\[ f(t_0) = \frac{1}{c} \int_{t_0 - \frac{v}{c}}^{t_0 + \frac{v}{c}} f(t) \, dt \]  
(Definition of \( f \))

Note also that: \( \overline{f f''} = \overline{f f''} = 0 \Rightarrow \overline{f f''} = 0 \)

Now if we focus on velocity, i.e., \( f = \frac{\dot{u}}{u} \), \( \overline{f} = \overline{\frac{\dot{u}}{u}} \left\{ \begin{array}{l} \dot{u} = \overline{u} + u' \\ f'' = u'' \end{array} \right\}

\[ \text{m} \text{omentum passing plane} \]

Y \[ \overline{u} \]

\[ \text{Wall} \rightarrow \overline{u} \text{ and } u \]

The average shear stress is written as:

\[ \overline{c} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u} \overline{v} \]

Laminar Component
Turbulent Component

(momentum leaving \( \xi \))

\[ \text{Aside: (Laminar flow)} \]

Y \[ \xi \]

\[ \mu \frac{\partial u}{\partial y} = \mu \frac{\partial u}{\partial y} \]

\[ \text{Control volume on a fluid element inside the boundary layer, momentum balance.} \]

\[ \text{See page 100 of notes.} \]

Note we typically write \( \overline{c} = \mu \frac{\partial u}{\partial y} \) for laminar fully developed flow over a flat plate since we evaluate \( \frac{\partial u}{\partial y} \) at the wall, where \( v = 0 \). Also, in the fluid, \( v^2 = 0 \), and \( v << \overline{u}, \overline{v} \approx 0 \).

\[ \overline{c} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u} \overline{v} \]

\[ \overline{u} = \overline{u} + u' \]
\[ v = \overline{v} + v' \]

For fully developed flow, \( \overline{v} = 0 \), \( \overline{v'} = 0 \)

\[ \overline{uv} = \frac{(\overline{u} + u')(\overline{v} + v')}{\overline{u} + u'} = \overline{uv} + \overline{u}v' + \overline{u'v} + \overline{v'}v' = \overline{uv} + \overline{u}v' + \overline{u'v} + \overline{v'}v' \]

\[ \overline{uv} = \overline{uv} + \overline{u}v' \Rightarrow \text{Not trivial to calculate, but we can model} \]

Note, \( \overline{u^2} = 0 \) and \( \overline{v^2} = 0 \), however \( \overline{u'v'} \neq 0 \).
Note also, $\bar{V} \ll \bar{U}$, and $\nabla \approx \bar{V}$ (for boundary layer flow) \[
\frac{\bar{u}^2 \bar{v}^2}{\bar{u} \bar{v}} \gg 1, \text{ so we can assume } \bar{u} \nabla \approx 0 \text{ since } \nabla \approx 0
\]
We see from before that: (Using some intuition)
\[
\bar{u}^2 \bar{v}^2 \rightarrow 0 \text{ as } \frac{\partial \bar{u}}{\partial y} \rightarrow 0 \text{ (Outside the b.l.)}
\]
\[
\bar{u}^2 \bar{v}^2 \uparrow \text{ as } \frac{\partial \bar{u}}{\partial y} \uparrow \text{ (Shear at the wall is highest)}
\]
To build a useful model by assuming $\bar{u}^2 \bar{v}^2$ is proportion to $\frac{\partial \bar{u}}{\partial y}$
\[
\bar{C} = \mu \frac{\partial \bar{u}}{\partial y} - \rho \bar{v} \bar{u}
\]
\[
= \mu \frac{\partial \bar{u}}{\partial y} - \left( \text{factor which reflects turbulent mixing} \right) \frac{\partial \bar{u}}{\partial y}
\]
\[
= -\rho \varepsilon
\]

or
\[
\bar{C} = \rho (\bar{u} + \varepsilon) \frac{\partial \bar{u}}{\partial y} \quad \Rightarrow \quad \varepsilon = \text{eddy diffusivity} \left[ \frac{m^2}{s} \right]
\]

This model is often used in computational fluid dynamics (CFD) software packages to deal with turbulent flows.

We can also re-write this as:
\[
\bar{C} = \rho \left( \bar{u} + \rho \varepsilon \right) \frac{\partial \bar{u}}{\partial y}
\]
\[
\mu = \mu_t \Rightarrow \text{Turbulent viscosity}
\]
Looking at our experimental velocity profiles:

Laminar $\frac{\partial \bar{u}}{\partial y}_\text{Lam}$

Turbulent $\frac{\partial \bar{u}}{\partial y}_\text{Turb}$
Since \( \tau = \rho u'v' \)

\[ u', v' \sim \sqrt{\frac{\tau}{\rho}} \Rightarrow \tau \approx \text{Turbulent shear stress or Reynolds stress} \]

We can now relate our viscosity to mixing length, \( \ell \), by creating an analogy with molecular viscosity.

![Diagram](image)

We know that shear stress is the time rate of change of momentum. Looking at our plane of interest, and considering the rate of molecular motion across the plane:

Assuming:
1) \( n \) molecules per unit volume
2) \( \frac{1}{3} \) of molecules have velocities along the \( y \)-dir.
3) Half of these, or \( \frac{1}{6} n \) molecules per unit volume have mean velocity \( \overline{v} \) in the +\( y \)-direction; the other half have a mean velocity \( -\overline{v} \) in the -\( y \) direction.

From this, we can say at any given time there are \( \frac{1}{6} n \overline{v} \) molecules that cross our plane from below, and \( \frac{1}{6} n \overline{v} \) from above.

Also, molecules that cross have, on average, experienced their last collisions at a distance \( \lambda \) (\( \lambda = \text{mean free path} \)) from the plane.
Since the mean velocity \( u_x = u_x(y) \) is a function of \( y \), we can write a balance equation for momentum.

The mean \( x \)-component of momentum transported per unit time per unit area across the plane upwards \( = \frac{1}{\delta} n \overline{\nu} [m u_x(y - \lambda)] \) \( \Rightarrow \) Taylor series exp.

The mean \( x \)-component of momentum transported per unit time per unit area across the plane downwards \( = \frac{1}{\delta} n \overline{\nu} [m u_x(y + \lambda)] \) \( \Rightarrow \) HOT. terms drop

\[
\mathcal{C} = \frac{1}{\delta} n \overline{\nu} m \left[ u_x(y - \lambda) - u_x(y + \lambda) \right]
\]

\( \Rightarrow \) Note if you substitute for \( \overline{\nu} \) & \( \lambda \), you will see that \( u \neq f(\rho, \text{or} \theta) \)

Now back to our turbulence derivation:

\( u_t \sim \rho v_t \ell \), where \( v_t \) = turbulent or friction velocity

\[
\frac{u_t}{\rho} = \mathcal{E} = v_t \ell \sim \sqrt{\frac{C_o}{\rho}} \ell, \quad v_t \sim v \text{or} \ u
\]

\[
\frac{C_o}{\rho} = \mathcal{E} \frac{\partial u}{\partial y}
\]

\[
\frac{C_o}{\rho} = \sqrt{\frac{C_o}{\rho}} \ell \frac{\partial u}{\partial y} = \sqrt{\frac{C_o}{\rho}} Ky \frac{\partial u}{\partial y} \quad \text{(1)}
\]

\[
\ell = Ky, \quad K = 0.41 \Rightarrow \text{Von Karman constant (from experiment)}
\]

This says that turbulent eddies at a location \( y \) must be no bigger than the distance to the wall. Makes sense since eddies cannot cross into the wall.
If we further non-dimensionalize:

\[ u^+ = \frac{u}{\sqrt{\frac{\nu^+}{c_p}}} \]
\[ y^+ = \frac{y}{\sqrt{\frac{\nu^+}{c_p}}} \]
\[ 2u^+ = \frac{\partial u^+}{\partial y^+} \]
\[ 2y^+ = \frac{\nu^+ y^+}{\sqrt{\frac{\nu^+}{c_p}}} \]

Back substitute into 1

\[ \frac{c_p}{\nu} = Ky^+ \sqrt{\frac{\nu^+}{c_p}} \cdot \frac{\partial u^+}{\partial y^+} \cdot \sqrt{\frac{\nu^+}{c_p}} \cdot \sqrt{\frac{\nu^+}{c_p}} \cdot \frac{1}{u} \]
\[ 1 = \frac{y^+}{\sqrt{\frac{\nu^+}{c_p}}} \cdot Ky^+ \frac{\partial u^+}{\partial y^+} \]
\[ Ky^+ \frac{\partial u^+}{\partial y^+} = 1 \]
\[ \int \partial u^+ = \frac{1}{K} \int \frac{dy^+}{y^+} \]
\[ U^+ = \frac{1}{K} \ln y^+ + C \]

\[ \Rightarrow \text{The log layer (away from the wall)} \]

Note, very near the wall, we can say that the turbulent eddies are very small since \( \ell = Ky \). So:

\[ \ell \to 0, \ \text{and} \ \varepsilon = \nu \ell \to 0 \Rightarrow \text{Viscosity dominates Turbulent viscosity} \to 0 \]

\[ c_p = \mu \frac{\partial u}{\partial y} \approx \mu \frac{u}{y} \]
\[ \approx \rho u \frac{u}{y} \to \frac{c_p}{\nu} = \nu \frac{u}{y} \to \frac{\nu^+}{c_p} \cdot \frac{y^+}{y} = \frac{u^+}{\sqrt{\frac{\nu^+}{c_p}}} \]
\[ U^+ = y^+ \]

\[ \Rightarrow \text{The viscous sublayer, } y^+ < 5. \]
Note people have plotted and measured these results experimentally:

\[ U^+ = \frac{U}{V^*} \]

Viscous Sublayer

Log Layer

Core Layer

\[ U^+ = \frac{1}{K} \ln y^+ + C \quad ; \quad K = 0.41 \]

(Valid for core layer)

\[ 0 < y^+ < 5 \quad U^+ = y^+ \]
\[ 5 < y^+ < 30 \quad U^+ = 5.0 \ln y^+ - 3.05 \]
\[ y^+ > 30 \quad U^+ = 2.5 \ln y^+ + 5.5 \]

\[ \Rightarrow \text{Experimentally Determined.} \]
\[ (\text{Note, core layer } K = 0.41) \]

But we as engineers care about pressure drop. For this, we cannot solve analytically due to turbulence, but we can solve experimentally. (Moody chart)

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\[ f = 0.184 \frac{Re_0}{\mu} \]

\[ f = (0.790 \ln Re_0 - 1.64)^{-2} \quad \Rightarrow 10^4 < Re_0 < 10^6 \]

\[ f = \frac{\Delta P}{\rho \frac{1}{2} u^2} \quad , \quad C_f = \frac{C_0}{\frac{1}{2} \rho u^2} \quad , \quad C_f = \frac{f}{4} \quad (\text{for smooth pipes}) \]

\[ \Rightarrow 4 \times 10^4 \leq Re_0 \leq 10^5 \]
Figure 7.6 Pipe friction factors.

*Adapted from Lienhard & Lienhard, "A Heat Transfer Textbook"