rightarrow \frac{\tau_o}{\rho} = \nu \frac{u}{y} \Rightarrow \underbrace{\sqrt{\frac{\tau_o}{\rho}} \cdot \frac{y}{\nu}}_{y^+} = \underbrace{\frac{u}{\sqrt{\tau_o/\rho}}}_{u^+}$$

$$\boxed{u^+ = y^+}$$ $\Rightarrow$ The Viscous sublayer, $y^+ < 5$.

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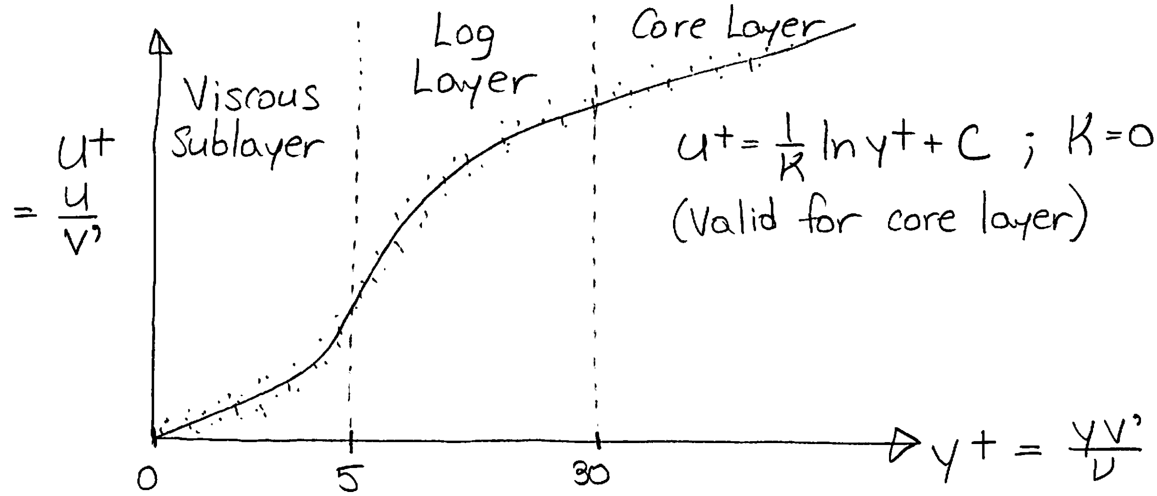
Note people have plotted and measured these results experimentally:

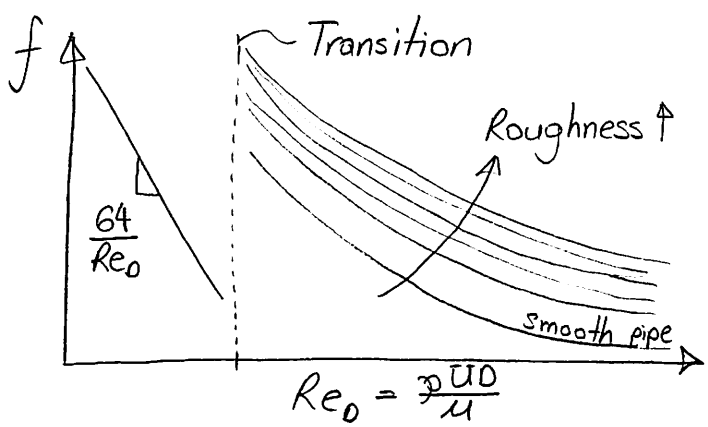


$u^+ = \frac{1}{K}\ln y^+ + C$; $K = 0.41$
(Valid for core layer)

$$\boxed{\begin{array}{ll} 0 < y^+ < 5 & u^+ = y^+ \\ 5 < y^+ < 30 & u^+ = 5.0 \ln y^+ - 3.05 \\ y^+ > 30 & u^+ = 2.5 \ln y^+ + 5.5 \end{array}}$$

$\Rightarrow$ Experimentally Determined. (Note, core layer $K = 0.41$)

But we as engineers care about pressure drop. For this, we cannot solve analytically due to turbulence, but we can solve experimentally. (Moody chart)



See page 368 of Mills

Good correlations to remember are: $\boxed{f = 0.184\, Re_D^{-0.2}}$ $\to 4.0 \times 10^4 < Re_D < 10^5$

$\boxed{f = (0.790 \ln Re_D - 1.64)^{-2}} \Rightarrow 10^4 < Re_D < 10^6$

$f = \dfrac{\Delta P}{(L/D)\frac{1}{2}\rho \bar{u}^2}$, $C_f = \dfrac{\tau_0}{\frac{1}{2}\rho \bar{u}^2}$, $C_f = \dfrac{f}{4}$ (for smooth pipes)

END OF LECTURE 17

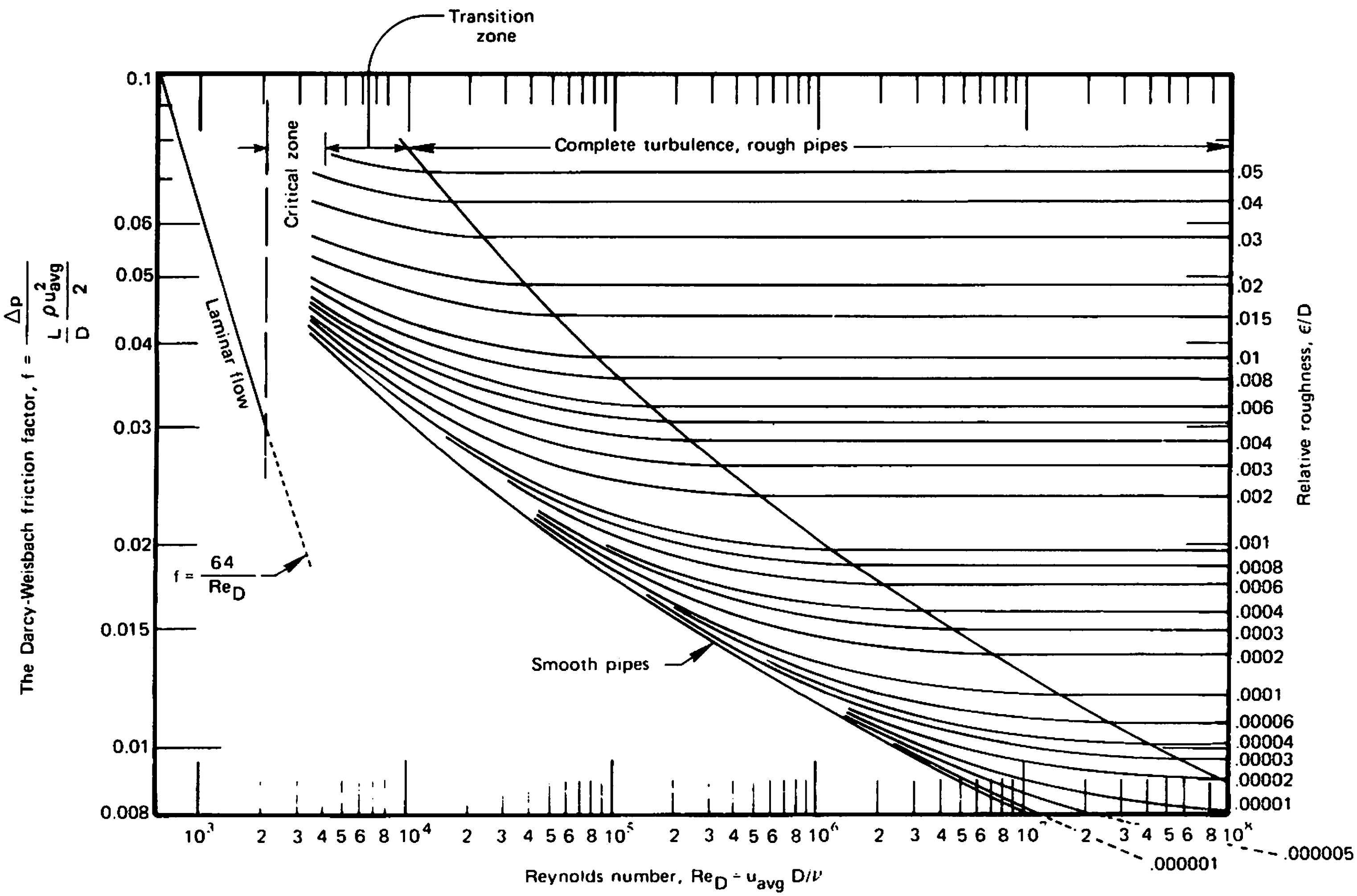


Figure 7.6 Pipe friction factors.

*Adapted from Lienhard & Lienhard, "A Heat Transfer Textbook"