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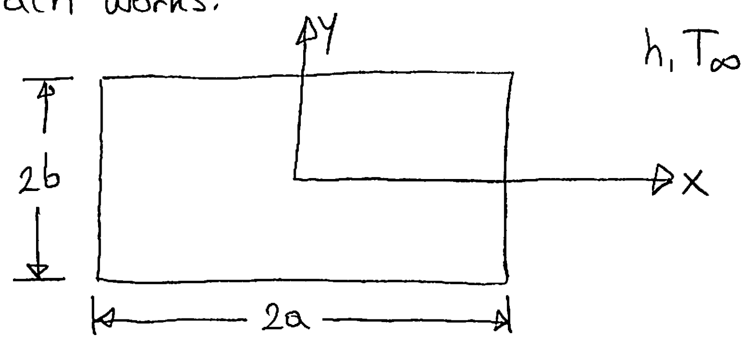
2-D Finite conduction (or 3-D)

Note, if we had a 2-D slab, the same separation of variables approach works:



$$\theta = f(\bar{x}, \bar{y}, Bi_x, Bi_y, Fo_x, Fo_y)$$

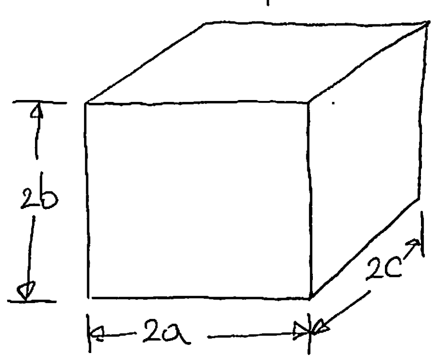
$$\boxed{\theta = \theta_{x,t} \cdot \theta_{y,t}}$$

$$\theta_{x,t} = f(\bar{x}, Fo_x, Bi_x)$$
$$\bar{x} = \frac{x}{a}$$
$$Fo_x = \frac{\alpha t}{a^2}$$
$$Bi_x = \frac{ha}{k}$$

$$\theta_{y,t} = f(\bar{y}, Fo_y, Bi_y)$$
$$\bar{y} = \frac{y}{b}$$
$$Fo_y = \frac{\alpha t}{b^2}$$
$$Bi_y = \frac{hb}{k}$$

Use charts to solve for each direction & then $\theta = \theta_{x,t} \cdot \theta_{y,t}$

For a 3-D body, it would be the same:



$$\boxed{\theta = \theta_{x,t} \cdot \theta_{y,t} \cdot \theta_{z,t}}$$

Use tabulated results.

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Steady Multi-Dimensional Heat Transfer (The Shape factor)

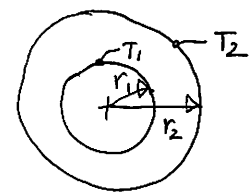
A heat conduction shape factor, S, may be defined for steady problems involving isothermal surfaces as follows:

$$Q = S k \Delta T$$

↳ Obtained analytically or numerically (for complex problems)

$$R = \frac{1}{kS}$$

For example, if we look at a cylinder:



$$S_{cyl} = \frac{1}{k R_{cyl}} \Rightarrow R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi k L}$$

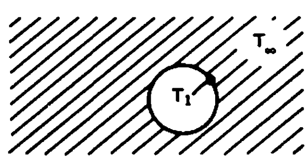
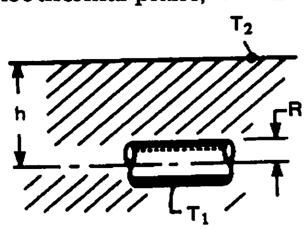
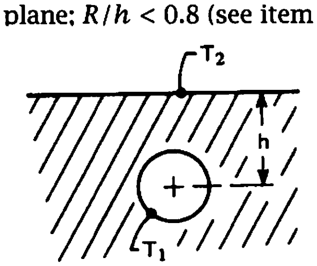
$$\boxed{S_{cyl} = \frac{2\pi L}{\ln(r_2/r_1)}}$$

In general, values are tabulated and solutions are in the form:

$$Q = S k \Delta T, \quad R_t = \frac{1}{kS}$$

See Mills Table 3.2, page 162-163

Table 5.4 Conduction shape factors: $Q = S\,k\Delta T$, $R_t = 1/(kS)$.

Situation	*Shape factor, S*	*Dimensions*	*Source*
1. Conduction through a slab	A/L	meter	Example 2.2
2. Conduction through wall of a long thick cylinder	$\dfrac{2\pi}{\ln(r_o/r_i)}$	none	Example 5.9
3. Conduction through a thick-walled hollow sphere	$\dfrac{4\pi(r_o r_i)}{r_o - r_i}$	meter	Example 5.10
4. The boundary of a spherical hole of radius R conducting into an infinite medium 	$4\pi R$	meter	Problems 5.19 and 2.15
5. Cylinder of radius R and length L, transferring heat to a parallel isothermal plane; $h \ll L$ 	$\dfrac{2\pi L}{\cosh^{-1}(h/R)}$	meter	[5.16]
6. Same as item 5, but with $L \longrightarrow \infty$ (two-dimensional conduction)	$\dfrac{2\pi}{\cosh^{-1}(h/R)}$	none	[5.16]
7. An isothermal sphere of radius R transfers heat to an isothermal plane; $R/h < 0.8$ (see item 4) 	$\dfrac{4\pi R}{1 - R/2h}$	meter	[5.16, 5.17]

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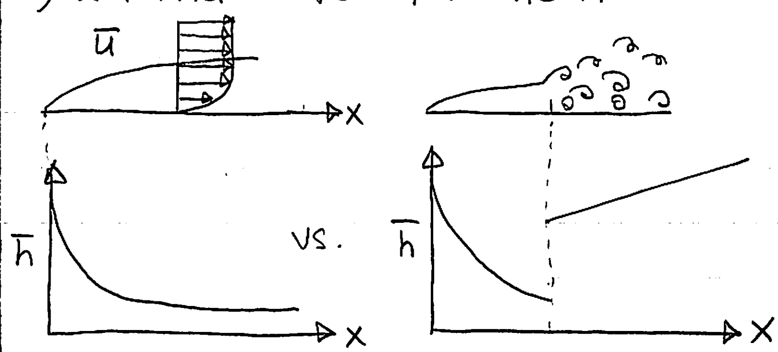
Convection Heat Transfer

Convection is the transfer of thermal energy due to both conduction and by bulk "carrying" of the energy through the velocity of the fluid.

Four Categories:

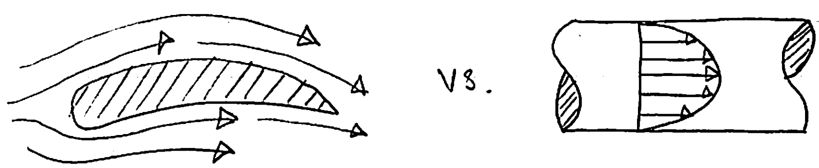
1) Forced vs. Free (Natural)

2) Laminar vs. Turbulent

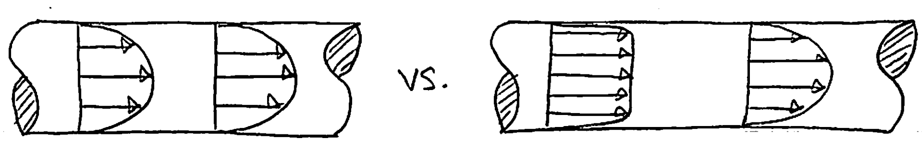


$\Rightarrow$ We will find out why.

3) External vs. Internal



4) Fully Developed vs. Developing



We assume: 1) Steady State
2) Constant properties
3) Incompressible

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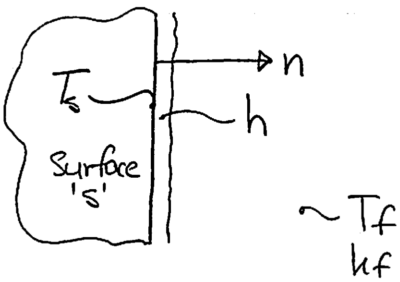


$$q'' = h\,(T_s - T_f) \quad ①$$

or

$$q'' = -k_f \left.\frac{\partial T}{\partial n}\right|_s \quad ②$$

where:
h = heat trans. coeff.
T_s = surface temp.
T_f = fluid temp faraway
k_f = fluid thermal cond.
T =

Non-dimensionalizing:
Let:

$$\theta = \frac{T - T_f}{T_s - T_f}, \quad n^* = \frac{n}{L}$$

Back substitute and equate ① & ②

$$-k_f \frac{\cancel{(T_s - T_f)}}{L} \cdot \left.\frac{\partial \theta}{\partial n^*}\right|_s = h\,\cancel{(T_s - T_f)}$$

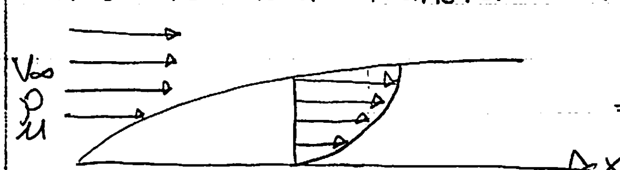
$$\boxed{-\frac{\partial \theta}{\partial n^*} = \frac{hL}{k_f} = Nu} \Rightarrow \text{Nusselt number}$$

$$\boxed{Nu = \frac{hL}{k_f} = \frac{\text{convective heat transfer rate}}{\text{conductive heat transfer rate}}}$$

Since $Nu \propto \frac{\partial \theta}{\partial n^*} \propto \theta$, it typically depends on:
1) Flow cond.
2) Fluid properties
3) Geometry
4) Boundary conditions

External Flow

Assuming ρ, μ, c = constant (density, viscosity, heat capacity)
Linked to heat transfer because of the boundary layer



$\Rightarrow$ Where all of the gradients happen in velocity

2-D Laminar Flow: (Flat Plate, steady)

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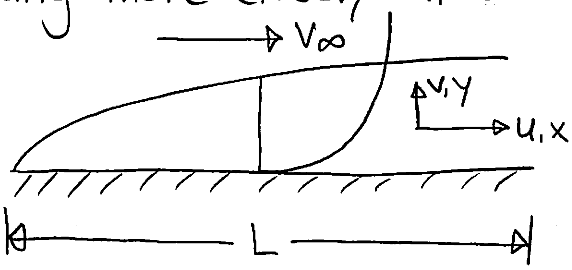
x-momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

y-momentum: $$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Looking more closely at our boundary layer:



Let: $\bar{u} = \frac{u}{V_\infty}$, $\bar{v} = \frac{v}{V_\infty}$, $\bar{P} = \frac{P}{\rho V_\infty^2}$

$\bar{x} = \frac{x}{L}$, $\bar{y} = \frac{y}{L}$

Now our equations become:

Reynolds Number

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{Re_L}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right], \quad Re_L = \frac{\rho V_\infty L}{\mu}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{y}} + \frac{1}{Re_L}\left[\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right]$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

So what can we say about these equations and their relative terms? $u \sim V_\infty$, $x \sim L$, $y \sim \delta$ (boundary layer thick.)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{V_\infty}{L} + \frac{v}{\delta} = 0 \Rightarrow \boxed{v \sim \left(\frac{\delta}{L}\right)V_\infty}$$

↳ Note dropped the negative sign since scaling analysis.

Looking back at our momentum equations

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Going to x-momentum

$$\underbrace{V_\infty \frac{V_\infty}{L} + V_\infty \left(\frac{\cancel{\delta}}{L}\right) \cdot \frac{V_\infty}{\cancel{\delta}}}_{\sim \frac{V_\infty^2}{L}} = \underbrace{-\frac{1}{\cancel{\rho}} \frac{\cancel{\rho} V_\infty^2}{L}}_{\sim \frac{V_\infty^2}{L}} + \nu \left[\overset{\approx 0 \text{ since } \frac{\delta}{L} \ll 1}{\cancel{\frac{V_\infty}{L^2}}} + \underbrace{\frac{V_\infty}{\delta^2}}_{\sim \frac{\nu V_\infty}{\delta^2}} \right]$$

Can rewrite as:

$$\frac{V_\infty}{L^2} = \frac{\delta^2}{L^2} \cdot \frac{V_\infty}{\delta^2} \approx 0 \text{ since } \frac{\delta}{L} \ll 1$$

Y - momentum:

$$\underbrace{V_\infty \frac{\delta}{L} V_\infty \cdot \frac{1}{L}}_{\sim \frac{V_\infty^2}{L}\left(\frac{\delta}{L}\right)} + \underbrace{V_\infty^2 \left(\frac{\delta}{L}\right)^2 \frac{1}{\delta}}_{\sim \frac{V_\infty^2}{L}\cdot\left(\frac{\delta}{L}\right)} = \underbrace{-\frac{1}{\cancel{\rho}} \cdot \frac{\cancel{\rho} V_\infty^2}{\delta}}_{= \frac{V_\infty^2}{\delta}} + \nu \frac{\delta}{L} V_\infty \left[\frac{1}{L^2} + \frac{1}{\delta^2}\right]$$

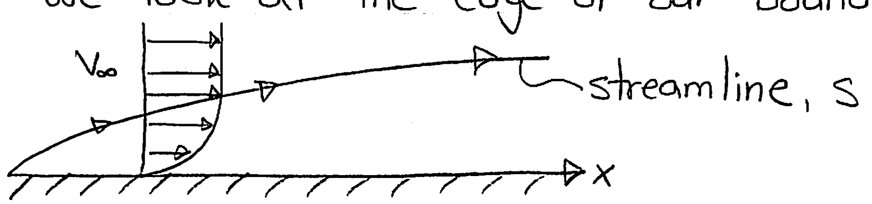
$$= \frac{V_\infty^2}{\delta} + \nu V_\infty \left(\frac{\delta}{L}\right)\left[\frac{1}{L^2} + \frac{1}{\delta^2}\right]$$

Since $\frac{\delta}{L} \ll 1$, we can drop the inertia terms, & viscous terms

Our y-momentum equation becomes:

$$\frac{\partial p}{\partial y} = 0 \Rightarrow \boxed{p(y) = \text{Constant}} \Rightarrow \text{Pressure is not a function of } y$$

If we look at the edge of our boundary layer



Our x-momentum equation along the streamline s becomes

$$u \frac{\partial u}{\partial x} + \underbrace{v \frac{\partial u}{\partial y}}_{0 \text{ at } s.} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \underbrace{\left(\frac{\partial^2 u}{\partial y^2}\right)}_{0 \text{ at } s.}$$

$$V_\infty \frac{\partial V_\infty}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \Rightarrow \boxed{p + \frac{1}{2} \rho V_\infty^2 = \text{constant}}$$

↳ Bernoulli equation

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Note from this, we can see that the pressure in the x-direction is constant ($P \neq f(x)$) for steady flow past a flat plate, since V_∞ = constant at the b.l. edge. Note, not true for cylinder, sphere, etc... where V_∞ changes.

$$\frac{\partial V_\infty}{\partial x} = 0 \quad \text{or} \quad \frac{1}{\rho}\frac{\partial P}{\partial x} = -V_\infty \frac{\partial V_\infty}{\partial x} = 0$$

So our boundary layer equations become:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (2)$$

$$P(y) = \text{constant} \quad (3)$$

Boundary conditions:

$u = v = 0$ @ $y = 0$

$u = U_\infty$ @ $y \to \infty$

So now we must convert our PDE into a solvable ODE. Cannot use separation of variables (infinite medium). Let's try a similarity solution.

Assume: $\dfrac{u}{V_\infty} = f(\eta)$ where $\eta = \dfrac{y}{\delta}$

So how do we define δ?

Looking back at our non-dimensional x-momentum:

$$\underbrace{\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}}_{\sim O(1)} + \underbrace{\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}}_{\sim O(1)} = \underbrace{\frac{1}{Re_L}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}_{\text{Has to be on order of 1.}},$$

$$\bar{u} = \frac{u}{V_\infty} \sim \frac{V_\infty}{V_\infty} \sim O(1)$$

$$\bar{x} = \frac{x}{L} \sim \frac{L}{L} \sim O(1)$$

$$\bar{y} = \frac{y}{L} \sim \frac{\delta}{L}$$

$$\frac{1}{Re_L}\frac{V_\infty}{V_\infty}\cdot\left(\frac{\delta}{L}\right)^2 \sim 1$$

$$\frac{1}{Re_L}\frac{\delta^2}{L^2} \sim 1 \Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}} \quad \text{or} \quad \frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}}$$

$$\boxed{\delta \sim \sqrt{\frac{\nu x}{V_\infty}}} \quad \text{so now we get:} \quad \boxed{\eta = y\sqrt{\frac{V_\infty}{\nu x}}}$$

↳ Will see an easier way to get this later.

So what about v? Looking at continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \Rightarrow \bar{u} = \phi(\eta) = \frac{u}{V_\infty}$$

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \bar{u}}{\partial x} = -\frac{1}{2x}\eta\phi'$$

Back substituting, we obtain:

$$\frac{\partial \bar{v}}{\partial y} = \frac{1}{2x}\phi'\eta$$

Aside:
We know $\eta = y\sqrt{\frac{V_\infty}{\nu x}}$

$d\eta = dy\sqrt{\frac{V_\infty}{\nu x}}$

$$\bar{v} = \frac{1}{2x}\int_0^\eta \phi'\eta\, dy = \underbrace{\frac{1}{2x}\sqrt{\frac{x\nu}{V_\infty}}}_{\frac{1}{2}\frac{1}{\sqrt{Re_x}}}\int_0^\eta \phi'\eta\, d\eta$$

Now we need to solve our integral:

$$\int_0^\eta \eta\phi'\, d\eta = uv - \int v\, du \quad \left(\text{Integration by parts: } \int u\, dv = uv - \int v\, du\right)$$

$$= \eta\phi - \int_0^\eta \phi\, d\eta \quad , \quad u = \eta, \; v = \phi$$

$$\text{let } \int_0^\eta \phi\, d\eta = F(\eta)$$

So now we have: $\bar{v} = \frac{1}{2}\frac{1}{\sqrt{Re_x}}\left(\eta F' - F\right)$

Putting everything together we get:

$$F''' + \frac{1}{2}FF'' = 0; \quad \eta = y\sqrt{\frac{V_\infty}{x\nu}}$$

At the wall: $\eta = 0$, $F' = F = 0$

Outside the b.l: $\eta \to \infty$, $F' = 1$

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Rewriting our equation:

$$\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{1}{2} F \frac{\partial \bar{u}}{\partial \eta} = 0$$

To solve, we need to assume an infinite series solution

$$F = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + \cdots +$$
$$F' = a_1 + 2a_2 \eta + 3a_3 \eta^2 + \cdots +$$
$$F'' = 2a_2 + 6a_3 \eta + \cdots +$$
$$F''' = 6a_3 + \cdots +$$

Back substitute and solve for your coefficients

$$F''' + \frac{1}{2} F F'' = 0$$

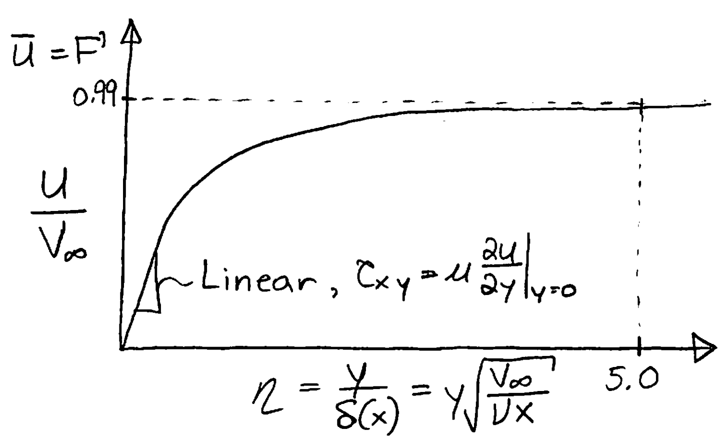
$$\underbrace{(\cdots)}_{0}\eta^0 + \underbrace{(\cdots)}_{0}\eta^1 + \underbrace{(\cdots)}_{0}\eta^2 + \underbrace{(\cdots)}_{0}\eta^3 + \cdots + \underbrace{(\cdots)}_{0}\eta^n = 0$$

You obtain a recursion formula relating your constants

$$\boxed{F = \frac{a_2 \eta^2}{2!} - \frac{a_2 \eta^5}{2 \cdot 5!} + \frac{11}{4} \frac{a_3{}^3 \eta^8}{8!} + \cdots}$$

$\boxed{a_2 = 0.332}$ ⇒ Heinrich Blasius solved this in 1911 for his Ph.D. work with Ludwig Prandtl.

So $F = \int_0^\eta \phi \, d\eta$, $F' = \phi = \bar{u} = \frac{u}{V_\infty}$



$$5.0 = \delta \sqrt{\frac{V_\infty}{x\nu}} = \frac{\delta}{x}\sqrt{Re_x}$$

$$\boxed{\delta = \frac{5x}{\sqrt{Re_x}}}$$

↳ Hydrodynamic boundary layer thickness.

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This is very usefull because now we can calculate things like shear stress

$$\tau(x) = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \rho \nu V_\infty \left.\frac{\partial \bar{u}}{\partial y}\right|_{y=0}$$

$$= \rho \nu V_\infty \left.\frac{\partial \bar{u}}{\partial \eta}\right|_{\eta=0} \cdot \left(\frac{\partial \eta}{\partial y}\right)$$

$$= \rho \nu V_\infty \, a_2 \sqrt{\frac{V_\infty}{x \nu}}$$

$$\boxed{\frac{\tau(x)}{\frac{1}{2}\rho V_\infty^2} = \frac{2a_2}{\sqrt{Re_x}} = \frac{0.664}{Re_x^{1/2}}}$$ ⇒ Skin friction coefficient for a flat plate in laminar flow conditions.

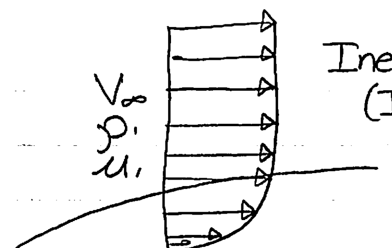
If we want the average:

$$\bar{\tau} = \frac{1}{L}\int_0^L \tau(x)\,dx \Rightarrow \tau(x) = C\frac{1}{\sqrt{x}}$$

$$= C \cdot \frac{1}{L}\int_0^L \frac{dx}{\sqrt{x}} = \frac{2C}{\sqrt{L}}$$

$$\boxed{C_\tau = \frac{\bar{\tau}}{\frac{1}{2}\rho V_\infty^2} = \frac{1.328}{Re_L^{1/2}}}$$ ⇒ Average skin friction coeff.

So is there an easier way to see some simple things about boundary layers?



Inertia Dominates (Inviscid Flow) ⇒ $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

} Inside the b.l. Inertia ~ Viscosity. Viscous effects become important.

$$\rho \frac{V_\infty^{\cancel{2}}}{L} \sim \mu \frac{V_\infty}{\delta^2} \Rightarrow \delta^2 = \frac{\mu L}{\rho V_\infty} \Rightarrow \delta \sim \sqrt{\frac{\nu L}{V_\infty}}\left(\frac{L}{L}\right)$$

$$\boxed{\delta \sim \frac{L}{\sqrt{Re_L}}}$$