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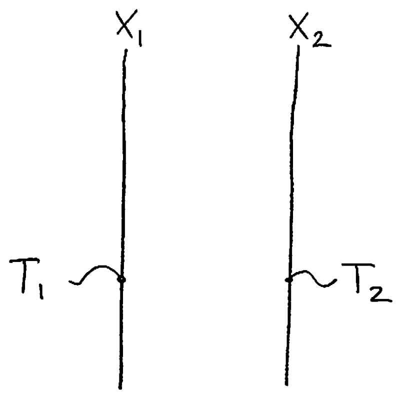
Wall with Variable Thermal Conductivity



1-D, S.S., no generation

When k weakly depends on T,

$k = a + bT \Rightarrow$ units $a \Rightarrow \frac{W}{m \cdot K}$

$b \Rightarrow \frac{W}{m \cdot K^2}$

No need to use heat equation, even simpler

$$q_x = -k \frac{\partial T}{\partial x} \Rightarrow \text{Integrate}$$

$$\int_{x_1}^{x_2} q_x \, dx = -\int_{T_1}^{T_2} (a + bT) \, dT$$

We know that the heat flux is independent of location, (from the conservation of energy)

$$q_x (x_2 - x_1) = -a(T_2 - T_1) - \frac{b(T_2^2 - T_1^2)}{2} \Rightarrow \text{Factor}$$

$$q_x \Delta x = -\Big(a + b \underbrace{\frac{T_2 + T_1}{2}}_{\overline{\overline{T}}}\Big)(T_2 - T_1)$$

So: $q_x \Delta x = -(a + b\overline{T}) \Delta T$

Rearanging, an interesting interpretation evolves

$$q_x = -\underbrace{(a + b\overline{T})}_{\overline{\overline{k}}} \frac{\Delta T}{\Delta x}$$

$$\boxed{q_x = -\overline{k} \frac{\Delta T}{\Delta x}}$$

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For the temperature distribution:

$$q_x = -k\frac{\partial T}{\partial x} = -(a+bT)\frac{\partial T}{\partial x}$$

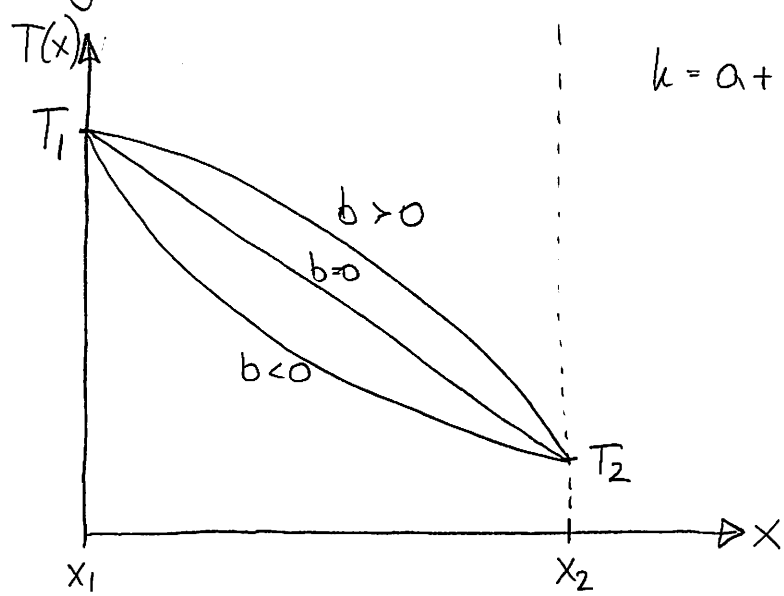
$$\int_{x_1}^{x} q_x\,dx = -\int_{T_1}^{T}(a+bT)\,dT$$ (Note bounds of integration are different)

$$q_x(x-x_1) = -a(T(x)-T_1) - \frac{b(T(x)^2-T_1^2)}{2}$$ ⇒ Quadratic equat.

Solving our quadratic equation above:

$$\boxed{T(x) = -\frac{a}{b} \pm \left\{\left(\frac{a}{b}\right)^2 - \frac{2}{b}q_x(x-x_1) - T_1^2 - 2\frac{a}{b}T_1\right)\right\}^{1/2}}$$

Plotting our result:



$k = a + bT$

Very usefull solution since many materials have a thermal conductivity temperature dependence that can be well approximated with a linear temperature dependent profile.

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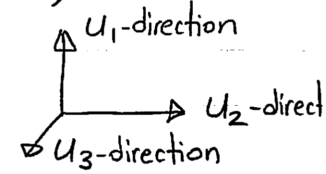
Conduction for General Shapes (An easier approach)

$\vec{V} = 0$ (no convection)

Steady State, $\dot{Q}''' = 0$, k = constant, 1-D



Define an orthogonal coordinate system u_1, u_2, u_3 (directions)

s_1, s_2, s_3 (lengths)

$ds_1 = h_1 du_1$

$ds_2 = h_2 du_2$

$ds_3 = h_3 du_3$

Note, we'll come to this later. and reconcile.

$$q'' = -k\frac{\partial T}{\partial s_1} \quad (\vec{q}'' = -k\vec{\nabla}T)$$

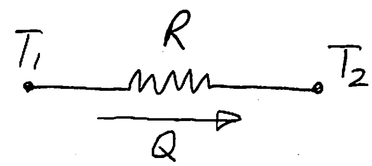
$$Q = Aq'' = -kA(u_1)\frac{\partial T}{\partial s_1}$$

Rearanging and integrating (assuming $ds_1 = du_1$)

$$Q\frac{1}{k}\int_{(u_1)_a}^{(u_1)_b}\frac{du_1}{A(u_1)} = -\int_{T_a}^{T_b} dT = T_a - T_b = \Delta T$$

$$Q = \frac{\Delta T}{\frac{1}{k}\int\frac{du_1}{A(u_1)}} = \frac{\Delta T}{R}$$

$$\boxed{R = \frac{1}{k}\int\frac{du_1}{A(u_1)}} \Rightarrow$$



Let's double check our previous solutions just to be sure:

① Slab; $u_1 = x$, $du_1 = dx$, $A(u_1)$ = constant

$$R_{slab} = \frac{1}{kA}\int du_1 = \frac{1}{kA}\int dx = \frac{L}{kA}$$

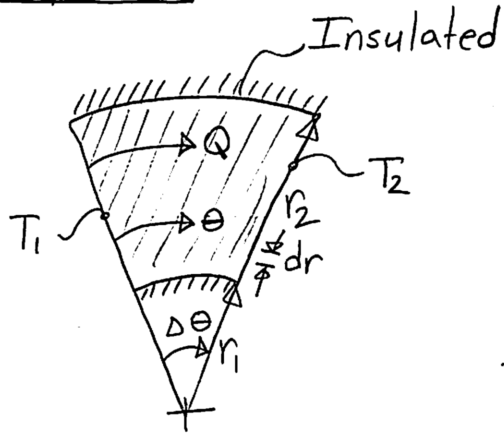
② Cylinder; $u_1 = r$, $du_1 = dr$, $A(u_1) = A(r) = 2\pi r l$

$$R_{cyl} = \frac{1}{k}\int_{r_1}^{r_2}\frac{dr}{2\pi r l} = \frac{1}{2\pi k l}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\ln(r_2/r_1)}{2\pi k l}$$

③ Sphere; $u_1 = r$, $du_1 = dr$, $A(u_1) = A(r) = 4\pi r^2$

$$R_{sph} = \frac{1}{k}\int_{r_1}^{r_2}\frac{dr}{4\pi r^2} = \frac{1}{4\pi k}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Example #4



$T = T(\theta)$, ℓ = thickness into page

Note, more complex since heat flux is also a function of r since heat flow through the region closer to the center happens easier than the outside (Lower length and thermal resistance)

Using our general formulation: $\overset{\text{direction}}{u_1 = \theta}$, $ds_1 = r\,d\theta$, $d(A(u_1)) = \ell\,dr$ } For radial Coordinates

$$dQ = -k\,d(A(u_1))\frac{\partial T}{\partial s_1} = -k\,d(A(u_1))\frac{1}{r}\cdot\frac{\partial T}{\partial u_1}$$

$$= -k\,\ell\,dr\frac{\partial T}{r\,d\theta} = -k\ell\cdot\frac{1}{r}\,dr\frac{\partial T}{\partial\theta}$$

$$\int dQ = \int -k\ell\underbrace{\frac{\partial T}{\partial\theta}}_{\text{Constant}}\cdot\frac{dr}{r}$$

$$Q = -k\ell\frac{\partial T}{\partial\theta}\int_{r_1}^{r_2}\frac{dr}{r} = -k\ell\frac{\partial T}{\partial\theta}\cdot\ln(r_2/r_1)$$

$$Q = -k\ell\ln(r_2/r_1)\frac{\partial T}{\partial\theta} = \underbrace{\frac{k\ell\ln(r_2/r_1)}{\Delta\theta}}_{R^{-1}}\cdot\Delta T$$

$$\boxed{R = \frac{\Delta\theta}{k\ell\ln(r_2/r_1)}}$$ ⇒ Thermal resistance of the arc

Let's check the bounds of our solution to see if it makes sense: What if $r_1 \approx r_2 = r_2 - \delta$

$$\ln\left(\frac{r_2}{r_1}\right) = \ln\left(\frac{r_2}{r_2-\delta}\right) = \ln\left(\frac{1}{1-\frac{\delta}{r_2}}\right) = \ln\left[\frac{1+\frac{\delta}{r_2}}{\left(1-\frac{\delta}{r_2}\right)\left(1+\frac{\delta}{r_2}\right)}\right]$$

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$$= \ln\left[\frac{1+\frac{\delta}{r_2}}{1+\cancel{\frac{\delta}{r_2}}-\cancel{\frac{\delta}{r_2}}-\frac{\delta^2}{r_2^2}}\right] \quad , \quad \text{since } \frac{\delta}{r_2} < 1$$

$$\frac{\delta^2}{r_2^2} << 1, \approx 0$$

$$= \ln\left(1+\frac{\delta}{r_2}\right)$$

Using series expansion:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \frac{(-x)^{N+1}}{N+1}, + \cdots$$

$$x = \frac{\delta}{r_2}$$

$$\ln\left(1+\frac{\delta}{r_2}\right) = \frac{\delta}{r_2} - \cancel{\frac{\delta^2}{2r_2^2}}^{\,0} + \cancel{\frac{\delta^3}{3r_2^3}}^{\,0} + \cdots$$

$$\ln\left(1+\frac{\delta}{r_2}\right) \approx \frac{\delta}{r_2}$$

So plugging back into our resistance calculation:

$$R = \frac{\Delta\Theta}{k\ell\ln(r_2/r_1)} = \underbrace{\frac{\Delta\Theta r_2}{k\ell\delta}}_{\frac{L}{kA}} \Rightarrow \begin{array}{l} \ell\delta = A \text{ (Area)} \\ \Delta\Theta r_2 = L \text{ (Length)} \end{array}$$

$\Rightarrow$ Works like a charm!

<u>General Orthogonal Coordinate System</u> (Pure Conduction)

Defining u_1, u_2, u_3 (3 coordinate dimensions)

$$\left.\begin{array}{l} ds_1 = h_1 du_1 \\ ds_2 = h_2 du_2 \\ ds_3 = h_3 du_3 \end{array}\right\} \text{For example} \Rightarrow \text{Cartesian} \quad \begin{array}{ll} h_1 = 1, & du_1 = dx \\ h_2 = 1, & du_2 = dy \\ h_3 = 1, & du_3 = dz \end{array}$$

We can generalize the following:

$$\boxed{q'' = -k\frac{1}{h_i}\cdot\frac{\partial T}{\partial u_i}}$$

$$Q = \iint q'' ds_2\, ds_3 = \iint q'' h_2 h_3\, du_2\, du_3$$

$$\boxed{Q = -k \frac{\partial T}{\partial u_1} \iint \frac{h_2 h_3}{h_1}\, du_2\, du_3}$$

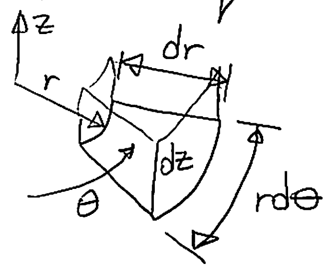
$$\boxed{R = \frac{1}{k} \int_{(u_1)_a}^{(u_1)_b} \frac{du_1}{\iint \frac{h_2 h_3}{h_1}\, du_2\, du_3}}$$

$\Rightarrow$ The most usefull & general form you'll learn. Very powerfull.

Let's check with two quick examples $\Rightarrow$ radial & spherical

① Radial:

$$\left.\begin{aligned} u_1 &= r \\ u_2 &= \theta \\ u_3 &= z \end{aligned}\right\}$$



$ds_1 = (1)\, dr \Rightarrow h_1 = 1$

$ds_2 = r\, d\theta \Rightarrow h_2 = r$

$ds_3 = (1)\, dz \Rightarrow h_3 = 1$

Back substituting into our formulation

$$Q = -k \frac{\partial T}{\partial r} \int_0^L \int_0^{2\pi} \frac{r\cancel{(1)}}{\cancel{(1)}}\, d\theta\, dz = -k \frac{\partial T}{\partial r} \int_0^L dz \int_0^{2\pi} r\, d\theta$$

$$= -k \underbrace{(2\pi r) L}_{A} \frac{\partial T}{\partial r} = -k A \frac{\partial T}{\partial r}$$

$$R = \frac{1}{k} \int_{r_1}^{r_2} \frac{dr}{\int_0^L \int_0^{2\pi} \frac{r(1)}{(1)}\, d\theta\, dz}$$

$$= \frac{1}{k} \int_{r_1}^{r_2} \frac{dr}{r \int_0^L dz \int_0^{2\pi} d\theta} = \frac{1}{2\pi k L} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\ln(r_2/r_1)}{2\pi k L} = R_{cyl}$$

We see that the formalism works. Try it for spherical.

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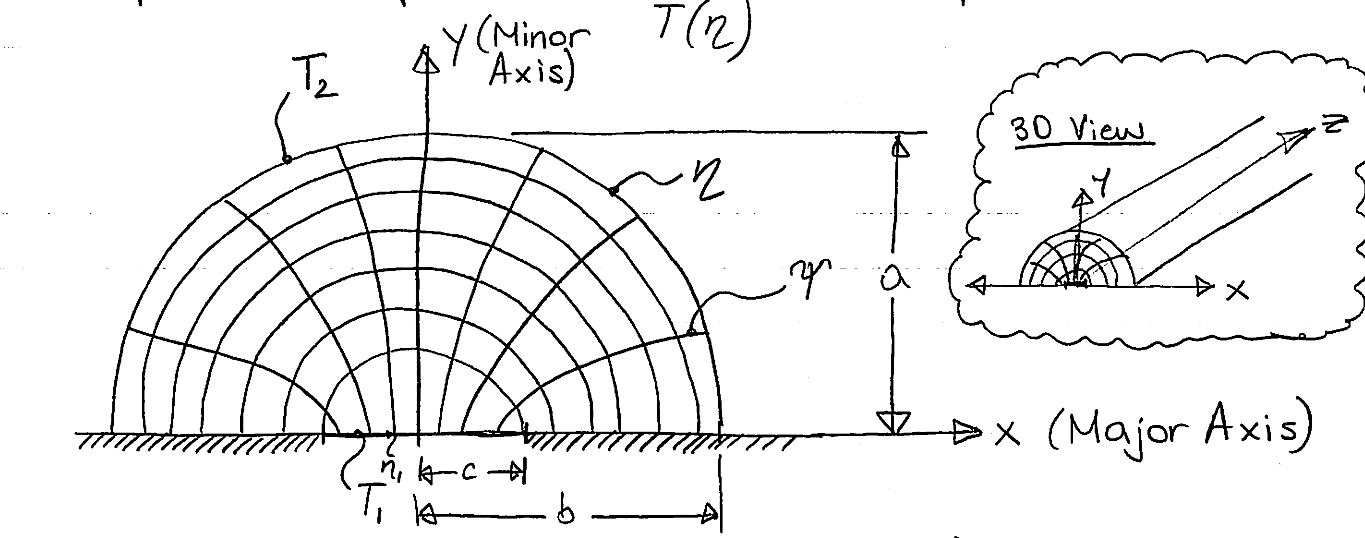
Example #5 Ellipse (Foci located 2c apart)



$u_1 = \eta \quad h_1$ = unknown yet (we will solve for it)
$u_2 = \psi \quad h_2$ = unknown yet
$u_3 = z, \quad h_3 = 1$

First we need to solve for h_1 & h_2. We can use some nice rules & identities for ellipses:

Rule #1: Coordinate Transformation:

$$b^2 - a^2 = c^2 \Rightarrow \begin{array}{ll} x = c\cosh\eta\cos\psi & ; \; b = c\cosh\eta_1 \\ y = c\cosh\eta\sin\psi & ; \; a = c\sinh\eta_1 \end{array}$$

$$(b^2 - a^2) = c^2 \underbrace{(\cosh^2\eta_1 - \sinh^2\eta_1)}_{1} \Rightarrow \text{so } b^2 - a^2 = c^2 \text{ (we are OK!)}$$

Some identities we'll need are

$$\cosh\eta = \frac{e^{\eta} + e^{-\eta}}{2}$$

$$\sinh\eta = \frac{e^{\eta} - e^{-\eta}}{2}$$

$\vec{r} = x\vec{i} + y\vec{j}$ (Coordinate Transformation) $\Rightarrow$

$$\left.\begin{array}{l} \left|\dfrac{d\vec{r}}{d\eta}\right| = h_1 = c\left(\sinh^2\eta\cos^2\psi + \cos^2\eta\sin^2\psi\right)^{1/2} \\ \left|\dfrac{d\vec{r}}{d\psi}\right| = h_2 = c\left(\cosh^2\eta\sin^2\psi + \sin^2\eta\cos^2\psi\right)^{1/2} \end{array}\right\} h_1 = h_2$$

This makes our lives a lot easier:

$$R = \frac{1}{k}\int_0^{\eta_1} \frac{d\eta}{\int_0^{\pi}\int_0^{L} d\varphi\, dz} = \frac{\eta_1}{\pi k L}$$

We know that: $\frac{a+b}{c} = \cosh \eta_1 + \sinh \eta_1$

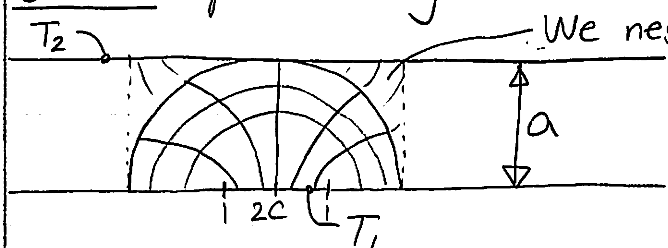
$$= \frac{e^{\eta_1}+e^{-\eta_1}}{2} + \frac{e^{\eta_1}-e^{-\eta_1}}{2} = e^{\eta_1}$$

$$\eta_1 = \ln\left(\frac{a+b}{c}\right)$$

So our solution becomes: $\boxed{Q = \frac{\Delta T \pi k L}{\ln\left(\frac{a+b}{c}\right)}}$ $\boxed{R = \frac{\ln\left(\frac{a+b}{c}\right)}{\pi k L}}$

Note, this solution is extremely usefull since it can be used to solve the following cases:

Case 1: Spot Welding

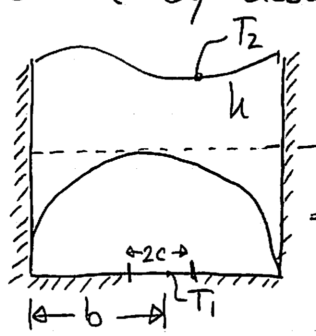


We neglect this resistance. But because spreading resistance here is high, safe approximation.

$$b^2 = \sqrt{a^2+c^2}$$

Using the above resistance puts an upper bound on our heat transfer Q by assuming cylindrical spreading.

Case 2:



$\Rightarrow R_1 = \frac{L}{kA}$

$\Rightarrow R_c = \frac{\ln\left(\frac{a+b}{c}\right)}{\pi k \ell} \Rightarrow a = \sqrt{b^2-c^2}$

Broken down into 2 resistances. Good approximation.

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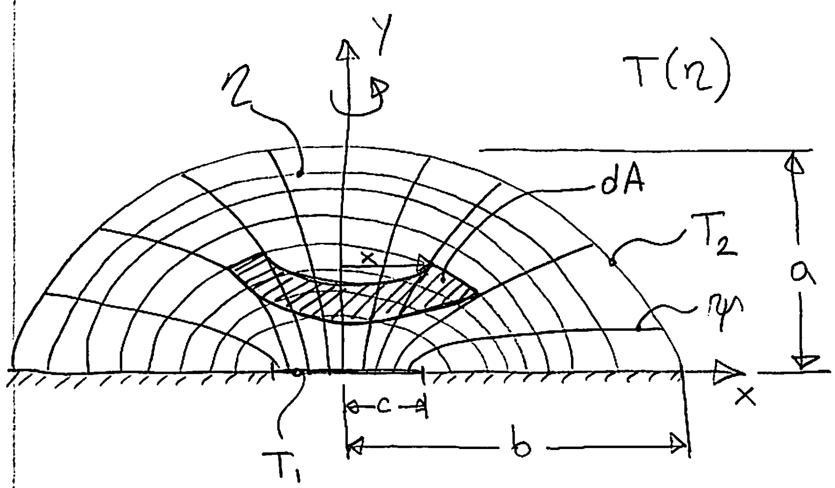
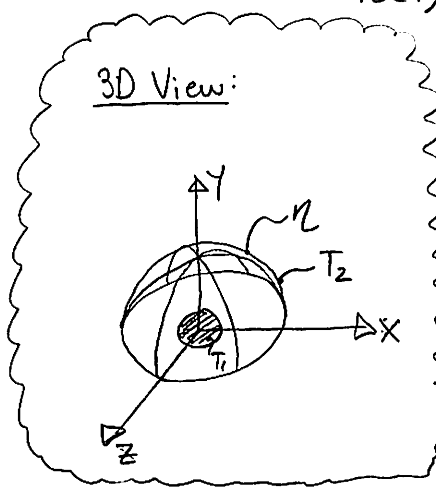
Revolved Ellipsoid b (Major axis), a (Minor axis), 2c (Distance between foci)





$$q'' = -k \frac{1}{h_1} \cdot \frac{\partial T}{\partial \eta}$$

$$dA = 2\pi x \cdot \underbrace{h_2 \cdot d\psi}_{ds_2}$$

$$Q = \iint q''\, dA = 2\pi k \frac{\partial T}{\partial \eta} \iint \frac{x h_2}{h_1} d\psi$$

$$Q = -2\pi k c \cosh \eta \frac{\partial T}{\partial \eta} \underbrace{\int_0^{\pi/2} \cos\psi \, d\psi}_{1}$$

$$= -2\pi k c \cosh \eta \frac{\partial T}{\partial \eta}$$

$$\Delta T = Q \underbrace{\frac{1}{2\pi k c} \int_0^{\eta_1} \frac{d\eta}{\cosh \eta}}_{R}$$

$$R = \frac{1}{\cancel{2}\pi k c} \int_0^{\eta_1} \frac{\cancel{2}\, d\eta}{e^{\eta} + e^{-\eta}} = \frac{1}{\pi k c} \int_0^{\eta_1} \frac{e^{\eta} d\eta}{e^{2\eta} + 1}, \quad \text{let } \lambda = e^{\eta}$$

$$R = \frac{1}{\pi k c} \int_0^{e^{\eta}} \frac{d\lambda}{\lambda^2 + 1}$$

$$R = \frac{1}{\pi k c} \left[\tan^{-1}(e^{\eta_1}) - \frac{\pi}{4} \right] \Rightarrow \text{we know that } \eta_1 = \ln\left(\frac{a+b}{c}\right)$$

$$e^{\eta_1} = \frac{a+b}{c}$$

$$\boxed{R = \frac{1}{\pi k c} \left[\tan^{-1}\left(\frac{a+b}{c}\right) - \frac{\pi}{4} \right]}$$

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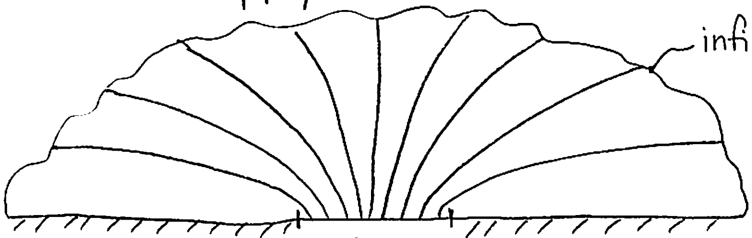
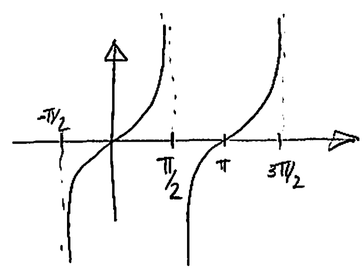
So if we apply our solution to a semi-infinite body:



$a \to \infty$

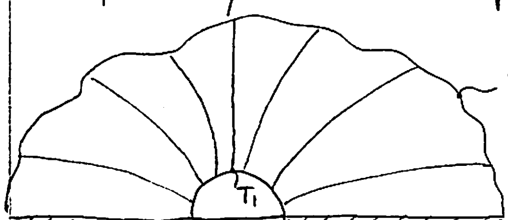
$b \to \infty$

$$R = \frac{1}{\pi k c}\left[\underbrace{\tan^{-1}\left(\frac{a+b}{c}\right)}_{\tan^{-1}(\infty) = \frac{\pi}{2}} - \frac{\pi}{4}\right]$$



$$\boxed{R = \frac{1}{\pi k c}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{1}{4kc}}$$

When NASA was working with this problem in the 50's and 60's, they used an approximation using spherical resistance.



$$R_{approx} \approx \frac{1}{2\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

→ note, divide by 2 not 4 since half-sphere.

$r_2 \to \infty$

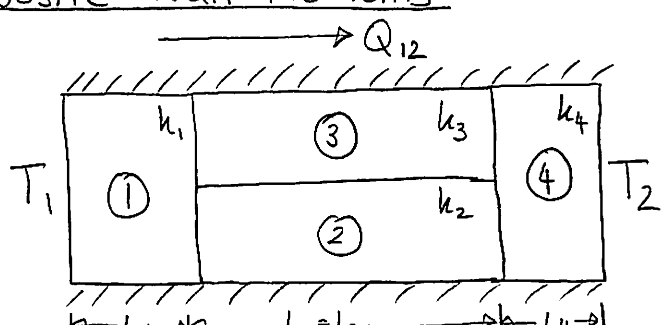
$r_1 \to c$

$$R_{approx} = \frac{1}{2\pi k c} \quad (\approx 50\% \text{ off})$$

$$R_{REAL} = \frac{1}{4kc} < R_{approx}$$

END OF LECTURE 5

Composite Wall Problems



There are two ways to analyze this problem (both giving different answers)