Wall with Variable Thermal Conductivity

When \( k \) weakly depends on \( T \),
\[
    k = a + bT \quad \Rightarrow \text{units} \quad a \rightarrow \frac{W}{m\cdot K}, \quad b \rightarrow \frac{W}{m\cdot K^2}
\]

No need to use heat equation, even simpler

\[
    q_x = -k \frac{\partial T}{\partial x} \quad \Rightarrow \text{Integrate}
\]

\[
    \int_{x_1}^{x_2} q_x \, dx = -\int_{T_1}^{T_2} (a + bT) \, dT
\]

We know that the heat flux is independent of location, (from the conservation of energy)

\[
    q_x (x_2 - x_1) = -a(T_2 - T_1) - \frac{b(T_2^2 - T_1^2)}{2} \quad \Rightarrow \text{Factor}
\]

\[
    q_x \Delta x = - \left( a + b \frac{T_2 + T_1}{2} \right) (T_2 - T_1)
\]

So:

\[
    q_x \Delta x = - (a + b \bar{T}) \Delta T
\]

Rearranging, an interesting interpretation evolves

\[
    q_x = - (a + b \bar{T}) \frac{\Delta T}{\Delta x} \quad \rightarrow \quad q_x = - \frac{\Delta T}{\Delta x} \frac{\Delta T}{\Delta x}
\]
For the temperature distribution:

\[ q_x = -k \frac{\partial T}{\partial x} = -(a + bT) \frac{\partial T}{\partial x} \]

\[ \int_{x_1}^{x} q_x \, dx = -\int_{T_i}^{T} (a + bT) \, dT \quad \text{(Note: bounds of integration are different)} \]

\[ q_x (x-x_i) = -\alpha (T(x) - T_i) - \frac{b(T(x)^2 - T_i^2)}{2} \Rightarrow \text{Quadratic equation.} \]

Solving our quadratic equation above:

\[ T(x) = -\frac{a}{b} \pm \left\{ \left( \frac{a}{b} \right)^2 - 2 \frac{b}{2} q_x (x-x_i) (T_i^2 - 2 \frac{a}{b} T_i) \right\}^{\frac{1}{2}} \]

Plotting our result:

Very useful solution since many materials have a thermal conductivity temperature dependence that can be well approximated with a linear temperature dependent profile.
Conduction for General Shapes (An easier approach)

\( \overrightarrow{V} = 0 \) (no convection)

Steady State, \( Q'' = 0 \), \( k = \text{constant} \), 1-D

Define an orthogonal coordinate system \( u_1, u_2, u_3 \)

\( q'' = -k \frac{\partial T}{\partial s}, \quad (\overrightarrow{q'} = -k \overrightarrow{\nabla} T) \)

\[ Q = A u_1 \frac{\partial T}{\partial s}, \]

Rearranging and integrating (assuming \( ds_t = du_t \))

\[ Q \frac{1}{k} \int_{u_1}^{(u_1)_b} \frac{du_1}{A(u_1)} = -\int_{T_a}^{T_b} dT = T_a - T_b = \Delta T \]

\[ Q = \frac{\Delta T}{\frac{1}{k} \int \frac{du_1}{A(u_1)}} = \frac{\Delta T}{R} \]

\[ R = \frac{1}{k} \int \frac{du_1}{A(u_1)} \]

\[ \Rightarrow \frac{R}{Q} = \frac{T_b - T_a}{T_2 - T_t} \]

Let's double check our previous solutions just to be sure:

1. Slab; \( u_1 = x \), \( du_1 = dx \), \( A(u_1) = \text{constant} \)

\[ R_{\text{slab}} = \frac{1}{\kappa A} \int du_1 = \frac{1}{\kappa A} \int dx = \frac{L}{\kappa A} \]

2. Cylinder; \( u_1 = r \), \( du_1 = dr \), \( A(u_1) = A(r) = 2\pi r l \)

\[ R_{\text{cyl}} = \frac{1}{\kappa} \int_{r_1}^{r_2} \frac{dr}{2\pi r l} = \frac{1}{2\pi \kappa l} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\ln \left( \frac{r_2}{r_1} \right)}{2\pi \kappa l} \]

3. Sphere; \( u_1 = r \), \( du_1 = dr \), \( A(u_1) = A(r) = 4\pi r^2 \)

\[ R_{\text{sph}} = \frac{1}{\kappa} \int_{r_1}^{r_2} \frac{dr}{4\pi r^2} = \frac{1}{4\pi \kappa} \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{1}{4\pi \kappa} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) \]
Example #4

\[ T = T(\theta), \quad l = \text{thickness into page} \]

Note, more complex since heat flux is also a function of \( r \) since heat flow through the region closer to the center happens easier than the outside (lower length and thermal resistance)

Using our general formulation: \( u_i = \theta, \quad ds_i = r d\theta \)

\[
dQ = -k d(A(u_i)) \frac{\partial T}{\partial s_i} = -k d(A(u_i)) \frac{1}{r} \frac{\partial T}{\partial u_i} \\
= -k l dr \frac{\partial T}{r d\theta} = -k l \frac{1}{r} dr \frac{\partial T}{d\theta} \\
\int dQ = \int -k l \frac{\partial T}{d\theta} \frac{dr}{r}
\]

\[ Q = -k l \frac{\partial T}{d\theta} \int_{r_i}^{r_2} \frac{dr}{r} = -k l \frac{\partial T}{d\theta} \cdot \ln\left(\frac{r_2}{r_i}\right) \]

\[ Q = -k l \ln\left(\frac{r_2}{r_i}\right) \frac{\partial T}{d\theta} = \frac{k l \ln\left(\frac{r_2}{r_i}\right)}{R} \cdot \Delta T \]

\[ R = \frac{\Delta \theta}{k l \ln\left(\frac{r_2}{r_i}\right)} \Rightarrow \text{Thermal resistance of the arc} \]

Let's check the bounds of our solution to see if it makes sense:

What if \( r_i \approx r_2 = r_2 - S \)

\[
\ln\left(\frac{r_2}{r_i}\right) = \ln\left(\frac{r_2}{r_2-S}\right) = \ln\left(\frac{1}{1-\frac{S}{r_2}}\right) = \ln\left[\frac{1+\frac{S}{r_2}}{(1-\frac{S}{r_2})(1+\frac{S}{r_2})}\right]
\]

38
\[ \ln \left( \frac{1 + \frac{S}{r_2^2}}{1 + \frac{S}{r_2^2} - \frac{S}{r_2^2} - \frac{S}{r_2^2}} \right) , \text{ since } \frac{S}{r_2^2} < 1 \]
\[ = \ln \left( 1 + \frac{S}{r_2^2} \right) \]

Using series expansion:
\[ \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots \frac{(-x)^{N+1}}{N+1} + \ldots \]
\[ x = \frac{S}{r_2^2} \]
\[ \ln \left( 1 + \frac{S}{r_2^2} \right) = \frac{S}{r_2^2} - \frac{S^2}{2r_2^4} + \frac{S^3}{3r_2^6} + \ldots \]
\[ \ln \left( 1 + \frac{S}{r_2^2} \right) \approx \frac{S}{r_2^2} \]

So plugging back into our resistance calculation:
\[ R = \frac{\Delta \theta}{L \ln \left( \frac{r_2}{r_1} \right)} = \frac{\Delta \theta r_2^2}{L \ln \left( \frac{r_2}{r_1} \right)} \Rightarrow L\delta = A \text{ (Area)} \]
\[ \frac{\Delta \theta r_2}{L} = L \text{ (Length)} \]
\[ \frac{L}{hA} \Rightarrow \text{Works like a charm!} \]

General Orthogonal Coordinate System (Pure Conduction)
Defining \( u_1, u_2, u_3 \) (3 coordinate dimensions)
\[ ds_1 = h_1 du_1 \]
\[ ds_2 = h_2 du_2 \]
\[ ds_3 = h_3 du_3 \]

For example \( \Rightarrow \) Cartesian
\[ h_1 = 1, \ du_1 = dx \]
\[ h_2 = 1, \ du_2 = dy \]
\[ h_3 = 1, \ du_3 = dz \]

We can generalize the following:
\[ q'' = -k \frac{1}{h_1} \cdot \frac{\partial T}{\partial u_1} \]
\[ Q = \iint q'' \, ds_2 \, ds_3 = \iint q'' \, h_2 h_3 \, du_2 \, du_3 \]

\[ Q = -k \frac{2T}{du_1} \iint \frac{h_2 h_3}{h_1} \, du_2 \, du_3 \]

\[ R = \frac{1}{k} \int_{(u_1)_b}^{(u_1)_a} \frac{du_1}{\iint \frac{h_2 h_3}{h_1} \, du_2 \, du_3} \]

\[ \Rightarrow \text{The most useful & general form you'll learn. Very powerful.} \]

Let's check with two quick examples \( \Rightarrow \) radial & spherical

1. Radial:
   \[ u_1 = r \]
   \[ u_2 = \theta \]
   \[ u_3 = z \]

   \[ ds_1 = (1) \, dr \Rightarrow h_1 = 1 \]
   \[ ds_2 = r \, d\theta \Rightarrow h_2 = r \]
   \[ ds_3 = (1) \, dz \Rightarrow h_3 = 1 \]

Back substituting into our formulation

\[ Q = -k \frac{2T}{ar} \int_0^L \int_0^{2\pi} \frac{r(\theta)}{(r)} \, d\theta \, dz = -k \frac{2T}{ar} \int_0^L d\theta \int_0^{2\pi} r \, d\theta \]

\[ = -k (2\pi r) L \frac{2T}{ar} = -k A \frac{2T}{ar} \]

\[ R = \frac{1}{k} \int_{r_i}^{r_f} \frac{dr}{\iint_{r_i}^{r_f} \frac{r(\theta)}{(r)} \, d\theta \, dz} \]

\[ = \frac{1}{k} \int_{r_i}^{r_f} \frac{dr}{r \int_{\theta_i}^{\theta_f} d\theta} = \frac{1}{2\pi k L} \int_{r_i}^{r_f} \frac{dr}{r} = \frac{\ln \left( \frac{r_f}{r_i} \right)}{2\pi k L} = R_{\text{cyl}} \]

We see that the formalism works. Try it for spherical.
Example #5  Ellipse (Foci located 2c apart)

\[ \begin{align*}
U_1 &= h_1 \quad \text{(we will solve for it)} \\
U_2 &= h_2 \quad \text{(unknown yet)} \\
U_3 &= z_1, \
\end{align*} \]

First we need to solve for \( h_1 \) \& \( h_2 \). We can use some nice rules \& identities for ellipses:

**Rule #1:**

\[ b^2 - a^2 = c^2 \quad \Rightarrow \quad x = c \cosh \eta \cos \varphi; \quad b = c \cosh \eta_1, \]
\[ y = c \cosh \eta \sin \varphi; \quad a = c \sinh \eta, \]

\[ (b^2 - a^2) = c^2 \left( \cosh^2 \eta_1 - \sinh^2 \eta_1 \right) \]

\[ 1 \quad \Rightarrow \quad b^2 - a^2 = c^2 \quad \text{(we are OK!)} \]

Some identities we'll need are

\[ \cosh \eta = \frac{e^\eta + e^{-\eta}}{2} \]
\[ \sinh \eta = \frac{e^\eta - e^{-\eta}}{2} \]

\[ \vec{r} = x \hat{i} + y \hat{j} \quad \Rightarrow \quad \left| \frac{dx}{d\eta} \right| = h_1 = C (\cosh^2 \eta \cos^2 \varphi + \cos^2 \eta \sin^2 \varphi)^{1/2} \]
\[ \left| \frac{dy}{d\eta} \right| = h_2 = C (\cosh^2 \eta \sin^2 \varphi + \sin^2 \eta \cos^2 \varphi)^{1/2} \]

\[ h_1 = h_2 \]
This makes our lives a lot easier:

\[ R = \frac{1}{\pi} \int_0^{n_1} \int_0^{\pi} \int_0^L d\eta d\theta dz = \frac{n_1}{\pi k L} \]

We know that:

\[ \frac{a + b}{c} = \cosh n_1 + \sinh n_1 = \frac{e^{n_1} + e^{-n_1}}{2} + \frac{e^{n_1} - e^{-n_1}}{2} = e^{n_1} \]

\[ n_1 = \ln \left( \frac{a + b}{c} \right) \]

So our solution becomes:

\[ Q = \frac{\Delta T \pi k L}{\ln \left( \frac{a + b}{c} \right)} \quad \text{and} \quad R = \frac{\ln \left( \frac{a + b}{c} \right)}{\pi k L} \]

Note, this solution is extremely useful since it can be used to solve the following cases:

Case 1: Spot Welding

We neglect this resistance. But because spreading resistance here is high, safe approximation.

\[ b^2 = \sqrt{a^2 + c^2} \]

Using the above resistance puts an upper bound on our heat transfer \( Q \) by assuming cylindrical spreading.

Case 2:

\[ R_i = \frac{L}{kA} \quad \Rightarrow \quad R_i = \frac{\ln \left( \frac{a + b}{c} \right)}{\pi k L} \]

\[ R_c = \frac{\ln \left( \frac{a + b}{c} \right)}{\pi k L} \Rightarrow a = \sqrt{b^2 - c^2} \]

Broken down into 2 resistances, Good approximation.
Revolved Ellipsoid  \( b \) (Major axis), \( a \) (Minor axis), \( 2c \) (Distance between foci)

- \( q'' = -k \frac{1}{h_1} \frac{2T}{\partial q} \)
- \( dA = 2\pi x \cdot h_2 \cdot d\psi \)
- \( Q = \int \int q'' dA = 2\pi k \frac{2T}{\partial q} \int \int \frac{xh_2}{h_1} d\psi \)
- \( Q = -2\pi k c \cosh \eta \frac{2T}{\partial q} \int_{0}^{\eta_2} \cos \psi d\psi \)
  \[ = -2\pi kc \cosh \eta \frac{2T}{\partial q} \]
- \( \Delta T = Q \left( \frac{1}{2\pi kc} \int_{0}^{\eta_1} \frac{dn}{\cosh \eta} \right) \)
- \( R = \frac{1}{2\pi kc} \int_{0}^{\eta_1} \frac{dn}{e^n + e^{-n}} \)
  \[ = \frac{1}{\pi kc} \int_{0}^{\eta_1} \frac{e^n dn}{e^{2n} + 1} \]
  \[ = \frac{1}{\pi kc} \int_{0}^{\eta_1} \frac{e^n dn}{\lambda^2 + 1} \]
  \[ = \frac{1}{\pi k c} \left[ \tan^{-1} (e^{\eta_1}) - \frac{\pi}{4} \right] \]
- We know that \( \eta_1 = \ln \left( \frac{a+b}{c} \right) \)
- \( e^{\eta_1} = \frac{a+b}{c} \)
So if we apply our solution to a semi-infinite body:

\[
R = \frac{1}{\pi k c} \left[ \tan^{-1} \left( \frac{a+b}{c} \right) - \frac{\pi}{4} \right]
\]

\[
\tan^{-1}(\infty) = \frac{\pi}{2}
\]

\[
R = \frac{1}{\pi k c} \left[ \frac{\pi}{2} - \frac{\pi}{4} \right] = \frac{1}{4k c}
\]

When NASA was working with this problem in the 50's and 60's, they used an approximation using spherical resistance.

\[
R_{\text{approx}} \approx \frac{1}{2\pi k c} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)
\]

\[
r_2 \to \infty, \text{ divide by 2}
\]

\[
r_1 \to c
\]

\[
R_{\text{approx}} \approx \frac{1}{2\pi k c} \left( \approx 50\% \text{ off} \right)
\]

\[
R_{\text{real}} = \frac{1}{4k c} < R_{\text{approx}}
\]

**Composite Wall Problems**

There are two ways to analyze this problem (both giving different answers)