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Two Dimensional Conduction (x,t)

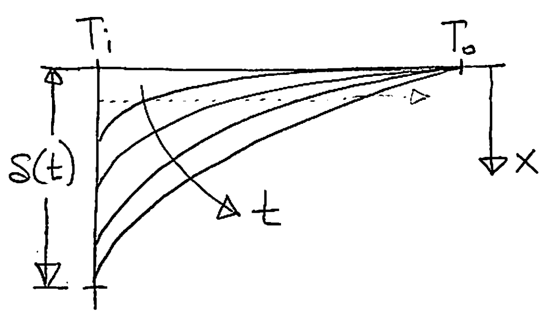
$T(x,t)$, $\dot{Q}''' = 0$, $V = 0$, $k = \text{constant}$, $\alpha = \text{constant}$

We derived previously that:

$$\frac{\partial^2 T}{\partial x^2} + \cancel{\frac{\partial^2 T}{\partial y^2}}^{0} + \cancel{\frac{\partial^2 T}{\partial z^2}}^{0} + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Here we need 2 boundary conditions and 1 initial condition



T_o = hot body temperature
T_i = initial temperature

So our governing PDE is: $\theta = \dfrac{T - T_o}{T_i - T_o}$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

$x = 0$, $\theta = 0$
$x \to \infty$, $\theta = 1$
$t = 0$, $\theta = 1$

Separation of variables won't work since the medium is infinite.

The only way to solve this is to look for a similarity variable, that relates x, t. One that works and is convenient is:

$$\eta = \frac{x}{f(t)}, \quad \text{so } \theta(\eta), \quad f(t) = \text{function of time}$$

Note, if you use Buckingham PI theorem you will get $\pi_1 = \theta$, $\pi_2 = \frac{x}{\sqrt{\alpha t}}$.

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{1}{f}\frac{\partial \theta}{\partial \eta}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial x}\right) = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{\partial \theta}{\partial \eta}\right) = \frac{1}{f}\frac{\partial}{\partial \eta}\left(\frac{\partial \theta}{\partial \eta}\right)\frac{\partial \eta}{\partial x} = \frac{1}{f^2}\frac{\partial^2 \theta}{\partial \eta^2}$$

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$$\frac{\partial\theta}{\partial t} = \frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial t} = -\frac{\partial\theta}{\partial\eta}\cdot x\cdot\frac{f'}{f^2} = -\eta\frac{\partial\theta}{\partial\eta}\frac{f'}{f}$$

Now if we back substitute into our PDE

$$\frac{1}{f^2}\frac{\partial^2\theta}{\partial\eta^2} + \frac{f'}{\alpha f}\eta\cdot\frac{\partial\theta}{\partial\eta} = 0$$

$$\frac{\partial^2\theta}{\partial\eta^2} + \underbrace{\left(\frac{f\cdot f'}{\alpha}\right)}_{\text{Let this term equal to a constant}}\eta\frac{\partial\theta}{\partial\eta} = 0$$

$$\frac{ff'}{\alpha} = 2 \quad \text{((Can choose any arbitrary number)}$$

$$f\frac{\partial f}{\partial t} = 2\alpha \Rightarrow \int f\,\partial f = \int 2\alpha\,\partial t$$

$$\frac{f^2}{2} = 2\alpha t$$

$$\boxed{f = 2\sqrt{\alpha t}}$$

$$\frac{ff'}{\alpha}\cdot\eta = 2\eta$$

$$\boxed{\eta = \frac{x}{2\sqrt{\alpha t}}} \Rightarrow \frac{\partial^2\theta}{\partial\eta^2} + 2\eta\frac{\partial\theta}{\partial\eta} = 0 \Rightarrow \text{ODE}$$

Our new B.C.'s become: $\left.\begin{array}{ll}\eta = 0, & \theta = 0\\ \eta\to\infty, & \theta = 1\end{array}\right\}$ Turned 2BC's & IC into 2BC's only

Rewriting our ODE

Aside

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}\cdot\frac{d}{dx}(f(x))$$

$$\underbrace{\frac{\theta''}{\theta'}}_{\text{Rewrite as: } \frac{\partial}{\partial\eta}(\ln\theta')} = -2\eta$$

$$\frac{\partial}{\partial\eta}\ln(\theta') = -2\eta$$

$$\int\partial\ln(\theta') = \int -2\eta\,\partial\eta$$

$$\ln(\theta') = -\cancel{2}\frac{\eta^2}{\cancel{2}} + C = -\eta^2 + C \quad \text{(Take ln(...) of both sides)}$$

$$\theta' = Ce^{-\eta^2}$$

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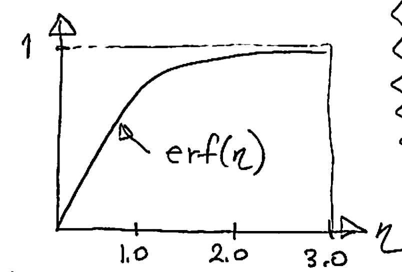
Integrating again,

$$\theta = c\int_0^{\eta} e^{-\eta^2}d\eta + C_2$$

Aside: Error function



We know $\theta(\eta=0)=0 \Rightarrow C_2 = 0$
$\theta(\eta \Rightarrow \infty) = 1$

$$1 = \underbrace{c\int_0^{\infty} e^{-\eta^2}d\eta}_{I_o = \frac{\sqrt{\pi}}{2}} \quad \text{(Gaussian integral)}$$

$$\Rightarrow C = \frac{2}{\sqrt{\pi}}$$

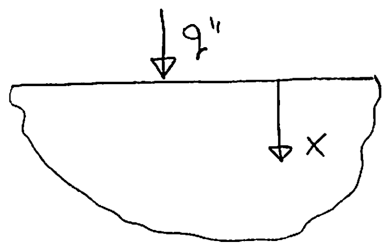
This integral cannot be solved analytically. Must be done numerically.

$$\therefore \boxed{\theta = \frac{T-T_o}{T_i-T_o} = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}d\eta = \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)}$$

→ error function, tabulated results

For the error function, see Table B4, pg 924 in Mills. Also: $1-\mathrm{erf}(\eta) = \underbrace{\mathrm{erfc}(\eta)}_{\text{complimentary error function.}}$

So what about heat transfer during transient conduction?



$$q''_{x=0} = -k\frac{\partial T}{\partial x}\bigg|_{x=0} \Rightarrow \text{let } \theta = \frac{T-T_o}{T_i-T_o}, \quad d\theta = \frac{1}{T_i-T_o}dT$$

$$= -k(T_i-T_o)\frac{\partial \theta}{\partial x}\bigg|_{x=0} = k\underbrace{(T_o-T_i)}_{\Delta T}\underbrace{\frac{\partial \eta}{\partial x}}_{\frac{1}{2\sqrt{\alpha t}}}\underbrace{\left(\frac{\partial \theta}{\partial \eta}\right)\bigg|_{x=0}}_{\frac{2}{\sqrt{\pi}}}$$

$$\boxed{q'' = \frac{k\Delta T}{\sqrt{\pi \alpha t}}}, \quad \boxed{\alpha = \frac{k}{\rho c}} \Rightarrow \text{Thermal diffusivity}$$

Note, to get total energy transfered, you need to integrate q''.

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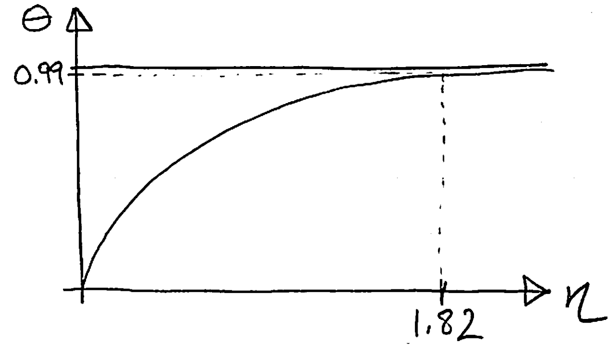
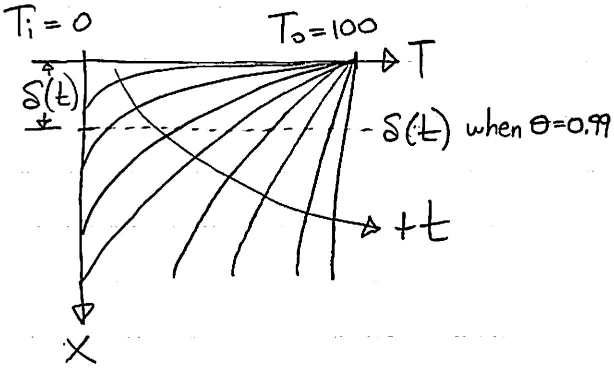
Lets analyze our solution: $\theta = \dfrac{T - T_o}{T_i - T_o}$





$$\frac{\delta}{2\sqrt{\alpha t}} = 1.82 \Rightarrow \boxed{\delta(t) = 3.04\sqrt{\alpha t}} \Rightarrow \text{Thermal penetration depth}$$

Lets look at some other cases:

1. Specified surface heat flux, $q'' = \text{constant}$

$$T(x,t) - T_i = \frac{q''}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\, e^{\left(-\frac{x^2}{4\alpha t}\right)} - x\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

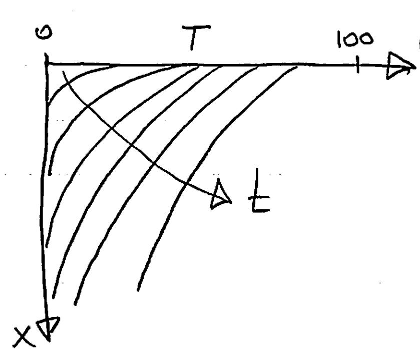
2. Convection on the surface, $q'' = h(T_\infty - T(0,t))$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - e^{\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)} \cdot \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

3. Energy pulse at surface, $e_s = \text{constant}$ $\left[\frac{J}{m^2}\right]$ @ $t = 0$
Like a laser pulse, no losses, all heat goes into the solid

$$T(x,t) - T_i = \frac{e_s}{k\sqrt{\pi t/\alpha}}\, e^{\left(-\frac{x^2}{4\alpha t}\right)}$$

2



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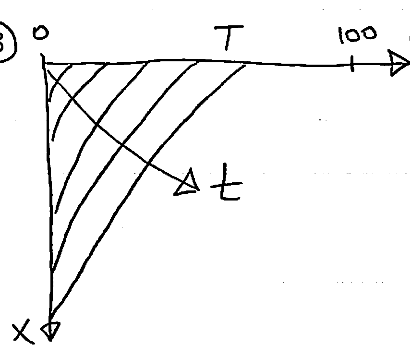

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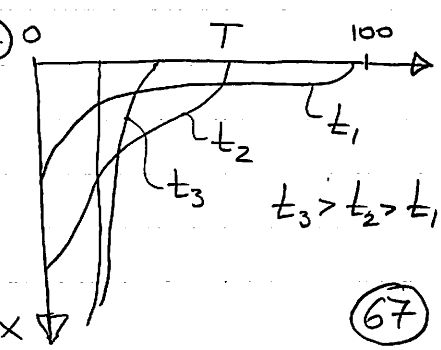

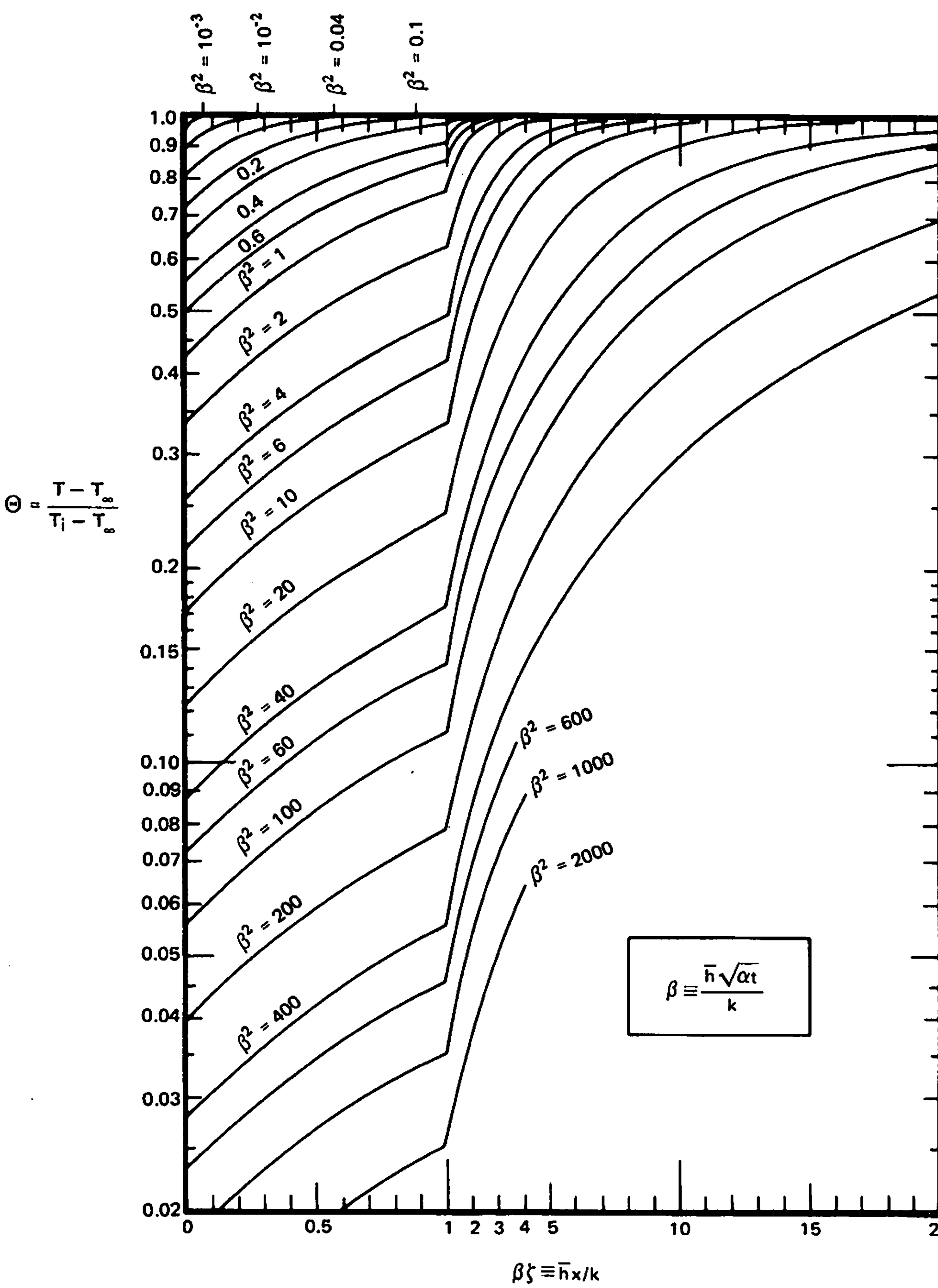

Figure 5.16 The cooling of a semi-infinite region by an environment at T_∞, through a heat transfer coefficient, $\bar{h}$.

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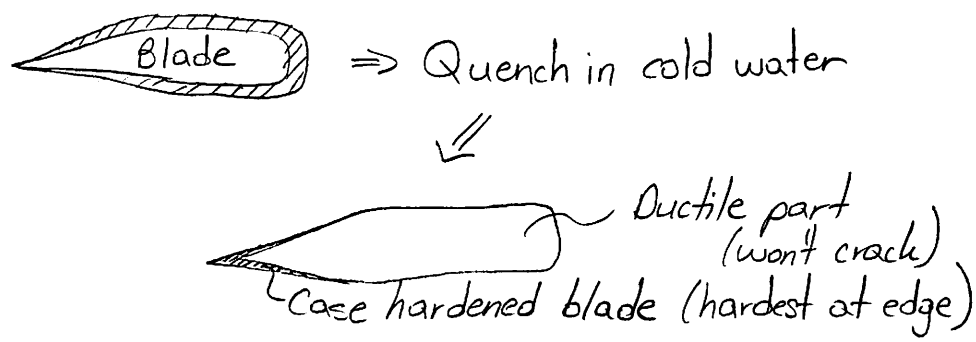
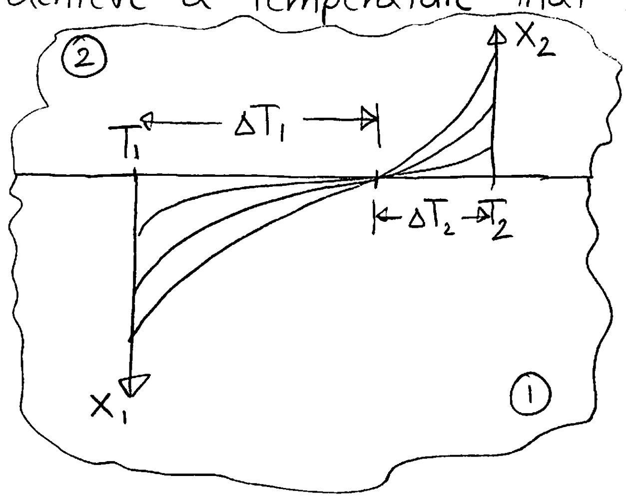
Application: Samurai in Japan used to make their swords with transient heat conduction in mind. They would coat the blade with clay sheath & quench



Contact of Two Semi-Infinite Solids

Two bodies at T_1 & T_2 brought into contact, they instantly achieve a temperature that is constant at their interface



Note, $T_{interface} \neq f(t)$ = constant

$T_{interface}$ will develop very quickly after the two bodies touch, & remain the same.

Applying an energy balance, $\Delta T = T_1 - T_2$

$$q'' = \frac{k\Delta T}{\sqrt{\pi \alpha t}} = \frac{(k\rho c)^{1/2}}{\sqrt{\pi}} \cdot \frac{\Delta T}{\sqrt{t}}$$

$$q''_1 = q''_2 \Rightarrow \frac{(k_1\rho_1 c_1)^{1/2}}{\cancel{\sqrt{\pi}}} \cdot \frac{\Delta T_1}{\cancel{\sqrt{t}}} = \frac{(k_2\rho_2 c_2)^{2}}{\cancel{\sqrt{\pi}}} \cdot \frac{\Delta T_2}{\cancel{\sqrt{t}}}$$

$$(k_1\rho_1 c_1)^{1/2}\Delta T_1 = (k_2\rho_2 c_2)^{1/2}\Delta T_2 \quad ①$$

$$\Delta T_1 + \Delta T_2 = \Delta T \quad ②$$

Back substituting ② into ①

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$$\Delta T_2 = \frac{(k\rho c)_1^{1/2}}{(k\rho c)_2^{1/2}} \cdot \Delta T_1$$

$$\Delta T = \left(1 + \sqrt{\frac{(k\rho c)_1}{(k\rho c)_2}}\right) \Delta T_1$$

$$\boxed{\Delta T_1 = \frac{\Delta T}{1 + \frac{(k\rho c)_1^{1/2}}{(k\rho c)_2^{1/2}}} = \frac{(k\rho c)_2^{1/2} \cdot \Delta T}{(k\rho c)_2^{1/2} + (k\rho c)_1^{1/2}}} \neq f(t), \therefore T_{interface} = \text{const.}$$

$$\boxed{q''_{x=0} = \frac{(k\rho c)_1^{1/2} (k\rho c)_2^{1/2}}{(k\rho c)_1^{1/2} + (k\rho c)_2^{1/2}} \cdot \frac{\Delta T}{\sqrt{\pi t}}} \Rightarrow \text{Heat flux between the two contacting bodies}$$

Note, this is fundamentally why when you touch certain objects in a room, they feel "colder" than others, even though they are at the same temperature.

Example | Touching a brass & wooden doorknob.

Brass: $\underbrace{k_1 = 109\ W/m\cdot K \quad \rho_1 = 8730\ kg/m^3 \quad c_1 = 380\ J/kg\cdot K}_{(k_1\rho_1c_1)^{1/2} = 19016}$

Wood: $\underbrace{k_2 = 0.17\ W/m\cdot K \text{ (Oak)} \quad \rho_2 = 750\ kg/m^3 \quad c_2 = 1700\ J/kg\cdot K}_{(k_2\rho_2c_2)^{1/2} = 466}$

Lets say $\Delta T = T_{body} - T_{Room} \approx 20°C$

Now we need human flesh properties.

Human Hand: $\left.\begin{aligned} k_3 &\cong 0.6\ W/m\cdot K \\ \rho_3 &\cong 1000\ kg/m^3 \\ c_3 &\cong 4190\ J/kg\cdot K \end{aligned}\right\} (k_3\rho_3c_3)^{1/2} = 1586$

Case 1: Brass Doorknob

$$q''_{3\to 1} = \frac{(19016)(1586)}{(19016)+(1586)} \cdot \frac{\Delta T}{\sqrt{\pi t}} = 1463 \frac{\Delta T}{\sqrt{\pi t}}$$

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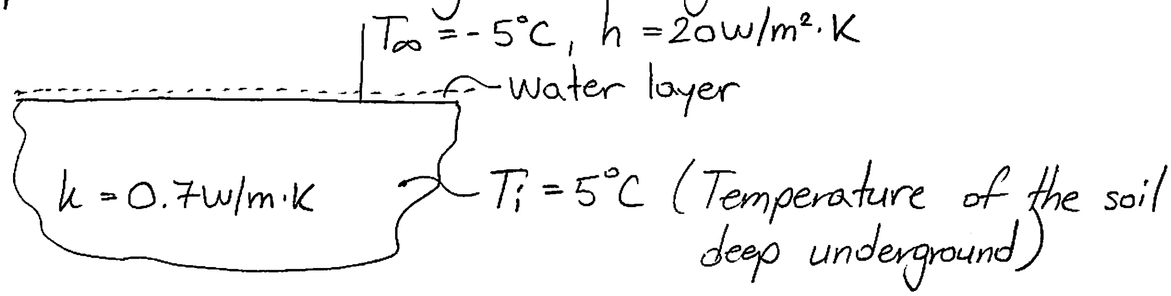
Case 2: Wooden Doorknob

$$q''_{3\to 2} = \frac{(466)(1586)}{466+1586} \cdot \frac{\Delta T}{\sqrt{\pi t}} = 360 \frac{\Delta T}{\sqrt{\pi t}}$$

We see that: $\dfrac{q''_{3\to 1}}{q''_{3\to 2}} \approx 4.1 \Rightarrow$ The brass doorknob will feel about 4 times "colder"

Your body senses heat by monitoring heat flux, not temperature, so the feeling of relative coldness depends on how fast you lose heat.

Example | Water freezing on the ground



$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \frac{0°C - (-5°C)}{5°C - (-5°C)} = 0.5$$

, note $T = 0°C$ because of water freezing temp.

$x = 0$ (top of soil)

Using our chart: 5.16 from Lienhard & Lienhard, 2008

$$\beta^2 = 0.6 = \left(\frac{h}{k}\right)^2 \alpha t \Rightarrow \boxed{t = 24 \text{ minutes}}$$

Therefore, ice will freeze in $\approx$ 24 minutes on the ground.

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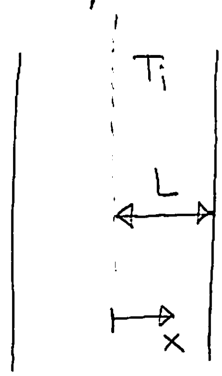
Transient Heat Conduction in Finite Bodies

Previously, we covered lumped capacitance : $T = f(t) \neq f(x)$
and transient heat conduction in infinite media : $0 \leq x \leq \infty$
What about finite bodies : $0 \leq x \leq L$?

Plane wall problem: Wall initially at T_i placed in medium at T_∞ & h_1 on outside.

h_1, T_∞



h_1, T_∞

What is $T(x,t)$

Writing out our heat equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$T(x, t=0) = T_i \quad \text{(initial condition)}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad \text{(1'st B.C.)}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h(T - T_\infty) \quad \text{(2'nd B.C.)}$$

To make things simpler, its helpfull to non-dimensionalize

Let $\bar{x} = \frac{x}{L}$, $\partial \bar{x} = \frac{1}{L}\partial x$ (Dimensionless distance from center)

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} , \quad \partial\theta = \frac{1}{T_i - T_\infty}\cdot \partial T \quad \text{(Dimensionless temperature)}$$

What about dimensionless time τ ?

$$\frac{\partial T}{\partial x} = \frac{(T_i - T_\infty)}{L}\frac{\partial \theta}{\partial \bar{x}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{(T_i - T_\infty)}{L^2}\frac{\partial \theta}{\partial \bar{x}}$$

$$\frac{\partial T}{\partial t} = (T_i - T_\infty)\frac{\partial \theta}{\partial t}$$

Back substituting

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$$\frac{\cancel{(T_i - T_\infty)}}{L^2}\frac{\partial^2\theta}{\partial\bar{x}^2} = \cancel{(T_i - T_\infty)}\frac{\partial\theta}{\partial t}\cdot\frac{1}{\alpha}$$

$$\frac{\partial^2\theta}{\partial\bar{x}^2} = \frac{L^2}{\alpha}\frac{\partial\theta}{\partial t}$$

So this helps us solve for τ: $\int \partial t = \int \frac{L^2}{\alpha}\partial\tau$

$$\boxed{\tau = \frac{\alpha t}{L^2}} \Rightarrow \text{Fourier number}$$

Aside: Fourier # simplified

$$F_o = \frac{\alpha t}{L^2} = \frac{kt}{\rho c L^2}\cdot\frac{\Delta T}{\Delta T} = \frac{\left(k\frac{\Delta T}{L}\right)}{\left(\frac{\rho c \Delta T L}{t}\right)} = \frac{q''_{cond}}{q''_{stored}}$$

$$\boxed{\tau = F_o = \frac{\alpha t}{L^2} = \frac{\text{diffusive heat conduction rate}}{\text{heat storage rate (transient)}}} = \text{dimensionless time in h.t. problems.}$$

Now if we non-dimensionalize our second boundary condition

$$-k\frac{\partial T}{\partial x}\bigg|_{x=L} = h(T-T_\infty)$$

$$-k\frac{\cancel{(T_i - T_\infty)}}{L}\frac{\partial\theta}{\partial\bar{x}}\bigg|_{x=1} = h\,\theta\,\cancel{(T_i - T_\infty)}$$

We can rewrite this as:

$$\frac{\partial\theta(1,\tau)}{\partial\bar{x}} = \underbrace{-\frac{hL}{k}}_{-Bi_L}\theta(1,\tau)$$

$-Bi_L \Rightarrow$ Biot number $=$ dimensionless h.t.c. $= \dfrac{\text{conduction resistance}}{\text{convection resistance}}$

Our other B.C. & I.C. are:

$$\frac{\partial\theta(0,\tau)}{\partial\bar{x}} = 0$$

$$\theta(\bar{x},0) = 1$$

Note we've taken the problem from: $T(x,t) = f(x, L, t, \alpha, h, T_\infty, T_i)$

to: $\theta = f(\bar{x}, Bi, F_o) \Rightarrow$ Very Powerfull!

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To solve, we need to use separation of variables:

$$\theta(\bar{x},\tau) = F(\bar{x}) \cdot G(\tau)$$

Back substituting & dividing by $F \cdot G$

$$\underbrace{\frac{1}{F}\frac{\partial^2 F}{\partial \bar{x}^2}}_{f(\bar{x})\text{ only}} = \underbrace{\frac{1}{G}\frac{\partial G}{\partial \tau}}_{f(\tau)\text{ only}} = \text{Constant}$$

(The only valid solution) $\neq 0$ since that says no time dependence. $G \neq f(\tau)$.

($\bar{x}$, & τ can be varied independently)

We will assume a solution of the form $-\lambda^2$

$$\frac{\partial^2 F}{\partial \bar{x}^2} + \lambda^2 F = 0 \quad (1)$$

$$\frac{\partial G}{\partial \tau} + \lambda^2 G = 0 \quad (2)$$

Solving (1) first: $\lambda'^2 + \lambda^2 = 0$

$$\lambda' = \pm\sqrt{-1}\,\lambda = \lambda i$$

So our solution becomes:

$$F = C_1 e^{+i\lambda\bar{x}} + C_2 e^{-i\lambda\bar{x}} \Rightarrow \text{since} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Can rewrite our solution as:

$$\boxed{F = C_1 \cos(\lambda\bar{x}) + C_2 \sin(\lambda\bar{x})}$$

For G:

$$\lambda' + \lambda^2 = 0 \Rightarrow \lambda' = -\lambda^2$$

$$\boxed{G = C_3 e^{-\lambda^2\tau}}$$

Combining our solution, we obtain:

$$\theta = F \cdot G = C_3 e^{-\lambda^2\tau}\left(C_1 \cos(\lambda\bar{x}) + C_2 \sin(\lambda\bar{x})\right) = e^{-\lambda^2\tau}\left[A\cos(\lambda\bar{x}) + B\sin(\lambda\bar{x})\right]$$

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Applying our B.C.'s:

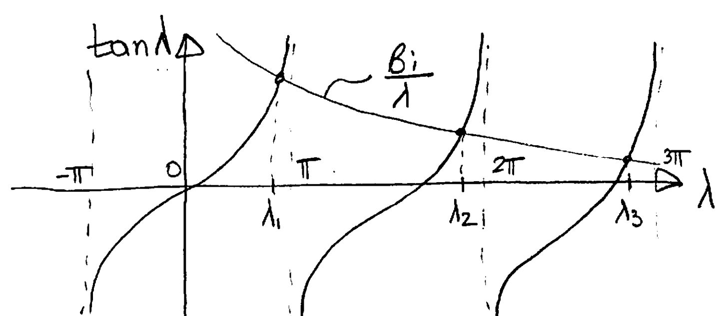
$$\frac{\partial \Theta(0,\tau)}{\partial \bar{x}} = 0 \Rightarrow -e^{-\lambda^2\tau}(A\lambda\sin(0) + B\lambda\cos(0)) = 0 \rightarrow B = 0$$

$$\Theta = Ae^{-\lambda^2\tau}\cos(\lambda\bar{x})$$

$$\frac{\partial \Theta(1,\tau)}{\partial \bar{x}} = -Bi\,\Theta(1,\tau) \Rightarrow -A\cancel{e^{-\lambda^2\tau}}\lambda\sin\lambda = -Bi\,A\cancel{e^{-\lambda^2\tau}}\cos\lambda$$

$$\lambda\tan\lambda = Bi$$

We know $\tan\lambda$ is a periodic function with period π. The solution can lie anywhere between 0 & π, π & 2π, ...



$\Rightarrow$ Multiple roots exist

So we have multiple solutions

$\lambda_n \tan\lambda_n = Bi$ (Eigenfuction with eigenvalues λ_n)

There exist an infinite number of solutions of the form $Ae^{-\lambda^2\tau}\cos(\lambda\bar{x})$. The final solution is a linear combination of all of them:

$$\boxed{\Theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2\tau}\cos(\lambda_n\bar{x})}$$

Where the constants A_n are determined from our I.C.

$$\Theta(\bar{x},0) = 1 \rightarrow 1 = \sum_{n=1}^{\infty} A_n\cos(\lambda_n\bar{x})$$

Using orthoganality $\Rightarrow$ multiply both sides by $\cos(\lambda_m\bar{x})$ & integrate

$$\int_0^1 \cos(\lambda_m\bar{x})d\bar{x} = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos(\lambda_n\bar{x})\cos(\lambda_m\bar{x})d\bar{x}}_{=0 \text{ if } n\neq m}$$

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So our solution becomes

$$\int_0^1 \cos(\lambda_n \bar{x})\, d\bar{x} = A_n \int_0^1 \cos^2(\lambda_n \bar{x})\, d\bar{x}$$

$$\boxed{A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)}} \Rightarrow \text{Constants in the } \theta \text{ solution series}$$

So now what? We can do the same for a cylinder & sphere.

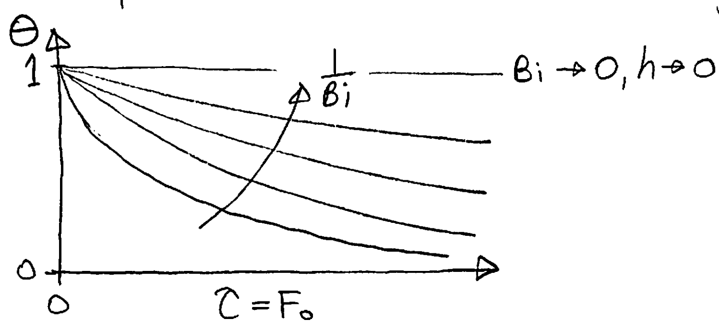
Since solution involves an infinite series, not very usefull to solve analytically. However, good approximation is made with the first few terms since rest decay rapidly due to the $e^{-\lambda_n^2 \tau}$ term.

END OF LECTURE 8

Tabular Solutions

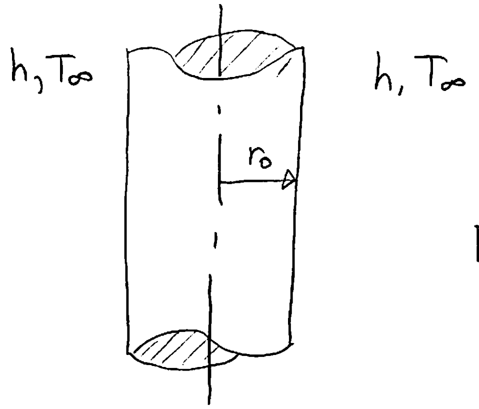
We know our solution is : $\theta = f(\bar{x}, Bi, Fo)$

We can plot our results as: (for a given $\bar{x}$)



$Bi_L = \dfrac{hL}{k}$

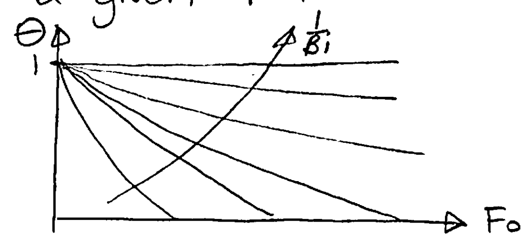
Similar if we had a cylinder :



$\bar{r} = \dfrac{r}{r_o}$, $\theta = f(\bar{r}, Bi_{r_o}, Fo)$

$Bi = \dfrac{hr_o}{k}$

For a given $\bar{r}$:



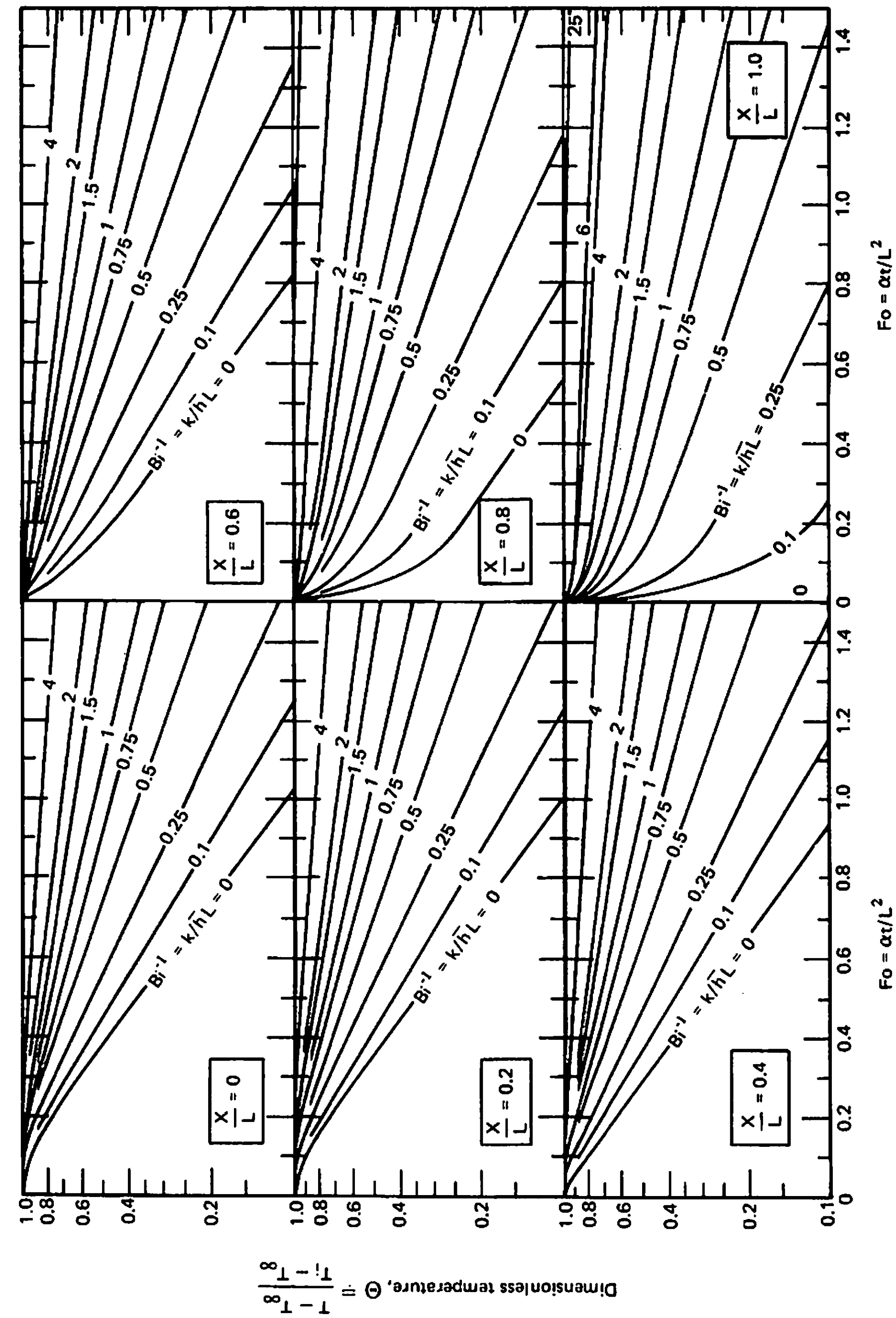


Figure 5.7 The transient temperature distribution in a *slab* at six positions: $x/L = 0$ is the center, $x/L = 1$ is one outside boundary.

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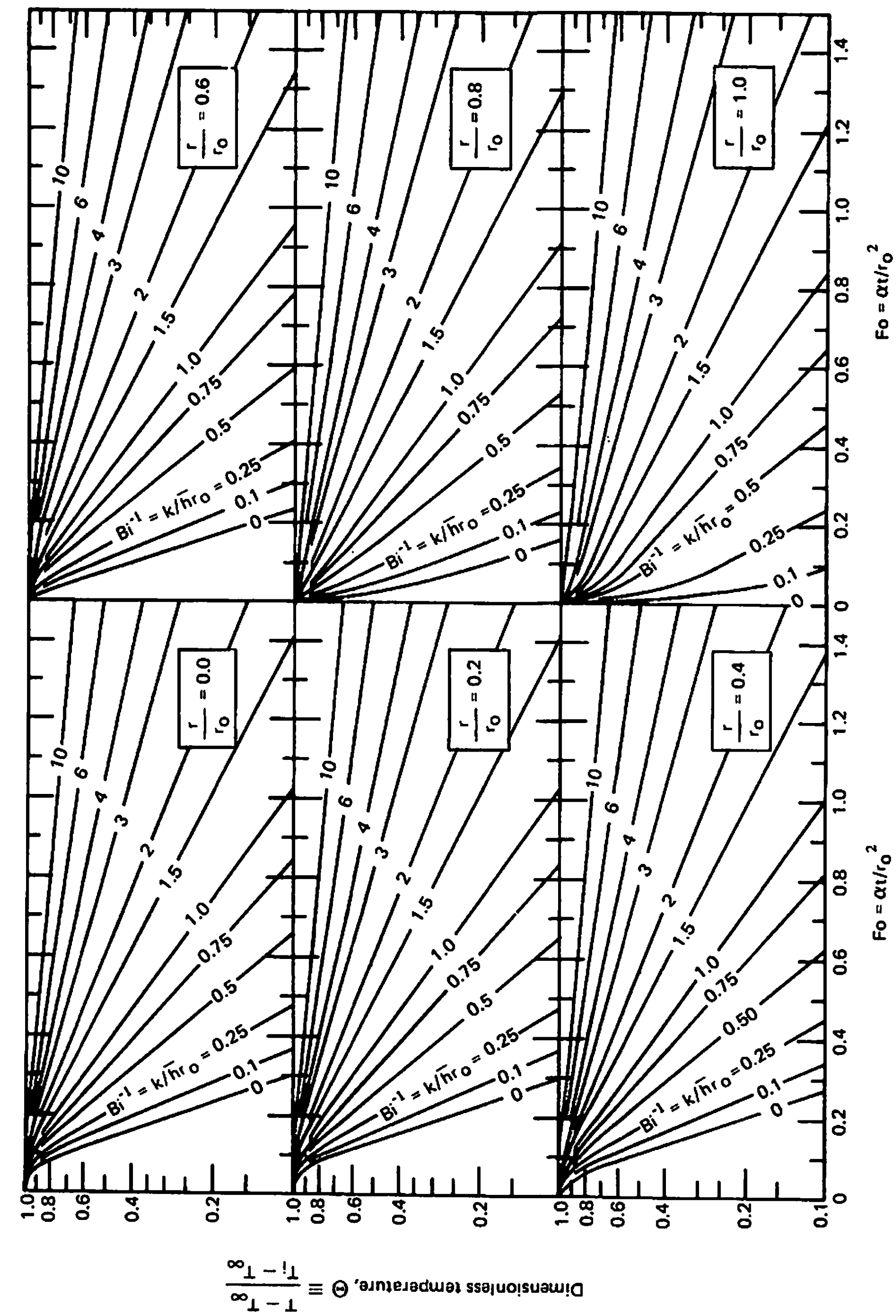

Figure 5.9 The transient temperature distribution in a *sphere* of radius r_o at six positions: $r/r_o = 0$ is the center; $r/r_o = 1$ is the outside boundary.

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