2-D Finite conduction (or 3-D)

Note, if we had a 2-D slab, the same separation of variables approach works:

\[ T = f(x, y, B_{ix}, B_{iy}, F_{ox}, F_{oy}) \]

\[ T = \Theta_{x,t} \cdot \Theta_{y,t} \]

\[ \Theta_{x,t} = f(\bar{x}, F_{ox}, B_{ix}) \]

\[ \bar{x} = \frac{x}{a} \]

\[ F_{ox} = \frac{ax + t}{a^2} \]

\[ B_{ix} = \frac{h_{a}}{k} \]

\[ \Theta_{y,t} = f(\bar{y}, F_{oy}, B_{iy}) \]

\[ \bar{y} = \frac{y}{b} \]

\[ F_{oy} = \frac{ax + t}{b^2} \]

\[ B_{iy} = \frac{hb}{k} \]

Use charts to solve for each direction & then \( T = \Theta_{x,t} \cdot \Theta_{y,t} \)

For a 3-D body, it would be the same:

\[ T = \Theta_{x,t} \cdot \Theta_{y,t} \cdot \Theta_{z,t} \]

Use tabulated results.
Steady Multi-Dimensional Heat Transfer (The Shape factor)
A heat conduction shape factor, $S$, may be defined for steady problems involving isothermal surfaces as follows:

$$Q = Sk\Delta T$$

$S$ obtained analytically or numerically (for complex problems)

$$R = \frac{1}{kS}$$

For example, if we look at a cylinder:

$$S_{\text{cyl}} = \frac{1}{kR_{\text{cyl}}} \Rightarrow R_{\text{cyl}} = \frac{1}{ln\left(\frac{r_2}{r_1}\right)}$$

$$S_{\text{cyl}} = \frac{2\pi L}{ln\left(\frac{r_2}{r_1}\right)}$$

In general, values are tabulated and solutions are in the form:

$$Q = Sh\Delta T, \quad R_\pm = \frac{1}{kS}$$

See Mills Table 3.2, page 162-163
Table 5.4 Conduction shape factors: $Q = S k \Delta T$, $R_t = 1/(kS)$.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Shape factor, $S$</th>
<th>Dimensions</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Conduction through a slab</td>
<td>$A/L$</td>
<td>meter</td>
<td>Example 2.2</td>
</tr>
<tr>
<td>2. Conduction through wall of a long thick cylinder</td>
<td>$\frac{2\pi}{\ln(r_o/\tau_1)}$</td>
<td>none</td>
<td>Example 5.9</td>
</tr>
<tr>
<td>3. Conduction through a thick-walled hollow sphere</td>
<td>$\frac{4\pi (r_o \tau_i)}{r_o - \tau_i}$</td>
<td>meter</td>
<td>Example 5.10</td>
</tr>
<tr>
<td>4. The boundary of a spherical hole of radius $R$ conducting into an infinite medium</td>
<td>$4\pi R$</td>
<td>meter</td>
<td>Problems 5.19 and 2.15</td>
</tr>
<tr>
<td>5. Cylinder of radius $R$ and length $L$, transferring heat to a parallel isothermal plane; $h \ll L$</td>
<td>$\frac{2\pi L}{\cosh^{-1}(h/R)}$</td>
<td>meter</td>
<td>[5.16]</td>
</tr>
<tr>
<td>6. Same as item 5, but with $L \to \infty$ (two-dimensional conduction)</td>
<td>$\frac{2\pi}{\cosh^{-1}(h/R)}$</td>
<td>none</td>
<td>[5.16]</td>
</tr>
<tr>
<td>7. An isothermal sphere of radius $R$ transfers heat to an isothermal plane; $R/h &lt; 0.8$ (see item 4)</td>
<td>$\frac{4\pi R}{1 - R/2h}$</td>
<td>meter</td>
<td>[5.16, 5.17]</td>
</tr>
</tbody>
</table>

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Convection Heat Transfer

Convection is the transfer of thermal energy due to both conduction and by bulk "carrying" of the energy through the velocity of the fluid.

Four Categories:

1) Forced vs. Free (Natural)

2) Laminar vs. Turbulent

3) External vs. Internal

4) Fully Developed vs. Developing

We assume: 1) Steady State
            2) Constant properties
            3) Incompressible
\[ q'' = h (T_s - T_f) \quad \text{or} \quad q'' = -k_f \frac{dT}{dn}\bigg|_s \]

Non-dimensionalizing:

Let:
\[ \Theta = \frac{T - T_f}{T_s - T_f}, \quad n^* = \frac{n}{L} \]

Back substitute and equate (1) & (2)
\[ -k_f \left( \frac{T_s - T_f}{L} \right) \frac{d\Theta}{dn^*}\bigg|_s = h (T_s - T_f) \]

\[ -\frac{d\Theta}{dn^*} = \frac{hL}{k_f} = \text{Nu} \Rightarrow \text{Nusselt number} \]

\[ \text{Nu} = \frac{hL}{k_f} = \frac{\text{convective heat transfer rate}}{\text{conductive heat transfer rate}} \]

Since \( \text{Nu} \propto \frac{d\Theta}{dn^*} \propto \Theta \), it typically depends on:

1) Flow cond.
2) Fluid properties
3) Geometry
4) Boundary conditions

External Flow

Assuming \( \rho, \mu, C = \text{constant} \) (density, viscosity, heat capacity)
Linked to heat transfer because of the boundary layer
\[ \nabla \Rightarrow \text{Where all of the gradients happen in velocity} \]

2-0 Laminar Flow: (Flat Plate, steady)
x-momentum: \( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial \rho}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \)

y-momentum: \( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial \rho}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \)

continuity:
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

Looking more closely at our boundary layer:

Let:
\[
\bar{u} = \frac{u}{V_\infty}, \quad \bar{v} = \frac{v}{V_\infty}, \quad \bar{\rho} = \frac{\rho}{\rho V_\infty^2}
\]

\[
\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}
\]

Now our equations become:
\[
\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{\rho}}{\partial \bar{x}} + \frac{1}{Re_L} \left[ \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]
\]
\[
\bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{\rho}}{\partial \bar{y}} + \frac{1}{Re_L} \left[ \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right]
\]
\[
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0
\]

So what can we say about these equations and their relative terms? \( u \sim V_\infty, \bar{x} \sim L, \bar{y} \sim S \) (boundary layer thick)
\[
\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \Rightarrow \frac{V_\infty}{L} + \frac{\bar{v}}{S} = 0 \Rightarrow \bar{v} = \frac{S}{L} V_\infty
\]

Looking back at our momentum equations

Note: dropped the negative sign since scaling analysis.
Going to x-momentum
\[ \frac{V_\infty}{L} + V_\infty \left( \frac{S}{L} \right) \cdot \frac{V_\infty}{S} \sim \frac{V_\infty^2}{L} \sim \frac{V_\infty^2}{L} \]
\[ \approx 0 \text{ since } \frac{S}{L} \ll 1 \]

Y-momentum:
\[ \frac{V_\infty}{L} \frac{S}{L} \frac{V_\infty}{1} + V_\infty^2 \left( \frac{S}{L} \right)^2 \frac{1}{S} = - \frac{1}{\rho} \frac{\partial \rho}{\partial y} + U \frac{S}{L} V_\infty \left[ \frac{1}{L^2} + \frac{1}{S^2} \right] \]
\[ \sim \frac{V_\infty^2}{L} \left( \frac{S}{L} \right) \sim \frac{V_\infty^2}{L} \left( \frac{S}{L} \right) = \frac{V_\infty^2}{S} + UV_\infty \left( \frac{S}{L} \right) \left[ \frac{1}{L^2} + \frac{1}{S^2} \right] \]

Since \( \frac{S}{L} \ll 1 \), we can drop the inertia terms, & viscous terms.

Our y-momentum equation becomes:
\[ \frac{\partial \rho}{\partial y} = 0 \Rightarrow \rho(y) = \text{Constant} \Rightarrow \text{Pressure is not a function of } y \]

If we look at the edge of our boundary layer

Our x-momentum equation along the streamline \( S \) becomes
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial \rho}{\partial x} + u \left( \frac{\partial^2 u}{\partial y^2} \right) \]
\[ \text{at } \rho \text{, } \frac{\partial \rho}{\partial x} = 0 \]
\[ V_\infty \frac{\partial V_\infty}{\partial x} + \frac{1}{\rho} \frac{\partial \rho}{\partial x} = 0 \Rightarrow \frac{\rho}{2} V_\infty^2 = \text{constant} \]

\( \rho \) Bernoulli equation
Note from this, we can see that the pressure in the x-direction is constant \((\partial P/\partial x) = 0\) for steady flow past a flat plate, since \(v_\infty = \text{constant}\) at the b.e. edge.

So our boundary layer equations become:

\[
\begin{align*}
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= u \frac{\partial^2 u}{\partial y^2} & (1) \\
\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 0 & (2) \\
\rho(y) &= \text{constant} & (3)
\end{align*}
\]

Boundary conditions:

\[
\begin{align*}
u = v_\infty @ y = 0 \\
u = u_\infty @ y \to \infty.
\end{align*}
\]

So now we must convert our PDE into a solvable ODE. Cannot use separation of variables (infinite medium). Let's try a similarity solution.

Assume: \(\frac{u}{v_\infty} = f(\eta)\) where \(\eta = \frac{y}{S}\)

So how do we define \(S\)?

Looking back at our non-dimensional x-momentum:

\[
\begin{align*}
\bar{u} \frac{\partial \bar{u}}{\partial x} + v \frac{\partial \bar{u}}{\partial y} &= \frac{1}{Re_L} \frac{\partial^2 \bar{u}}{\partial \eta^2}, \\
\bar{u} &= \frac{u}{v_\infty} \sim \frac{v_\infty}{v_\infty} \sim O(1) \\
\sim O(1) \to \sim O(1) \\
\text{Has to be on order of 1.} \\
\bar{x} &= \frac{x}{L} \sim \frac{L}{L} \sim O(1) \\
\bar{y} &= \frac{y}{L} \sim \frac{S}{L}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{Re_L} \frac{v_\infty}{v_\infty} \left(\frac{S}{L}\right)^2 \sim 1 \\
\frac{1}{Re_L} \frac{S^2}{L^2} \sim 1 \Rightarrow \frac{S}{L} \sim \frac{1}{\sqrt{Re_L}} \quad \text{or} \quad \frac{S}{x} \sim \frac{1}{\sqrt{Re_x}}
\end{align*}
\]

\[
S \sim \sqrt{\frac{v_\infty}{v_\infty}} \quad \text{so now we get:} \quad \eta = \sqrt{\frac{v_\infty}{v_\infty}} x
\]

\(\)Will see an easier way to get this later.
So what about \( v \)? Looking at continuity:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \bar{u} = \phi(n) = \frac{u}{V_\infty}
\]

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}
\]

\[
\frac{\partial u}{\partial x} = -\frac{1}{2x} n \phi'
\]

Back substituting, we obtain:

\[
\frac{\partial v}{\partial y} = \frac{1}{2x} \phi' n
\]

\[
\bar{v} = \frac{1}{2x} \int_0^n \phi' n \, dy = \frac{1}{2x} \sqrt{\frac{X}{V_\infty}} \int_0^n \phi' n \, dn
\]

\[
= \frac{1}{2} \sqrt{\frac{1}{Re_x^*}}
\]

Now we need to solve our integral:

\[
\int_0^n n \phi' \, dn = uv - \int v \, du \quad \text{(Integration by parts: } \int udv = uv - \int v \, du) \]

\[
= n \phi - \int_0^n \phi' \, dn , \quad u = n , \quad v = \phi
\]

let \( \int_0^n \phi' \, dn = F(n) \)

So now we have:

\[
\bar{v} = \frac{1}{2} \sqrt{\frac{1}{Re_x^*}} \left( n F' - F \right)
\]

Putting everything together we get:

\[
F'' + \frac{1}{2} FF'' = 0; \quad n = \sqrt{\frac{V_\infty}{xV}}
\]

At the wall: \( n = 0 \), \( F' = F = 0 \)

Outside the b.l: \( n \to \infty \), \( F' = 1 \)
Rewriting our equation:
\[
\frac{d^2 \bar{U}}{d\eta^2} + \frac{1}{2} F \frac{d\bar{U}}{d\eta} = 0
\]
To solve, we need to assume an infinite series solution
\[
F = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + \ldots +
F' = a_1 + 2a_2 \eta + 3a_3 \eta^2 + \ldots +
F'' = 2a_2 + 6a_3 \eta + \ldots +
F''' = 6a_3 + \ldots +
\]
Back substitute and solve for your coefficients
\[
F'' + \frac{1}{2} FF' = 0
\]
\[
\begin{bmatrix}
\cdots \eta^0 & \cdots \eta^1 & \cdots \eta^2 & \cdots \eta^3 & \cdots \eta^4 & \cdots \eta^5 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
You obtain a recursion formula relating your constants
\[
F = \frac{a_2 \eta^2}{2!} - \frac{a_2 \eta^5}{2 \cdot 5!} + \frac{11 a_3 \eta^8}{8!} + \ldots
\]
\[
a_2 = 0.332 \Rightarrow \text{Heinrich Blasius solved this in 1911 for his Ph.D. work with Ludwig Prandtl.}
\]
So
\[
F = \int_0^\eta \phi d\eta, \quad F' = \phi = \bar{U} = \frac{U}{V_\infty}
\]
\[
\bar{U} = F'
\]

5.0 = \delta \sqrt{\frac{V_\infty}{\chi U}} = \frac{\delta}{\chi} \sqrt{Re_x}
\]
\[
\delta = \frac{5 \chi}{\sqrt{Re_x}}
\]
\[\text{Hydrodynamic boundary layer thickness.}\]
This is very useful because now we can calculate things like shear stress

$$\tau(x) = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \rho u V_\infty \frac{\partial u}{\partial y} \bigg|_{y=0} = \rho u V_\infty \frac{\partial u}{\partial n} \bigg|_{n=0} \left(\frac{\partial n}{\partial y}\right) = \rho u V_\infty a_2 \sqrt{\frac{V_\infty}{x U}}$$

$$\frac{\tau(x)}{\frac{1}{2} \rho V_\infty^2} = \frac{2a_2}{\sqrt{Re_x}} = \frac{0.664}{Re_x^{1/2}} \Rightarrow \text{Skin friction coefficient for a flat plate in laminar flow conditions.}$$

If we want the average:

$$\overline{\tau} = \frac{1}{L} \int_0^L \tau(x) \, dx \Rightarrow \tau(x) = C \frac{1}{x^2}$$

$$= C \cdot \frac{1}{L} \int_0^L \frac{dx}{\sqrt{x}} = \frac{2C}{\sqrt{L}}$$

$$C_c = \frac{\overline{\tau}}{\frac{1}{2} \rho V_\infty^2} = \frac{1.328}{Re_x^{1/2}} \Rightarrow \text{Average skin friction coeff.}$$

So is there an easier way to see some simple things about boundary layers?

Inviscid Flow

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x}$$

Inside the b.l. Inertia ~ Viscosity

$$\rho \frac{V_\infty^2}{L} \sim \mu \frac{V_\infty}{S^2} \Rightarrow S^2 = \frac{\mu L}{\rho V_\infty} \Rightarrow S \sim \sqrt{\frac{UL}{V_\infty} \left(\frac{L}{L}\right)}$$

$$S \sim \frac{1}{\sqrt{Re_x}}$$