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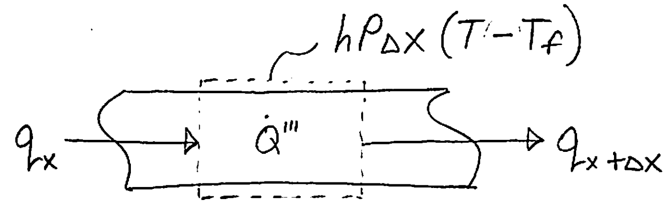
Fins with Heat Generation ($\dot{Q}'''$)

T_f = fluid temperature
$T = T(x)$



$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\Delta x + \cdots$$

Writing out our energy balance

$$A\cancel{q_x} - A\left(\cancel{q_x} + \frac{\partial q_x}{\partial x}\cancel{\Delta x} + \frac{1}{2}\frac{\partial^2 q_x}{\partial x^2}\Delta x^{\cancel{2}} + \text{H.O.T.}\right) + \dot{Q}'''A\cancel{\Delta x} - hP\cancel{\Delta x}(T - T_f) = 0$$

We know from Fourier's Law: $q_x = -k\dfrac{\partial T}{\partial x}$

$$+kA\left(\frac{\partial^2 T}{\partial x^2} + \frac{1}{2}\frac{\partial^3 T}{\partial x^3}\Delta x + \text{H.O.T.}\right) + \dot{Q}'''A - hP(T - T_f) = 0$$

Letting $\Delta x \to 0$, and $\theta = T - T_f$
$d\theta = dT$

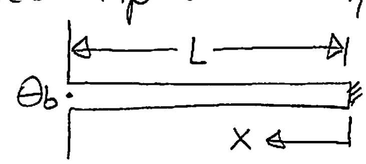
We obtain:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\dot{Q}'''}{k} - m^2\theta = 0\,, \qquad m^2 = \frac{hP}{kA}$$

Assuming the insulated tip condition, our B.C.'s become

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0$$

$$\theta(x = L) = \theta_b$$



To solve, we can do the particular and homogeneous solution:

<u>Particular</u>:

$$\underbrace{\frac{\partial^2\theta_p}{\partial x^2}}_{\text{set this to zero (quick solution for the particular)}} + \frac{\dot{Q}'''}{k} - m^2\theta_p = 0$$

$$\theta_p = \frac{\dot{Q}'''}{km^2} = \frac{\dot{Q}'''\cancel{k}A}{\cancel{k}hP} = \frac{\dot{Q}'''A}{hP} \Rightarrow \boxed{\theta_p = \frac{\dot{Q}'''A}{hP}}$$

Now we move on to the homogeneous part.

<u>Homogeneous:</u> $\frac{\partial^2 \theta_h}{\partial x^2} - m^2 \theta_h = 0$

We've already solved this problem before. Nothing but

$$\theta_h = C_1 \cosh(mx) + C_2 \sinh(mx)$$

Now we combine $\theta_p + \theta_h = \theta(x)$ and apply our B.C.'s

$$\theta(x) = \theta_p(x) + \theta_h(x) = \frac{\dot{Q}'''A}{hP} + C_1 \cosh(mx) + C_2 \sinh(mx)$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0 = -C_1 m \overset{0}{\cancel{\sinh}}(m(0)) + C_2 m \overset{1}{\cancel{\cosh}}(m(0)) \Rightarrow C_2 = 0$$

$$\theta(x=L) = \theta_b = \frac{\dot{Q}'''A}{hP} + C_1 \cosh(mL)$$

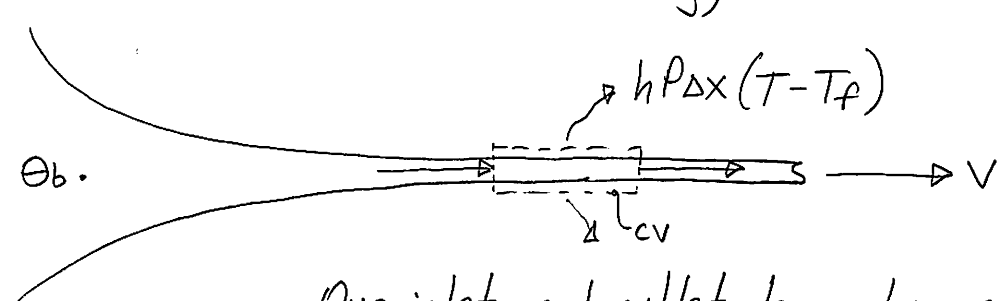
$$C_1 = \frac{\theta_b - \frac{\dot{Q}'''A}{hP}}{\cosh(mL)}$$

$$\therefore \quad \theta(x) = \frac{\dot{Q}'''A}{hP} + \frac{\theta_b - \frac{\dot{Q}'''A}{hP}}{\cosh(mL)} \cdot \cosh(mx)$$

$$\boxed{\theta(x) = \theta_b \frac{\cosh(mx)}{\cosh(mL)} + \frac{\dot{Q}'''A}{hP}\left(1 - \frac{\cosh(mx)}{\cosh(mL)}\right)}$$

↳ Temperature profile in the fin with heat generat.

<u>Moving Fins</u> (For example, metal extrusion and solidification ⇒ continuous casting)



Our inlet and outlet terms become a bit more complicated.

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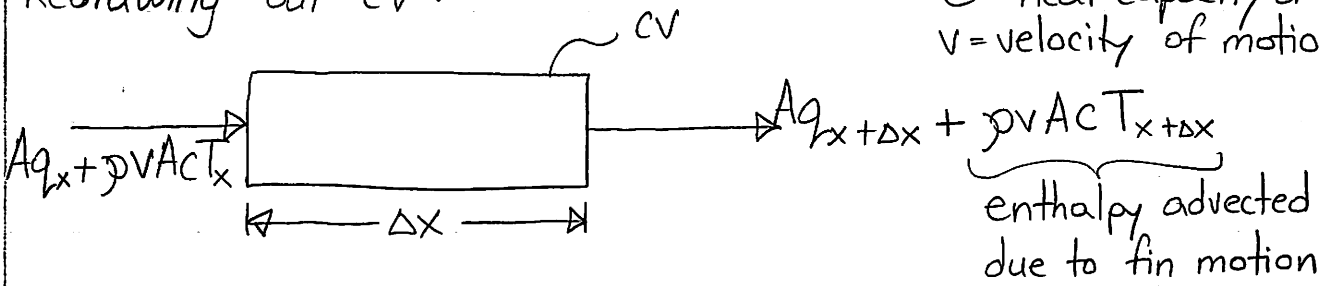
Redrawing our CV:

C = heat capacity of fin
v = velocity of motion



Energy balance on our control volume:

$$Aq_x + \rho v A c T_x - \left(Aq_{x+\Delta x} + \rho v A c T_{x+\Delta x}\right) - hP\Delta x\,(T - T_f) = 0$$

Substituting in Fourier's Law and doing a Taylor series expans.

$$\cancel{Aq_x} + \cancel{\rho v A c T_x} - \left(\cancel{Aq_x} + A\frac{\partial q_x}{\partial x}\Delta x + A\frac{1}{2}\frac{\partial^2 q_x}{\partial x^2}\Delta x^2 + H.O.T + \cancel{\rho v A c T_x} + \rho v A c\frac{\partial T}{\partial x}\Delta x + \rho v A c\frac{\partial^2 T}{\partial x^2}\frac{\Delta x^2}{2} + H.O.T\right) - hP\Delta x(T - T_f) = 0$$

$$kA\frac{\partial^2 T}{\partial x^2}\cancel{\Delta x} + \frac{1}{2}kA\frac{\partial^3 T}{\partial x^3}\Delta x^{\cancel{2}} + H.O.T. - \rho v A c\frac{\partial T}{\partial x}\cancel{\Delta x} - \rho v A c\frac{\partial^2 T}{\partial x^2}\frac{\Delta x^{\cancel{2}}}{2} + H.O.T. - hP\cancel{\Delta x}(T - T_f) = 0$$

Let $\Delta x \to 0$, all H.O.T. terms drop out, we are left with:

$$kA\frac{\partial^2 T}{\partial x^2} - \rho v A c\frac{\partial T}{\partial x} - hP(T - T_f) = 0$$

$$\frac{\partial^2 T}{\partial x^2} - \frac{\rho v c}{k}\frac{\partial T}{\partial x} - \underbrace{\frac{hP}{kA}}_{m^2}(T - T_f) = 0$$

Let $\theta = T - T_f$
$d\theta = dT$

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{\rho v c}{k}\frac{\partial \theta}{\partial x} - m^2\theta = 0$$

Our B.C.'s are (for infinite fin) $\Rightarrow \begin{cases} 1)\ \theta(x=0) = \theta_b \\ 2)\ \theta(x \to \infty) = 0 \end{cases}$

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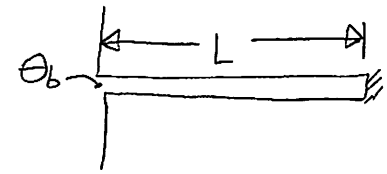
Note, if we non dimensionalize, it becomes more powerfull

$$\frac{\partial^2\theta}{\partial x^2} - \frac{\rho V c}{k}\frac{\partial\theta}{\partial x} - m^2\theta = 0$$



Let $\theta^* = \frac{\theta}{\theta_b}$, $x^* = \frac{x}{L}$

$d\theta = \theta_b\, d\theta^*$, $dx = L\,dx^*$ $\Rightarrow$ Back substituting

$$\frac{\partial^2(\cancel{\theta_b}\,\theta^*)}{L^2\partial x^{*2}} - \frac{\rho V c}{k}\frac{\partial(\cancel{\theta_b}\,\theta^*)}{L\,dx^*} - m^2(\cancel{\theta_b}\,\theta^*) = 0$$

$$\frac{1}{L^2}\frac{\partial^2\theta^*}{\partial x^{*2}} - \frac{\rho V c}{kL}\frac{\partial\theta^*}{\partial x^*} - m^2\theta^* = 0$$

Multiply through by L^2 on both sides

$$\frac{\partial^2\theta^*}{\partial x^{*2}} - \underbrace{\frac{L\rho V c}{k}}\frac{\partial\theta^*}{\partial x^*} - (mL)^2\theta^* = 0$$

This is interesting: $\alpha = \frac{k}{\rho c}$

This becomes: $\frac{LV}{\alpha}$ $\xrightarrow{}$ Thermal advection, $\alpha \longrightarrow$ Thermal diffusion

Note, a side definition: Convection = motion of fluid in response to heat. (Diffusion + Advection)

Advection = motion of fluid, mass, momentum or heat due to the motion of the fluid itself.

Back to our problem:

$$\boxed{Pe = \text{Peclet Number} = \frac{\text{advective transport rate}}{\text{diffusive transport rate}} = \frac{LV}{\alpha}}$$

Let's first test the limits of our equation

$$\frac{LV}{\alpha} = \frac{\rho c L V}{k} = \frac{\rho c V}{\frac{k}{L}}\left(\frac{\Delta T}{\Delta T}\right)$$

$$= \frac{\rho c V \Delta T}{k\frac{\Delta T}{L}} \begin{matrix}\Rightarrow q''_{adv} \\ \}\, q''_{cond}\end{matrix}$$

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If advection dominates, $\frac{LV}{\alpha} \gg 1$
Our diffusion term drops out.

$$\underbrace{\frac{\partial^2 \theta}{\partial x^2}}_{0\text{ (negligible)}} - \frac{\rho V C}{k}\frac{\partial \theta}{\partial x} - m^2\theta = 0$$

$$\frac{V}{\alpha}\frac{\partial \theta}{\partial x} + m^2\theta = 0 \quad \text{(Rearanging)}$$

$$\frac{1}{\theta}\frac{\partial \theta}{\partial x} = -\frac{m^2\alpha}{V} = -\frac{hP\cancel{k}}{\cancel{k}AV\rho C} = -\frac{hP}{\rho CVA}$$

$$\int \frac{d\theta}{\theta} = -\frac{hP}{\rho CVA}\int dx$$

$$\ln(\theta) = -\frac{hP}{\rho CVA}x + C_1$$

Constant of integration, not heat capacity C.

$$e^{\ln(\theta)} = e^{-\frac{hP}{\rho CVA}x + C_1} = C_1' e^{-\frac{hP}{\rho CVA}x}$$

$$\theta(x) = C_1' e^{-\frac{hPx}{\rho CVA}}$$

We know that $\theta(x=0) = \theta_b = C_1' \cancel{e^{0}}^{\,1} \Rightarrow C_1' = \theta_b$
Let's check our second b.c.

$$\theta(x \to \infty) = 0 \Rightarrow \theta_b e^{-\infty} \to 0 \text{ OK!}$$

$$\therefore \boxed{\theta(x) = \theta_b e^{-\frac{hPx}{\rho CVA}}} \Rightarrow \text{When advection dominates diffusion.}$$

But what if conduction is non-negligible? (i.e. $Pe \sim 1$)

Now we have to solve the full differential equation
Start with the characteristic equation:

Let. $\theta = e^{\lambda x}$

$$\left.\begin{aligned} \frac{\partial \theta}{\partial x} &= \lambda e^{\lambda x} \\ \frac{\partial^2 \theta}{\partial x^2} &= \lambda^2 e^{\lambda x} \end{aligned}\right\} \text{Substitute into our ODE}$$

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$$\lambda^2 \cancel{e^{\lambda x}} - \frac{V}{\alpha}\lambda \cancel{e^{\lambda x}} - m^2 \cancel{e^{\lambda x}} = 0$$

$$\lambda^2 - \frac{V}{\alpha}\lambda - m^2 = 0 \quad \text{(Quadratic equation)}$$

$$\lambda = \frac{V}{2\alpha} - \sqrt{\frac{V^2}{4\alpha^2} + m^2}$$ ⇒ Only use negative root since we need $\theta(x \to \infty) = 0$

$$\boxed{\lambda = -\frac{V}{2\alpha}\left[\sqrt{1 + \frac{4m^2\alpha^2}{V^2}} - 1\right]} \Rightarrow \boxed{\theta = \theta_b e^{\lambda x}}$$

To check our solution, if $\left(\frac{\alpha}{V}\right)^2 \ll 1$ (Advection Dominates)

Then $\frac{4m^2\alpha^2}{V^2} \ll 1$, we can do a Taylor series expansion

$$\sqrt{1+x} = 1 + \frac{x}{2} \underbrace{- \frac{x^2}{4} + \frac{3x^3}{8} - \dots}_{\text{H.O.T. can be neglected if } x \ll 1.}$$

$f(x) = (1+x)^{1/2}$
$\frac{\partial f}{\partial x} = \frac{1}{2}(1+x)^{-1/2}(1)$
$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{4}(1+x)^{-3/2}(1$
$\vdots$

So our solution becomes:

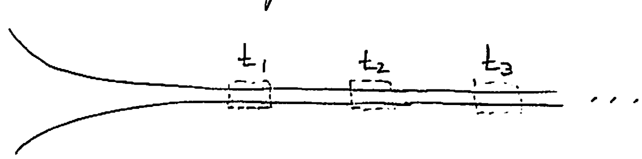
$$\lambda = -\frac{V}{2\alpha}\left[\sqrt{1 + \frac{4m^2\alpha^2}{V^2}} - 1\right]$$

$$= -\frac{V}{2\alpha}\left(\cancel{1} + \frac{2m^2\alpha^2}{V^2} - \cancel{1}\right) = -\frac{m^2\alpha}{V}$$

$$\theta = \theta_b e^{\left(-\frac{hP\alpha}{kAV}x\right)}$$

$$\boxed{\theta = \theta_b e^{-\frac{hPx}{\rho cAV}}}$$ ⇒ Matches our previous solution

So what if we move with our element and map out our temperature with respect to time.



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$$\theta = \theta_b e^{-\frac{hPx}{\rho c A V}} \Rightarrow \text{Note, } t = \frac{x}{V}$$

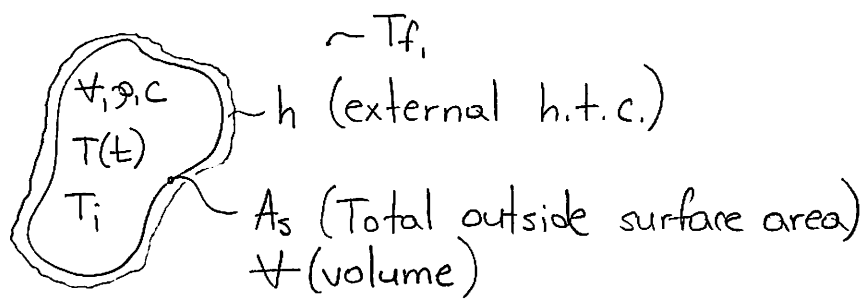
$$\theta = \theta_b e^{-\frac{hP}{\rho C A}t} \Rightarrow \text{Multiply by } \frac{\Delta x}{\Delta x} \text{ in the exponent}$$

$$\theta = \theta_b e^{-\frac{h(P\Delta x)}{\rho C(A\Delta x)}\cdot t} \Rightarrow P\Delta x = A_s \text{ (Total outside surface area)}$$

$$A\Delta x = \forall \text{ (Total fin volume in element)}$$

$$\boxed{\theta = \theta_b e^{-\frac{hA_s}{\rho c \forall}t}} \Rightarrow \text{Looks familiar (Lumped capacitance model)}$$

<u>Lumped Capacitance</u> Why do we care? Forensic analysis of dead bodies.



Energy ballance: $\dot{E}_{STORED} + \cancel{\dot{E}_{in}}^{0} - \dot{E}_{OUT} + \cancel{\dot{E}_{gen}}^{0} = 0$

$$\frac{\partial}{\partial t}(\forall \rho c T) = hA_s(T - T_f)$$

$$\forall \rho c \frac{\partial T}{\partial t} - hA_s(T - T_f) = 0$$

$$\frac{\partial T}{\partial t} - \frac{hA_s}{\forall \rho c}(T - T_f) = 0 \Rightarrow \text{Let } \theta = T - T_f$$

$$\frac{\partial \theta}{\partial t} - \frac{hA_s}{\forall \rho c}\theta = 0 \Rightarrow \text{Let } \lambda = \frac{hA_s}{\forall \rho c}$$

$$\frac{\partial \theta}{\partial t} - \lambda\theta = 0 \Rightarrow \int \frac{\partial \theta}{\theta} = \int \lambda dt$$

$$\ln\theta + \ln C_1 = \lambda t + C_2 \Rightarrow \text{Combine constants}$$

$$\ln\theta = \lambda t + C_3$$

$$e^{\ln\theta} = e^{\lambda t + C_3} \Rightarrow \theta = C_4 e^{\lambda t}$$

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Our B.C. is $\Theta(t=0) = \Theta_i = T_i - T_f$

$$\Theta(t=0) = C_4 \cancel{e^{0}}^{1} = \Theta_i \Rightarrow C_4 = \Theta_i$$

$$\therefore \boxed{\Theta = \Theta_i e^{-\lambda t} = \Theta_i e^{-\frac{hA_s t}{\forall \rho C}}} \Rightarrow \text{Lumped capacitance model.}$$

Note, only valid if the whole body is isothermal: $\frac{ht}{k} < 0.1$

Non-constant Fluid Temperature ($T(t)$ = body temperature)
Our previous equation would become: ($T_f(t)$ = fluid temp.)

$$\left[\frac{\partial T}{\partial t} + \lambda T = \lambda T_f(t)\right] \cdot e^{\lambda t} \quad \text{(Multiply by } e^{\lambda t}\text{, integrating factor method)}$$

$$\underbrace{e^{\lambda t}\frac{\partial T}{\partial t} + \lambda e^{\lambda t} T}_{} = \lambda e^{\lambda t} T_f(t)$$

$$\frac{\partial}{\partial t}\left(e^{\lambda t} T\right) = \lambda e^{\lambda t} T_f(t) \quad \text{(Integrate both sides)}$$

$$\int_0^t \partial\left(e^{\lambda t} T\right) = \lambda \int_0^t e^{\lambda t} T_f(t)\, dt$$

$$\boxed{e^{\lambda t} T \Big|_0^t = \lambda \int_0^t e^{\lambda t} T_f(t)\, dt} \Rightarrow \text{General solution, } \lambda = \frac{hA_s}{\forall \rho C}$$

Example Let $T_f = B \cdot t$ where B = constant
I.C. $\Rightarrow t = 0,\ T_f = T = 0$

$$e^{\lambda t} T \Big|_0^t = \lambda \int_0^t e^{\lambda t} (Bt)\, dt$$

$$e^{\lambda t} T(t) - \cancel{e^{\lambda(0)} T(0)}^{0} = e^{\lambda t} T(t) = \lambda B \int_0^t t e^{\lambda t}\, dt$$

So how do we solve this?

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Let $s = \lambda t$, $ds = \lambda dt$

$$e^{\lambda t}T(t) = \cancel{\lambda}B\int_0^{\frac{s}{\lambda}} \frac{s}{\cancel{\lambda}} e^s \frac{ds}{\lambda}$$

$$= \frac{B}{\lambda}\int_0^{\frac{s}{\lambda}} s e^s ds = \frac{B}{\lambda}\left(se^s - e^s\right)\Big|_0^{\frac{s}{\lambda}}$$

$$e^{\lambda t}T(t) = \frac{B}{\lambda} e^{\lambda t}(\lambda t - 1)\Big|_0^t$$

$$e^{\lambda t}T(t) = \frac{B}{\lambda}\left[e^{\lambda t}(\lambda t - 1) - e^0(0-1)\right]$$

$$= \frac{B}{\lambda}\left[e^{\lambda t}(\lambda t - 1) + 1\right]$$

Dividing through by $e^{\lambda t}$

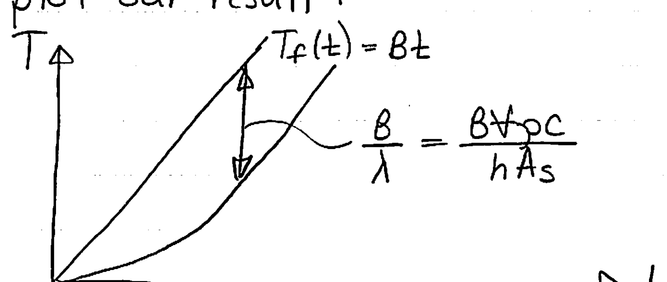
$$T(t) = \frac{B}{\lambda}\left[(\lambda t - 1) + e^{-\lambda t}\right]$$

$$= \underbrace{Bt}_{T_f(t)} - \frac{B}{\lambda}\left(1 - e^{-\lambda t}\right)$$

$$\boxed{T(t) = T_f(t) - \underbrace{\frac{B}{\lambda}\left(1 - e^{-\lambda t}\right)}_{\text{Lag term}}} \; ; \; \boxed{\lambda = \frac{hA_s}{\forall \rho c}}$$

Lag term. This is why its difficult to measure temperature changes in a fluid since measurement lags if B is high.

If we plot our result:



For a thermocouple:

$$\frac{B}{\lambda} = \frac{B\forall\rho c}{hA_s} = \frac{BR^{\cancel{3}}\rho c}{h\cancel{R^2}}$$

$$= \frac{BR\rho c}{h} \Rightarrow \begin{array}{l} R \sim 0.001 \\ h \sim 2500 \\ \rho \sim 1000 \\ c \sim 150 \\ B \sim 2 \end{array}$$

$$\boxed{\frac{B}{\lambda} = \frac{2}{25}\,^\circ C}$$

↳ For Water w/ TC.

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Example | Time of death.

Person is found dead at 5PM in a room with $T_f = 20°C$. The body temperature is $T = 25°C$ when found.
$h = 8\ W/m^2\cdot K$
Geometry: $D \approx 30\,cm$, $L = 1.7\,m$
Estimate the time of death.

A healthy persons temperature is $T_i = 37°C$
Assume radiation is negligible: Check:

$$h_{rad} = \varepsilon\sigma\left(T_s^2 + T_\infty^2\right)\left(T_s + T_\infty\right)$$
$$= (1.0)\left(5.67\times10^{-8}\,W/m^2\cdot K^4\right)\left((37+273)^2 + (20+273)^2\right)\left((37+273) + (20+273)\right)$$
$$h_{rad} = 6.26\ W/m^2\cdot K$$

Add another $\bar{h}_{rad} \approx 5\ W/m^2\cdot K$ to our convection h.t.c.

$$\bar{h}_{TOT} = h_c + h_{rad} \approx 13\,W/m^2\cdot K$$

Now lets assume a lumped capacitance model.

$$L_c = \frac{V}{A_s} = \frac{\pi r_o^2 L}{2\pi r_o L + 2\pi r_o^2} = 0.069\,m$$

Check the Biot number: $Bi_{L_c} = \dfrac{h L_c}{k}$

Assume $k = k_{water} = 0.6\ W/m\cdot K$ (since body is made up of 72% water by mass)

$$Bi_{L_c} \approx \frac{(13\,W/m^2\cdot K)(0.069\,m)}{0.6\,W/m\cdot K} = 1.5 > 0.1$$

So lumped capacitance is not applicable but we use it to get a rough estimate:

$$\lambda = \frac{hA_s}{\rho c V} = \frac{h}{\rho c L_c} = 4.53\times10^{-5}\,s^{-1} \Rightarrow \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\lambda t} \Rightarrow t = 7.5\,hr$$

END OF LECTURE 7 — ∴ Killed at 9:30 AM