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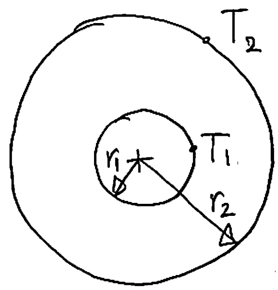
③ Spherical System



Using our spherical heat equation

$$\cancel{\frac{1}{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0$$

$$\int \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \int 0$$

$$r^2\frac{\partial T}{\partial r} = C_1 \Rightarrow \int \frac{\partial T}{\partial r} = \int \frac{C_1}{r^2}$$

$$T = -\frac{C_1}{r} + C_2$$

Our boundary conditions are:
$$T_1 = -\frac{C_1}{r_1} + C_2 \quad ①$$
$$T_2 = -\frac{C_1}{r_2} + C_2 \quad ②$$

① - ②

$$C_1 = \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

$$C_2 = T_2 + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\cdot\frac{1}{r_2}$$

$$T = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}\cdot\frac{1}{r} + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\cdot\frac{1}{r_2} + T_2$$

$$\boxed{\frac{T - T_2}{T_1 - T_2} = \frac{\left(\frac{1}{r} - \frac{1}{r_2}\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}} \Rightarrow \text{Spherical temperature profile}$$

Calculating our thermal resistance:

$$Q = -kA\frac{\partial T}{\partial r} \quad ; \quad \frac{\partial T}{\partial r} = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}\left(-\frac{1}{r^2}\right)$$

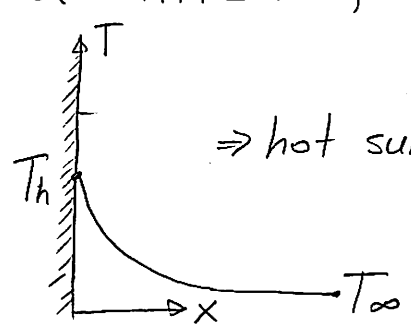
$$= -k(4\pi \cancel{r^2})\frac{T_1 - T_2}{\frac{1}{r_1} - \frac{1}{r_2}}\left(-\frac{1}{\cancel{r^2}}\right)$$

$$Q = \frac{T_1 - T_2}{\left\{\frac{\frac{1}{r_1} - \frac{1}{r_2}}{4\pi k}\right\}} = \frac{\Delta T}{R_{sph}}$$

$$\boxed{R_{sph} = \frac{\left|\frac{1}{r_1} - \frac{1}{r_2}\right|}{4\pi k}}$$ ⇒ Spherical thermal resistance.

We can do the same thermal resistance analogy for convection heat transfer:

$Q = hA\Delta T$; h = heat transfer coefficient $[W/m^2 \cdot K]$
A = area



⇒ hot surface exchanging heat with a fluid
Newton developed this concept in the 1700's. Known as Newton's law of cooling.

$$Q = hA\Delta T = \frac{\Delta T}{(1/hA)} \Rightarrow \boxed{R_{Th,conv} = \frac{1}{hA}}$$

↳ Thermal resistance associated with convection

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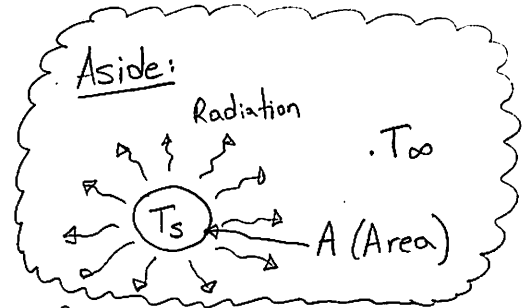


Radiation thermal resistance:

$$Q_{rad} = \varepsilon \sigma A \underbrace{(T_s^4 - T_\infty^4)}_{\text{Factoring}: (T_s^2 + T_\infty^2)(T_s + T_\infty)(T_s - T_\infty)}$$ ⇒ Stefan Boltzmann Law (Black Body)

Emissivity (ε)

Stefan-Boltzmann Constant (5.67×10^{-8}) (σ)

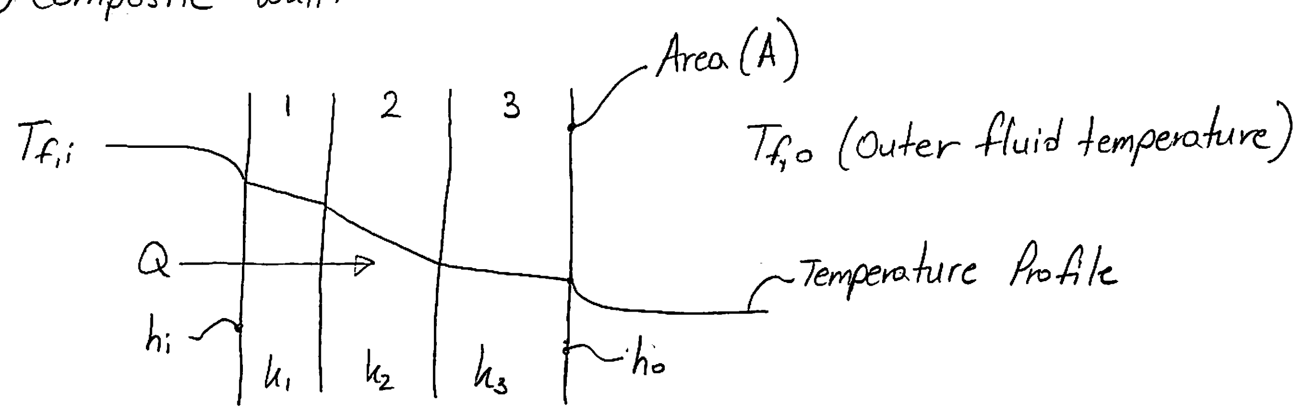
$$Q_{rad} = \underbrace{\varepsilon \sigma A (T_s^2 + T_\infty^2)(T_s + T_\infty)}_{h_{rad} \cdot A} \underbrace{(T_s - T_\infty)}_{\Delta T}$$

$$\boxed{h_{rad} = \varepsilon \sigma (T_s^2 + T_\infty^2)(T_s + T_\infty)}$$ ⇒ Radiation heat transfer coefficient.

$$\boxed{R_{Radiation} = \frac{1}{h_{rad} A}}$$ ⇒ Thermal resistance associated with radiation. (nonlinear)

Composite problems

① Composite wall:



$$\boxed{Q = \frac{A(T_{f,i} - T_{f,o})}{\frac{1}{h_i} + \frac{1}{h_o} + \sum_{j=1}^{n} \frac{L_j}{k_j}}}$$,

L_j = thickness of the j'th wall

k_j = thermal conductivity of the j'th wall

h_i = inner wall heat transfer coeff.

h_o = outer wall h.t.c.

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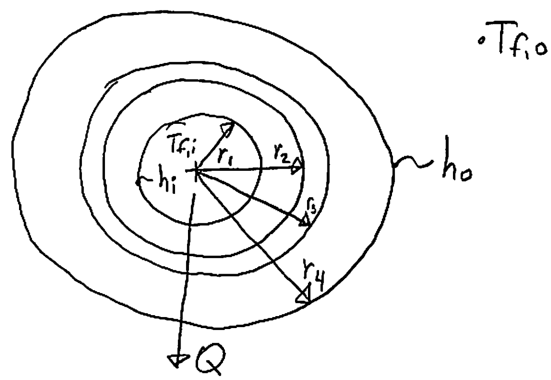
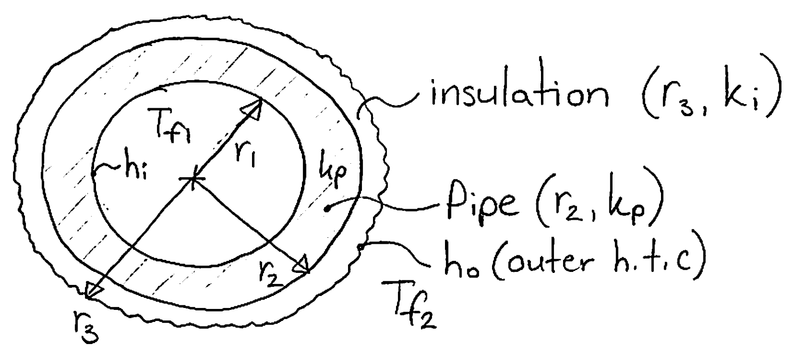
② Composite cylinder



$$\boxed{Q = \frac{2\pi l\,(T_{f,i} - T_{f,o})}{\dfrac{1}{h_i r_i} + \dfrac{1}{h_o r_{n+1}} + \displaystyle\sum_{j=1}^{n} \dfrac{\ln(r_{j+1}/r_j)}{k_j}}}$$

Critical Thickness of Insulation:



k_i = insulation thermal c.
k_p = pipe thermal cond.
T_{f_1} = inner fluid temp.
T_{f_2} = outer fluid temp.
h_o = outer h.t.c.
$r_1 < r_2 < r_3$

Note, this is an optimization problem since by adding insulation:

1) The outer area, for which h_o convects heat away increases $\Rightarrow Q_{loss} \uparrow$

2) The resistance of the insulation increases $\Rightarrow Q_{loss} \downarrow$

$$R_{TOT} = \underbrace{\frac{1}{2\pi r_1 h_i}}_{\text{inner convection}} + \underbrace{\frac{\ln(r_2/r_1)}{2\pi k_p L}}_{\text{pipe cond.}} + \underbrace{\frac{\ln(r_3/r_2)}{2\pi k_i L}}_{\text{insulation cond.}} + \underbrace{\frac{1}{2\pi r_3 h_o L}}_{\text{outer convection}}$$

We need to maximize R_{TOT} with respect to r_3

$$\frac{\partial R_{TOT}}{\partial r_3} = 0$$

$$\frac{1}{\cancel{2\pi} k_i \cancel{L}} \frac{\partial}{\partial r_3} \ln(r_3/r_2) + \frac{1}{\cancel{2\pi} h_o \cancel{L}} \frac{\partial}{\partial r_3}\left(\frac{1}{r_3}\right) = 0$$

$$\frac{1}{k_i}\frac{\partial}{\partial r_3}\ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_o}\frac{\partial}{\partial r_3}\left(\frac{1}{r_3}\right) = 0$$

$$\frac{1}{k_i}\cdot\frac{\cancel{r_2}}{\cancel{r_3}\cdot\cancel{r_2}} + \frac{1}{h_o}\left(-\frac{1}{r_3^{\cancel{2}}}\right) = 0$$

$$\frac{1}{k_i} - \frac{1}{h_o r_3} = 0$$

$$\boxed{r_{3,crit} = \frac{k_i}{h_o}}$$ $\Rightarrow$ Critical thickness of insulation for a pipe.

Now we need to check to make sure this is a minimum for Q_{TOT}

$$\frac{\partial^2 R_{TOT}}{\partial r_3^2} = \left[\frac{\partial}{\partial r}\left(\frac{1}{k_i r_3}\right) - \frac{\partial}{\partial r}\left(\frac{1}{h_o r_3^2}\right)\right]\frac{1}{2\pi L}$$

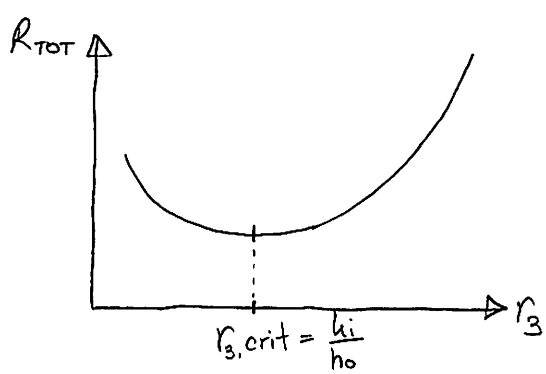
$$= \left(-\frac{1}{k_i r_3^2} + \frac{2}{h_o r_3^3}\right)\frac{1}{2\pi L}$$

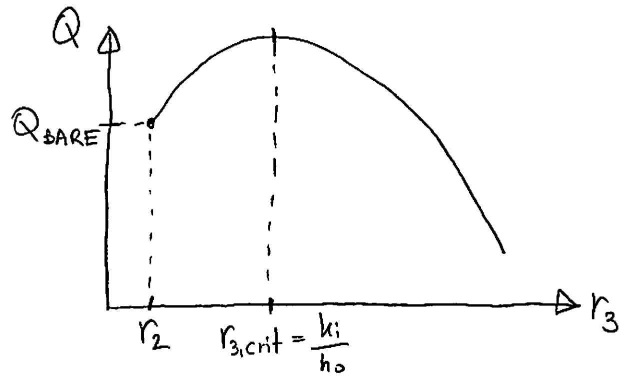
Substitute in $r_{3,crit} = \frac{k_i}{h_o}$ into equation 1

$$= -\frac{1}{k_i\left(\frac{k_i^2}{h_o^2}\right)} + \frac{2}{\cancel{h_o}\left(\frac{k_i^3}{h_o^{\cancel{3}}}\right)} = \left(-\frac{h_o^2}{k_i^3} + \frac{2h_o^2}{k_i^3}\right)\cdot\frac{1}{2\pi L}$$

$$= +\frac{h_o^2}{2\pi k_i^3} > 0 \quad \text{(Always)}$$

So our $R_{TOT}(r_{3,cr})$ is a global minimum.
No optimum thickness exists, only a critical insulation thickness.





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If we do a sample calculation

$$\left.\begin{aligned} k_i &= 0.1\ W/m\cdot K \\ h_o &= 5\ W/m^2\cdot K \end{aligned}\right\} r_{3,crit} = \frac{0.1\ \cancel{W}/\cancel{m}\cdot \cancel{K}}{5\ \cancel{W}/m^{\cancel{2}}\cdot \cancel{K}} = 0.02\,m = 2\,cm$$

this is a radius!

Note, here $r_{3,crit} < r_2$ meaning it's OK to insulate.

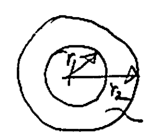
In general: $k_i \sim 0.1\ W/m\cdot K$ (common insulating materials)

$h_{o,minimum} \sim 5\ W/m^2 K$ (natural convection)

$r_{crit} \sim 1\,cm$ $\Rightarrow$ We usually operate well above this.

$\therefore$ We can insulate hot water and steam pipes without worrying about increasing external heat transfer losses.

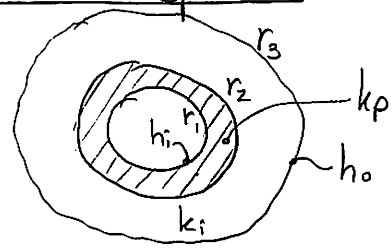
How about electrical wires? $\Rightarrow$



$\Rightarrow r_1 \sim 1\,mm$

Good to insulate to increase losses & cool the wire.

<u>For a sphere:</u>



$r_1 < r_2 < r_3$

$$R_{TOT} = \frac{1}{4\pi r_1^2 h_i} + \frac{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}{4\pi k_p} + \frac{\left(\frac{1}{r_2} - \frac{1}{r_3}\right)}{4\pi k_i} + \frac{1}{4\pi r_3^2 h_o}$$

$$\frac{\partial R_{TOT}}{\partial r_3} = \frac{1}{\cancel{4\pi} k_i}\left(\frac{1}{r_3^2}\right) + \frac{1}{\cancel{4\pi} h_o}\left(-\frac{2}{r_3^3}\right) = 0$$

$$\frac{1}{r_{3,cr}^{\cancel{2}} k_i} - \frac{2}{h_o r_{3,cr}^{\cancel{3}}} = 0 \Rightarrow \boxed{r_{3,cr} = \frac{2k_i}{h_o}}$$

Critical insulation thickness for a sphere.

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Heat Generation (Slab)

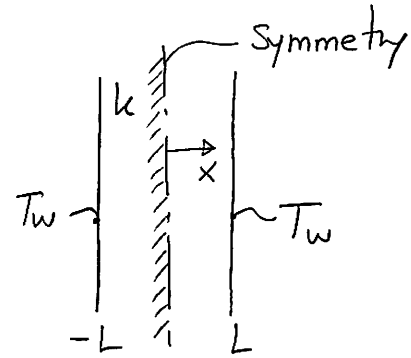
Steady State, 1D, constant properties, Uniform $\dot{Q}'''$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \cancelto{1D}{\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)} + \cancelto{1D}{\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)} + \dot{Q}''' = \cancelto{S.S.}{\rho c_p\frac{\partial T}{\partial t}}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{Q}'''}{k} = 0 \quad (1)$$

$$T(x=L) = T_w$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$



Integrating (1)

$$\frac{\partial T}{\partial x} = -\frac{\dot{Q}'''}{k}x + C_1$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 = -\frac{\dot{Q}'''}{k}(0) + C_1 = 0 \Rightarrow C_1 = 0$$

$$T(x) = -\frac{\dot{Q}'''x^2}{2k} + C_2$$

$$T(x=L) = T_w = -\frac{\dot{Q}'''L^2}{2k} + C_2 \Rightarrow C_2 = T_w + \frac{\dot{Q}'''L^2}{2k}$$

$$\boxed{T(x) = T_w + \frac{\dot{Q}'''L^2}{2k}\left(1-\left(\frac{x}{L}\right)^2\right)}$$

Where is T_{max}?

$$\frac{\partial T}{\partial x} = -\frac{\dot{Q}'''x}{k} = 0 \Rightarrow x = 0 \Rightarrow \frac{\partial^2 T}{\partial x^2} = -\frac{\dot{Q}'''}{k} < 0 \quad (T \text{ is a max at } x=0)$$

$$\boxed{T_{max} = T_w + \frac{\dot{Q}'''L^2}{2k}}$$

$$q'' = q(L) = -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = +\cancel{k}\frac{\dot{Q}'''L}{\cancel{k}} = \dot{Q}'''L$$

$$\boxed{q''_{OUT} = \dot{Q}'''L}$$

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In general, for heat generation we have a formulation:

$$-\frac{k}{r^n}\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right) = \dot{Q}''' \quad \begin{cases} n=0 & (\text{Slab}) \\ n=1 & (\text{Cylinder}) \\ n=2 & (\text{Sphere}) \end{cases}$$

This only works if $\dot{Q}''' = \text{constant}$

$$-k\frac{\partial}{\partial r}\left(r^n\frac{\partial T}{\partial r}\right) = \dot{Q}''' r^n$$

$$-kr^n\frac{\partial T}{\partial r} = \dot{Q}'''\int r^n dr = \frac{\dot{Q}'''}{n+1}r^{n+1} + C_1$$

$$q'' = -k\frac{\partial T}{\partial r} = \frac{\dot{Q}''' r}{n+1} + \frac{C_1}{r^n}$$

Choosing $q''(0) = 0 \rightarrow C_1 = 0$ for $n=0$
For $n = 1, 2$, $C_1 = 0$ if $|q''| < \infty$ (since $\frac{C_1}{r^1}$ or $\frac{C_1}{r^2}$ at $r \to 0$ blows up)

$$\frac{\partial T}{\partial r} = -\frac{\dot{Q}''' r}{(n+1)k}$$

$$T = -\frac{\dot{Q}'''}{(n+1)k}\int r\,dr = C_2 - \frac{\dot{Q}''' r^2}{2(n+1)k}$$

Letting $T(a) = T_a \Rightarrow T_a = C_2 - \frac{\dot{Q}''' a^2}{2(n+1)k} \Rightarrow C_2 = T_a + \frac{\dot{Q}''' a^2}{2(n+1)k}$

So our generalized result is:

$$\boxed{T(r) = T_a + \frac{\dot{Q}''' a^2}{2(n+1)k}\left(1 - \frac{r^2}{a^2}\right)} \Rightarrow \text{Temperature profile}$$

$$\boxed{q'' = \frac{\dot{Q}''' r}{n+1}} \Rightarrow \text{Heat flux}$$

$$\boxed{T_{max} = T(r=0) = T_a + \frac{\dot{Q}''' a^2}{2(n+1)k}} \Rightarrow \text{Maximum temperature}$$

where T_a is the boundary temperature on the outside.

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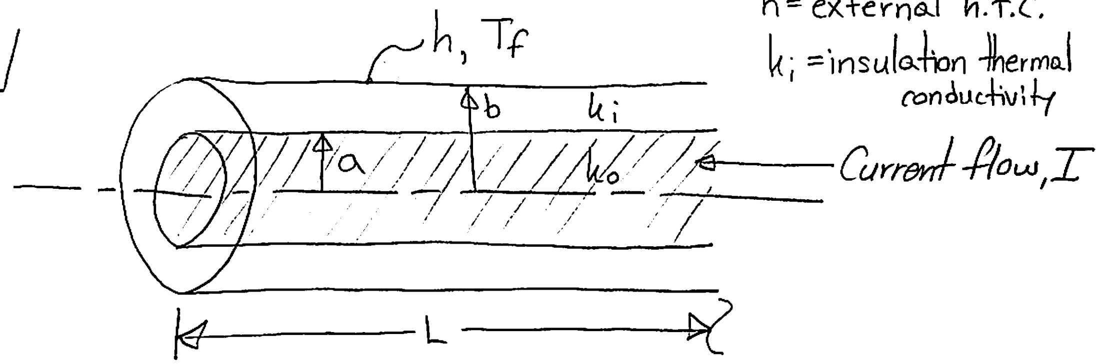
k_o = wire thermal cond.
h = external h.t.c.
k_i = insulation thermal conductivity

Example #3

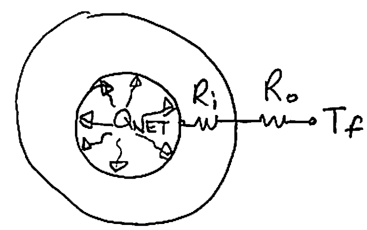


We have the wire electrical resistance, R_e, and the current, I, what is T_{max}?

$$\text{Energy loss (Joule Heating)} = I^2 R_e$$

$$\dot{Q}''' = \frac{I^2 R_e}{\pi a^2 L}$$

$$Q_{NET} = \pi a^2 L \dot{Q}''' = \frac{\Delta T}{R_{TOT}}$$



$$\pi a^2 L \dot{Q}''' = \frac{T_a - T_f}{\frac{\ln(b/a)}{2\pi L k_i} + \frac{1}{2\pi b L h}}$$

$$T_a = T_f + \frac{a^2 \dot{Q}'''}{2}\left[\frac{\ln(b/a)}{k_i} + \frac{1}{bh}\right] \quad (1)$$

We just learned that for a cylinder $(n=1)$

$$T_{max} = T_a + \frac{\dot{Q}''' a^2}{4k} \quad (2)$$

Substitute (1) into (2)

$$\boxed{T_{max} = T_f + \frac{a^2 \dot{Q}'''}{2}\left[\frac{1}{2k_o} + \frac{\ln(b/a)}{k_i} + \frac{1}{bh}\right]}$$

END OF LECTURE 4