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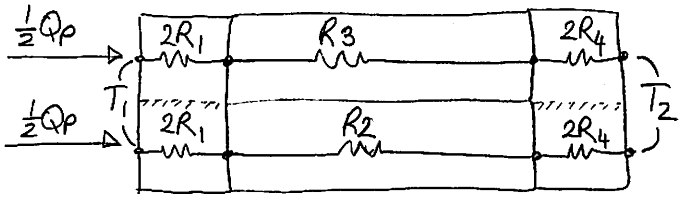
① Parallel Approach



$$R_L = \frac{L_1}{k_1 A_2} + \frac{L_2}{k_2 A_2} + \frac{L_4}{k_4 A_2}$$

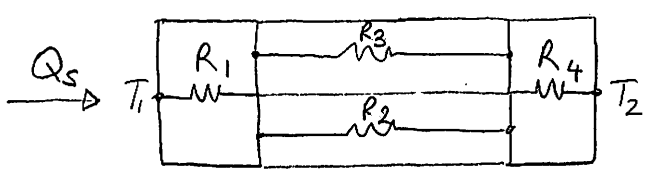
$$R_u = \frac{L_1}{k_1 A_3} + \frac{L_2}{k_3 A_3} + \frac{L_4}{k_4 A_3}$$

$$\frac{1}{R} = \frac{1}{R_L} + \frac{1}{R_u}$$

$$\frac{1}{R_p} = \left(\frac{A_2}{L_1/k_1 + L_2/k_2 + L_4/k_4} + \frac{A_3}{L_1/k_1 + L_3/k_3 + L_4/k_4} \right)$$

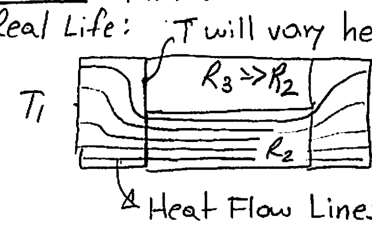
$$\boxed{Q_p = \frac{\Delta T}{R_p} = \left(\frac{A_2}{L_1/k_1 + L_2/k_2 + L_4/k_4} + \frac{A_3}{L_1/k_1 + L_3/k_3 + L_4/k_4} \right) \Delta T}$$

② Series Approach



Aside: Think of this way:

Real Life:



$$R_s = \frac{L_1}{k_1 A_1} + \left(\frac{k_2 A_2}{L_2} + \frac{k_3 A_3}{L_2} \right)^{-1} + \frac{L_4}{k_4 A_1}$$

$$\boxed{Q_s = \frac{\Delta T}{R_s} = \left(\frac{L_1}{k_1 A_1} + \frac{L_2}{k_2 A_2 + k_3 A_3} + \frac{L_4}{k_4 A_1} \right)^{-1} \cdot \Delta T}$$

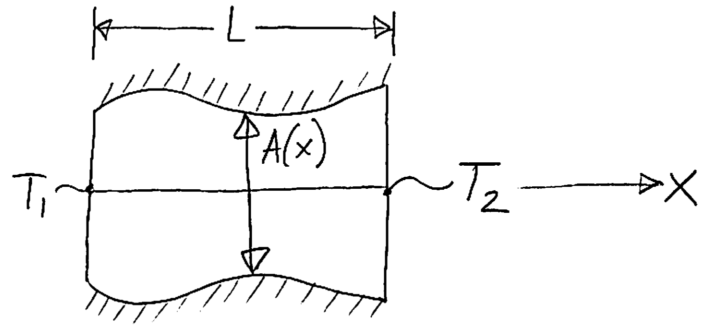
It is always true that: $\boxed{Q_p \le Q_{REAL} \le Q_s}$

Q_p → Lower Bound estimate; Q_s → Upper Bound estimate.

In general:

1) Adding adiabatic surfaces : $R \uparrow$, $Q \downarrow$
2) Allowing infinite conductance : $R \downarrow$, $Q \uparrow$

Example #6



$$dR = \frac{dx}{kA(x)} \longrightarrow R \geq \frac{1}{k}\int_0^L \frac{dx}{A(x)}$$

→ Assuming it's a 1D problem. In reality, 2D and added resistance will come into play due to longer path.

$$Q \leq \frac{\Delta T}{\frac{1}{k}\int_0^L \frac{dx}{A(x)}}$$

The Nievé approach: $\bar{A} = \frac{1}{L}\int_0^L A(x)\,dx \rightarrow R \simeq \frac{L^2}{k\int_0^L A(x)dx}$

$$Q \simeq \frac{k\Delta T}{L^2}\int_0^L A(x)\,dx$$

But, it's important to note that:

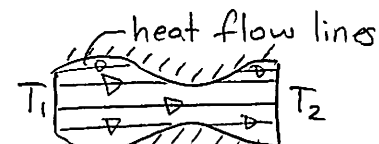
$$\boxed{\underbrace{\frac{1}{L}\int_0^L A(x)\,dx}_{\text{Arithmetic Mean}} \geq \underbrace{\left(\frac{1}{L}\int_0^L \frac{dx}{A(x)}\right)^{-1}}_{\text{Harmonic Mean}}}$$

$$\boxed{R_{Real} \geq \frac{1}{k}\int_0^L \frac{dx}{A(x)} \geq \frac{L}{k}\left(\frac{1}{L}\int_0^L A(x)\,dx\right)^{-1}}$$

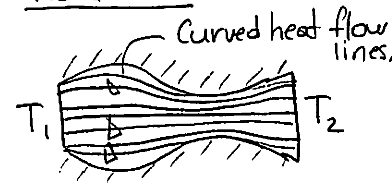
$$\boxed{Q_{Real} \leq \frac{\Delta T}{\frac{1}{k}\int_0^L \frac{dx}{A(x)}} \leq \frac{k\Delta T}{L^2}\int_0^L A(x)\,dx}$$

Aside:

1-D solution



Real Life:



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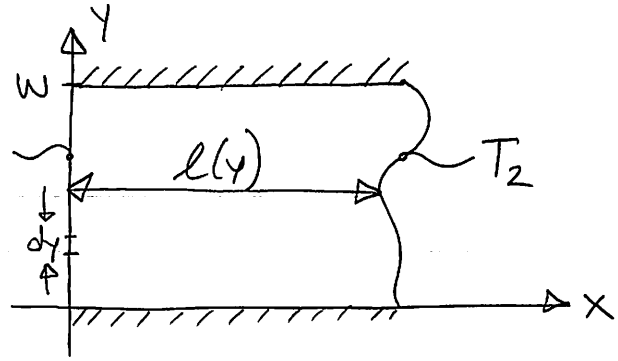
Example #7



Q' = heat transfer rate per unit length into board or paper.

$$dQ' \simeq k\,dy \frac{\Delta T}{\ell(y)}$$

$$Q' \geq k \Delta T \int_0^w \frac{dy}{\ell(y)}$$

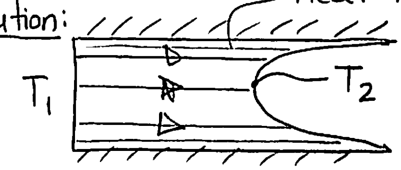
$$R' \leq \frac{1}{k}\left(\int_0^w \frac{dy}{\ell(y)}\right)^{-1}$$

⇒ note: $Q' \geq (\cdots)$ since in reality, heat can spread laterally and take the path of least resistance. This is not considered here.

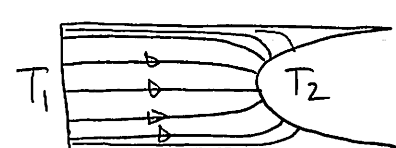
Real heat transfer.

i.e.

Solution:



Real Life:



The Nievé Approach:

$$\bar{\ell} \cong \frac{1}{w}\int_0^w \ell(y)\,dy$$

$$\bar{R} \cong \frac{1}{kw}\int_0^w \ell(y)\,dy$$

$$Q' \cong kw\Delta T\left(\int_0^w \ell(y)\,dy\right)^{-1}$$

From this we can say that:

$$\boxed{R'_{Real} \leq \frac{1}{k}\left(\int_0^w \frac{dy}{\ell(y)}\right)^{-1} \leq \frac{1}{kw}\int_0^w \ell(y)\,dy}$$

$$\boxed{Q'_{Real} \geq k\Delta T\int_0^w \frac{dy}{\ell(y)} \geq kw\Delta T\left(\int_0^w \ell(y)\,dy\right)^{-1}}$$

⇒ We've already learned that Arithmetic mean ≥ Harmonic mean.

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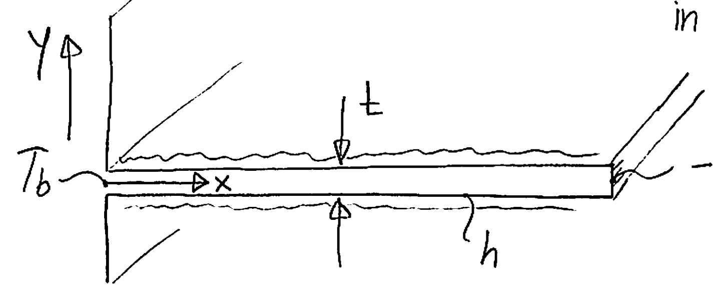
Quasi-1D Conduction: Fins

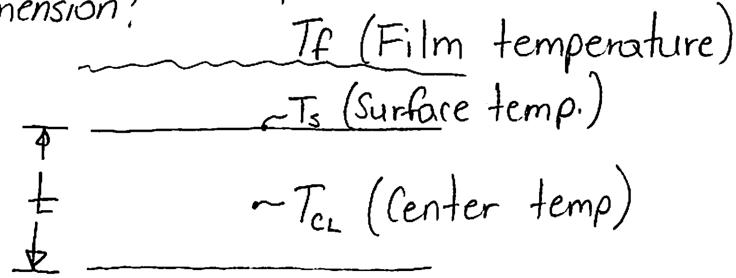
Assumptions; 1) P, A = constant (Perimiter and Area of Fin)
2) No heat generation
3) The heat transfer is 1-D

What does assumption 3) mean? No temperature variation in the y-direction.



$-kA\frac{\partial T}{\partial x} = 0$ (Adiabatic)
or
(Other B.C.'s)

So how do we quantify the temperature variation in the y-dimension?



Doing a heat balance across the fin, we obtain:

$$\underbrace{-k\frac{\partial T}{\partial y}(L\,dx)}_{\text{Conduction}} = \underbrace{h(L\,dx)(T_s - T_f)}_{\text{Convection}}$$

$$\frac{\partial T}{\partial y} = \frac{T_{CL} - T_s}{t/2} = h(T_s - T_f)$$

$$\frac{T_{CL} - T_s}{T_s - T_f} = -\frac{1}{2}\frac{ht}{k} \approx 0.05$$

$$\frac{1}{2}\left(\frac{ht}{k}\right) \leq \frac{1}{20} \Rightarrow \boxed{\frac{ht}{k} \leq \frac{1}{10} = Bi_t}$$

$\Rightarrow$ for fin heat transfer in the thickness direct. Biot Number.

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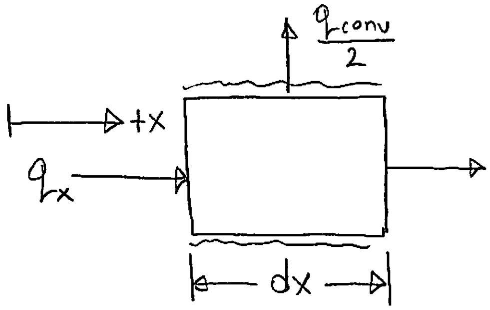
Interpretation: 1) Temp. changes across the fin thickness are small compared to those external to the fin. (isothermal in y, at a particular x)
2) Internal resistance to conduction across the thickness of the fin is small compared to resistance due to external convection. i.e. $R_{cond} \ll R_{conv}$

We can double check this right away:

$$\frac{R_{cond}}{R_{conv}} = \frac{\frac{t}{k\cancel{A}}}{\frac{1}{h\cancel{A}}} = \underbrace{\frac{ht}{k}}_{Bi_t} \ll 1$$

$$Bi_t \ll 1$$

Now we can model the temperature distribution:



$$q_{x+\Delta x} = q_x + \frac{\partial}{\partial x}(q_x)\Delta x + \frac{\partial^2}{\partial x^2}(q_x)\Delta x^2 \frac{1}{2} + \cdots$$

$q_{conv} = hP\,dx\,(T - T_f)$; P = perimeter or surface area/unit length

We also know that: $q_x = -kA\frac{\partial T}{\partial x}$

Energy balance:

$$q_x - \left\{ q_x + \frac{\partial q_x}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 q_x}{\partial x^2}\Delta x^2 + \cdots \right\} = hP\Delta x\,(T - T_f)$$

$$-\frac{\partial q_x}{\partial x}\cdot dx = hP\,dx\,(T - T_f)$$

$$\frac{d}{dx}\left(kA\frac{\partial T}{\partial x}\right) - hP(T - T_f) = 0$$

Defining $\theta = T - T_f$, $d\theta = dT$

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$$kA\frac{\partial^2\theta}{\partial x^2} - hP\theta = 0 \Rightarrow \frac{\partial^2\theta}{\partial x^2} - \underbrace{\frac{hP}{kA}}_{\text{Let}}\theta = 0$$

Let $\boxed{\frac{hP}{kA} = m^2}$

Works if: $a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$

$$\underbrace{\frac{\partial^2\theta}{\partial x^2} - m^2\theta = 0}$$

Linear, second order ODE

Our characteristic equation is: (ODE concept)
Euler did it!

$$\lambda^2 - m^2 = 0$$
$$\lambda^2 = m^2$$
$$\lambda = \pm m$$

So our general solution is:

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Aside: Proof of solution:

$$\theta = C_1 e^{mx} + C_2 e^{-mx}$$
$$\theta' = mC_1 e^{mx} - mC_2 e^{-mx}$$
$$\theta'' = m^2 C_1 e^{mx} + m^2 C_2 e^{-mx}$$
$$\theta'' - m^2\theta = m^2(C_1 e^{mx} + C_2 e^{mx}) - m^2(\theta)$$
$$= 0 \quad \text{QED}$$

We have four directions to go to:

1. Infinite Fin case ($x \to \infty$)

Our B.C.'s become:

$$\theta(x=0) = T_B - T_f = \theta_b$$
$$\theta(x = L \to \infty) = T_f - T_f = 0$$

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$
$$\theta(\infty) = 0 = \underbrace{C_1 e^{m(\infty)}}_{C_1 = 0 \text{ for solution to be valid}} + C_2 e^{-m(\infty)}$$

$$\theta(x=0) = \theta_b$$
$$\theta_b = T_b - T_f = C_2 \cancel{e^{-m(0)}}^{1} \Rightarrow C_2 = \theta_b$$

$$\theta(x) = \theta_b e^{-mx}$$

$$\boxed{\frac{T - T_f}{T_b - T_f} = e^{-\frac{hP}{kA}x}}$$

What about heat transfer from the base to the fin?

$$Q_{net} = Q(0) = -kA\frac{\partial\theta}{\partial x}\bigg|_{x=0} = +kA\theta_b m \cancel{e^{-m(0)}}^{1}$$

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$$\boxed{Q_{net} = +kA\theta_b\sqrt{\frac{hP}{kA}} = \sqrt{kAhP}\,\theta_b}$$

Note: $\dfrac{Q_{net}}{Q_o} = \dfrac{\sqrt{kAhP}\,\cancel{\theta_b}}{hA\cancel{\theta_b}} = \sqrt{\dfrac{kP}{hA}} = \sqrt{\dfrac{k}{ht}} = Bi_t^{-1/2} \gg 1$

Also note here that: $\dfrac{P}{A} = \dfrac{1}{t}$. In general $L_c = \dfrac{V_{body}}{A_{surface}} = \dfrac{A\cancel{V}}{P\cancel{V}} = \dfrac{A}{P} = t$

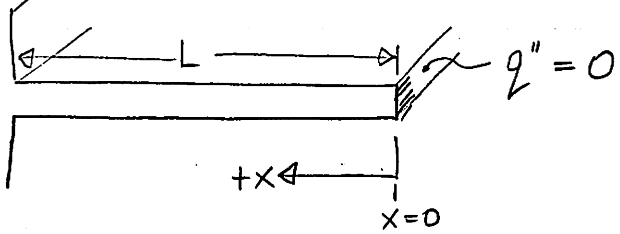
$\uparrow$ Characteristic Length.

<u>Sanity check:</u> $Q_{net} = \displaystyle\int_0^\infty hP\theta\,dx = \int_0^\infty hP\theta_b e^{-mx}\,dx$

$$= hP\theta_b \left.\frac{e^{-mx}}{m}\right|_0^\infty = \frac{hP\theta_b}{m} = \sqrt{kAhP}\,\theta_b$$

$\Downarrow$ Works!

(2.) <u>Insulated Tip case</u> $\left(-k\left.\dfrac{\partial T}{\partial x}\right|_{x=0} = 0\right)$



B.C.'s:

$-k\left.\dfrac{\partial\theta}{\partial x}\right|_{x=0} = 0$

$\theta(x=L) = \theta_b$

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

We can rewrite this as:

$$\theta(x) = C_1\cosh(mx) + C_2\sinh(mx)$$

Substitute in our B.C.'s

$$\frac{\partial\theta}{\partial x} = C_1 m\sinh(mx) + C_2 m\cosh(mx)$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = C_1 m\,\cancel{\sinh(0)}^{0} + C_2 m\,\cancel{\cosh(0)}^{1} \Rightarrow C_2 m = 0 \Rightarrow C_2 = 0$$

$\theta(x) = C_1\cosh(mx) \Rightarrow$ Apply our second B.C.

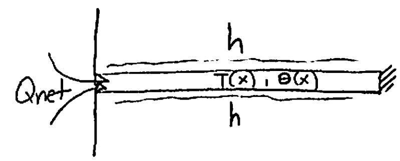
$\theta(x=L) = \theta_b = C_1\cosh(mL)$

$$C_1 = \frac{\theta_b}{\cosh(mL)}$$

Back substituting

$$\theta(x) = \frac{\theta_b \cosh(mx)}{\cosh(mL)}$$

$$\boxed{\frac{T-T_f}{T_b-T_f} = \frac{\cosh(mx)}{\cosh(mL)}}$$

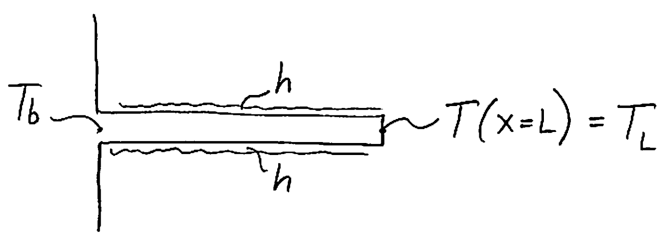


How about heat transfer?

$$Q_{net} = -kA\left.\frac{\partial\theta}{\partial x}\right|_{x=L} = +kA\theta_b m\frac{\sinh(mL)}{\cosh(mL)} = kA\theta_b\sqrt{\frac{hP}{kA}}\frac{\sinh(mL)}{\cosh(mL)}$$

$$\boxed{Q_{net} = \sqrt{kAhP}\,(T_b-T_f)\underbrace{\frac{\sinh(mL)}{\cosh(mL)}}_{\tanh(mL)}}$$ $\Rightarrow$ Heat transfer from the insulated tip fin.

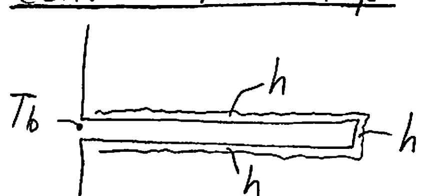
3. Perscribed Tip Temperature $\left(T(x=L)=T_L\right)$



I won't go through the math, (you can do this) but the solution is:

$$\boxed{\frac{T-T_f}{T_b-T_f} = \frac{T_L-T_f}{T_b-T_f}\cdot\frac{\sinh(mx)+\sinh[m(L-x)]}{\sinh(mL)}}$$

4. Convection on Tip $\left(h\left(T(x=L)-T_f\right) = -k\left.\frac{\partial T}{\partial x}\right|_{x=L}\right)$



$$\boxed{\frac{T-T_f}{T_b-T_f} = \frac{\cosh(m(L-x)) + \frac{h}{mk}\sinh(m(L-x))}{\cosh(mL)+\frac{h}{mk}\sinh(mL)}}$$

Fin Efficiency (Insulated tip)

$$\text{Fin Efficiency} = \eta_{fin} = \frac{\text{Actual Heat Transfer}}{\text{Heat Transfer if } \theta = \theta_b \text{ everywhere}}$$

$$= \frac{q_{actual}}{q_{ideal}}$$

We know: $q_{ideal} = h(PL)(T_b - T_f)$

$$q_{actual} = \int_0^L hP(T-T_f)\,dx = \underbrace{\sqrt{kAPh}\,\theta_b \tanh(mL)}_{\text{Insulated Tip } Q_{net}}$$

$$\eta_{fin} = \frac{\sqrt{kAPh}\,\cancel{\theta_b}\tanh(mL)}{hPL\cancel{\theta_b}} = \underbrace{\sqrt{\frac{kA}{hP}}}_{\frac{1}{m}} \cdot \frac{\tanh(mL)}{L}$$

$$\boxed{\eta_{fin} = \frac{\tanh(mL)}{mL}} = \frac{q_{actual}}{q_{ideal}}$$

For other configurations, see Mills pg. 104-105

Fin Resistance (R_{fin}) → For insulated tip case

$$R_{fin} = \frac{\theta_b}{q_{actual}} = \frac{T_b - T_f}{q_{actual}} = \frac{\cancel{\theta_b}}{\sqrt{kAPh}\tanh(mL)\cancel{\theta_b}} = \frac{1}{hA_{fin}\eta_{fin}}$$

$$\boxed{R_{fin} = \frac{1}{hA_{fin}\eta_{fin}}}$$

⇒ A_{fin} = outside area of fin (PL)
η_{fin} = fin efficiency
h = heat transfer coeff.

Fin Effectiveness (ε_{fin})

$$\varepsilon_f = \frac{\text{actual heat transfer}}{\text{heat transfer if no fin}}$$

$$= \frac{\cancel{q_{ideal}}\cdot\eta_f}{\cancel{q_{ideal}}\left(\frac{A}{PL}\right)} = \eta_f\left(\frac{PL}{A}\right) \Rightarrow \boxed{\varepsilon_f = \eta_f\frac{PL}{A}}$$

P = perimeter
L = fin length
A = C.S.A. of fin.

END OF LECTURE 6