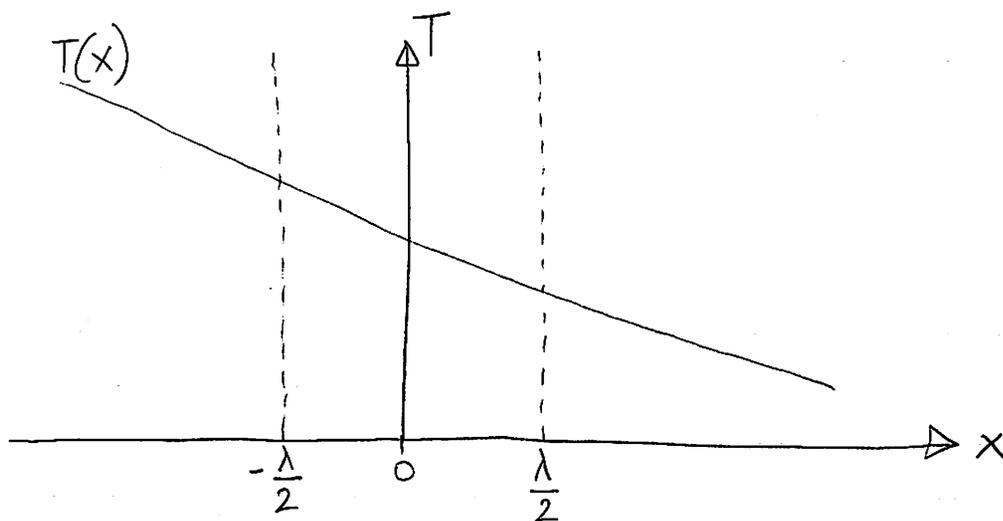


ME 420 - Intermediate Heat Transfer

Mechanisms of Heat Transfer $\left\{ \begin{array}{l} \text{Conduction} = \text{Convection} \\ \text{Radiation} \end{array} \right.$
 Convection with bulk fluid motion.

$$q_x'' = -k \frac{\partial T}{\partial x}, \text{ where } k = \text{thermal conductivity [W/m}\cdot\text{K]}$$

When we have gradients, we have diffusional transport



$$q_{x+}'' = nm c_p \bar{c} T(x - \frac{\lambda}{2}), \text{ where}$$

$n = \# \text{ of particles/m}^3$
 $m = \text{mass per particle}$
 $c_p = \text{specific heat capacity}$
 $\bar{c} = \text{average speed of a particle}$

We see that: $n \cdot m = \rho$

$$q_{x+}'' = + \rho c_p \bar{c} T(x - \frac{\lambda}{2})$$

$$q_{x-}'' = + \rho c_p \bar{c} T(x + \frac{\lambda}{2})$$

$$c_p \Rightarrow \left[\frac{\text{J}}{\text{kg}\cdot\text{K}} \right], n \Rightarrow \frac{\#}{\text{m}^3}, m = \frac{\text{kg}}{\#}$$

$$q_x'' = q_{x+}'' - q_{x-}'' = \rho c_p \bar{c} \left[T(x - \frac{\lambda}{2}) - T(x + \frac{\lambda}{2}) \right]$$

$$= - \rho c_p \bar{c} \left[T(x + \frac{\lambda}{2}) - T(x - \frac{\lambda}{2}) \right]$$

Now if we multiply our expression by $\frac{1}{\lambda}$

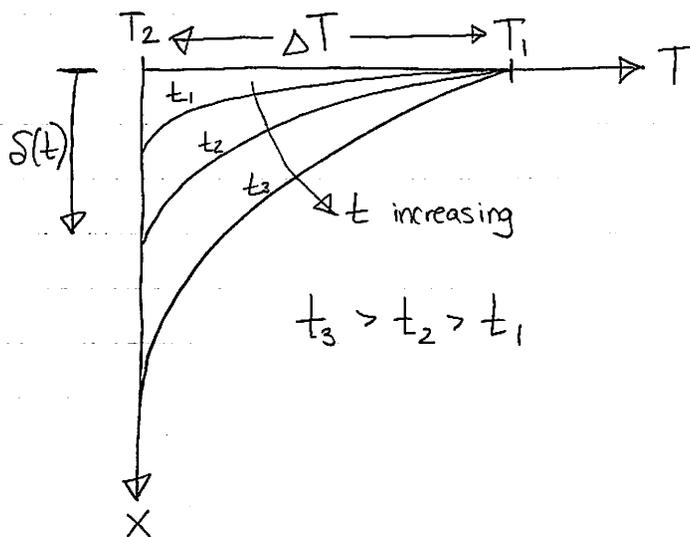
$$q'' = -\rho c_p \bar{c} \lambda \underbrace{\left[\frac{T(x + \frac{\Delta}{2}) - T(x - \frac{\Delta}{2})}{\Delta} \right]}_{\frac{\partial T}{\partial x}}$$

$$q'' = -\rho c_p \bar{c} \lambda \frac{\partial T}{\partial x} \Rightarrow \text{Gradient dependent}$$

We can also see that: $k = \rho c_p \bar{c} \lambda \Rightarrow \text{Thermal conductivity}$

Remember, this derivation is valid for gases, however the kinetic approach is also often used analogously in solid state physics to study phonons.

Now let's look more closely at time for heat transfer



$S(t)$ = thermal penetration depth
 ΔT = temperature difference
 t = time

on the order of

$$E \sim \rho \delta c_p \Delta T \quad \left[\frac{\text{J}}{\text{m}^2} \right]$$

$$\dot{E} \sim \rho \delta c_p \frac{\partial T}{\partial t}$$

$$\dot{E} = \dot{Q}_{in} = -k \left(\frac{\partial T}{\partial x} \right)_0 \sim k \frac{\Delta T}{\delta}$$

Equating our two expressions:

$$\rho c_p \delta \frac{\partial T}{\partial t} \sim k \frac{\Delta T}{\delta}$$

$$\frac{\partial T}{\partial t} = \frac{\partial \delta}{\partial t} \cdot \frac{\partial T}{\partial \delta}$$

$$\rho c_p \delta \frac{\partial \delta}{\partial t} \cdot \frac{\partial T}{\partial \delta} \sim k \frac{\partial T}{\partial \delta}$$

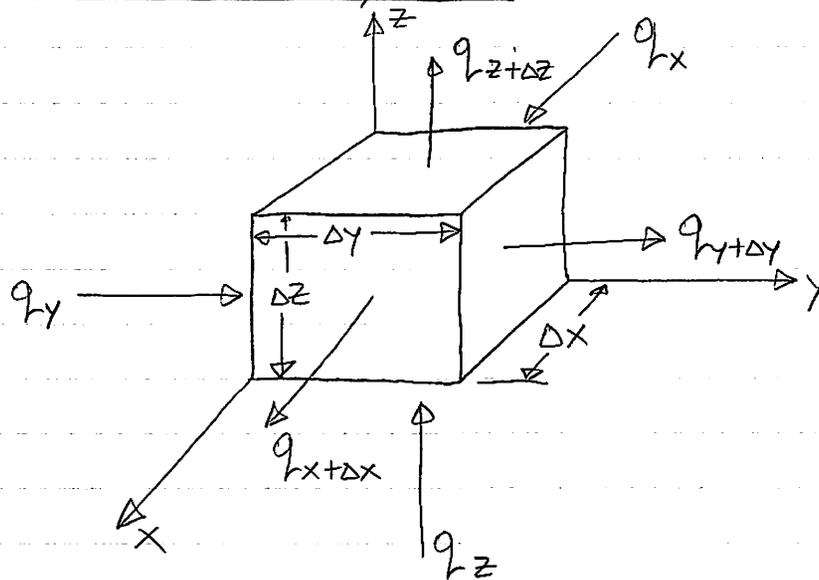
$$\rho c_p \delta \frac{\partial \delta}{\partial t} \sim k$$

$$\rho c_p \frac{\delta^2}{t} \sim k$$

$$\delta^2 \sim \frac{k}{\rho c_p} \cdot t$$

$$\delta \sim \sqrt{\frac{k}{\rho c_p} \cdot t} \sim \sqrt{\alpha t} \Rightarrow \text{Diffusion distance} \sim \sqrt{t}$$

$$\alpha = \frac{k}{\rho c_p} \Rightarrow \text{Thermal Diffusivity}$$

Heat Diffusion Equation

Note, q_x, q_y, q_z etc...
are heat fluxes
 $\left[\frac{W}{m^2}\right] \Rightarrow q''$

From 1st law of thermodynamics ($E_{in} = E_{out}$)

$$\dot{Q}_{in} - \dot{Q}_{out} + \dot{Q}_{gen} = \dot{Q}_{st}$$

Entering & leaving through the surfaces are

$$\dot{Q}_{in} = q_x A_x + q_y A_y + q_z A_z$$

$$\dot{Q}_{out} = q_{x+\Delta x} A_{x+\Delta x} + q_{y+\Delta y} A_{y+\Delta y} + q_{z+\Delta z} A_{z+\Delta z}$$

For the cartesian coordinate system

$$A_x = A_{x+\Delta x} = \Delta y \Delta z$$

$$A_y = A_{y+\Delta y} = \Delta x \Delta z$$

$$A_z = A_{z+\Delta z} = \Delta x \Delta y$$

Within the control volume, we have energy generation \dot{Q}'''

$$\dot{Q}_{gen} = \dot{Q}''' V = \dot{Q}''' \Delta x \Delta y \Delta z$$

Within the control volume, the change in stored energy is

$$\dot{Q}_{ST} = \frac{\partial U}{\partial t} = \frac{\partial (Mu)}{\partial t} = \frac{\partial (\rho \nabla u)}{\partial t}$$

Assuming a constant density ρ

$$\dot{Q}_{ST} = \rho \frac{\partial u}{\partial t} \Delta x \Delta y \Delta z$$

Adopting an equation of state (simple incompressible substance with constant specific heat)

$$\dot{Q}_{ST} = \rho c_p \frac{\partial T}{\partial t} \Delta x \Delta y \Delta z$$

Aside:

$$U = M c_p (T - T_{ref})$$

$$u = c_p (T - T_{ref})$$

Now we plug everything together

$$\dot{Q}_{in} - \dot{Q}_{out} = \Delta y \Delta z (q_x - q_{x+\Delta x}) + \Delta x \Delta z (q_y - q_{y+\Delta y}) + \Delta x \Delta y (q_z - q_{z+\Delta z})$$

Using Taylor series expansion

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 + \dots + \frac{1}{n!} \frac{\partial^n q_x}{\partial x^n} \Delta x^n$$

So,

$$q_x - q_{x+\Delta x} = -\frac{\partial q_x}{\partial x} \Delta x - \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 - \dots - \frac{1}{n!} \frac{\partial^n q_x}{\partial x^n} \Delta x^n$$

As $\Delta x \rightarrow 0$, Δx^2 really $\rightarrow 0$, so higher order terms can be dropped. Maybe not very rigorous but will reconcile later

$$q_x - q_{x+\Delta x} = -\frac{\partial q_x}{\partial x} \Delta x$$

$$q_y - q_{y+\Delta y} = -\frac{\partial q_y}{\partial y} \Delta y$$

etc...

Back substituting:

$$\dot{Q}_{in} - \dot{Q}_{out} = -\Delta y \Delta z \left(\frac{\partial q_x}{\partial x} \Delta x \right) - \Delta x \Delta z \left(\frac{\partial q_y}{\partial y} \Delta y \right) - \Delta x \Delta y \left(\frac{\partial q_z}{\partial z} \Delta z \right)$$

Note, could have kept the H.O.T. terms here.

Rearranging:

$$\dot{Q}_{in} - \dot{Q}_{out} = -\Delta x \Delta y \Delta z \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right)$$

$$\dot{Q}_{in} - \dot{Q}_{out} + \dot{Q}_{gen} = \dot{Q}_{st}$$

$$-\Delta x \Delta y \Delta z \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) + \dot{Q}''' \Delta x \Delta y \Delta z = \rho c_p \frac{\partial T}{\partial t} \Delta x \Delta y \Delta z$$

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t} \quad (\text{Let } \Delta x, \Delta y, \Delta z \rightarrow 0, \text{ H.O.T. terms drop out})$$

But we've already proven before that $q_x = -k \frac{\partial T}{\partial x}$, etc...

Substituting the constitutive equations into our first law

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}$$

Second order PDE is the conservation of thermal energy for an isotropic, incompressible substance, with density and specific heat independent of time.

A well-posed model requires two boundary conditions in each coordinate (x , y , and z), and an initial condition.

If thermal conductivity does not depend on location

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

or

$$\nabla^2 T + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

So now we have a new tool.

For constant properties, and 1-D conduction

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

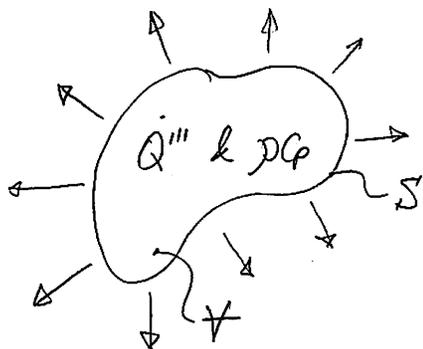
For 1-D conduction, and no generation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

For steady state (1-D, and no generation)

$$\frac{\partial^2 T}{\partial x^2} = 0$$

Note, there was an easier approach... but more abstract



$$\dot{Q}_{in} - \dot{Q}_{out} + \dot{Q}_{gen} = \dot{Q}_{ST}$$

$$\dot{Q}_{in} - \dot{Q}_{out} = - \iint_S \vec{q} \cdot \hat{n} dS$$

$$\dot{Q}_{gen} = \iiint_V \dot{Q}''' dV$$

$$\dot{Q}_{ST} = \iiint_V \rho c \frac{\partial T}{\partial t} dV$$

$$-\oint_S \vec{q} \cdot \hat{n} dS + \iiint_V \dot{Q}''' dV = \iiint_V \rho c \frac{\partial T}{\partial t} dV$$

Applying Green's Theorem (Divergence Theorem)

$$\oint_S \vec{v} \cdot d\vec{s} = \oint_S \vec{v} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{v} dV$$

So we obtain:

$$-\iiint_V \nabla \cdot \vec{q} dV + \iiint_V \dot{Q}''' dV = \iiint_V \rho c \frac{\partial T}{\partial t} dV$$

Rearranging:

$$-\iiint_V \left[-(\nabla \cdot \vec{q}) + \dot{Q}''' - \rho c \frac{\partial T}{\partial t} \right] dV = 0$$

Thus:

$$-(\nabla \cdot \vec{q}) + \dot{Q}''' = \rho c \frac{\partial T}{\partial t}$$

Using Fourier's Law in vector form $\vec{q} = -k \nabla T$

$$(\nabla \cdot k \nabla T) + \dot{Q}''' = \rho c \frac{\partial T}{\partial t}$$

Note, under the same assumptions used earlier, we get the following and did not assume a coordinate system

$$\boxed{\nabla^2 T + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}} \quad , \quad \text{Note } \boxed{(\nabla \cdot \nabla T) = \nabla^2 T}$$

↳ Divergence of a gradient is the Laplacian

For radial & spherical co-ordinates, we can use a more general formulation:

$$dV = ds_1 ds_2 ds_3 \quad , \quad s_1, s_2, s_3 \text{ are coordinates in consideration } \textcircled{8}$$

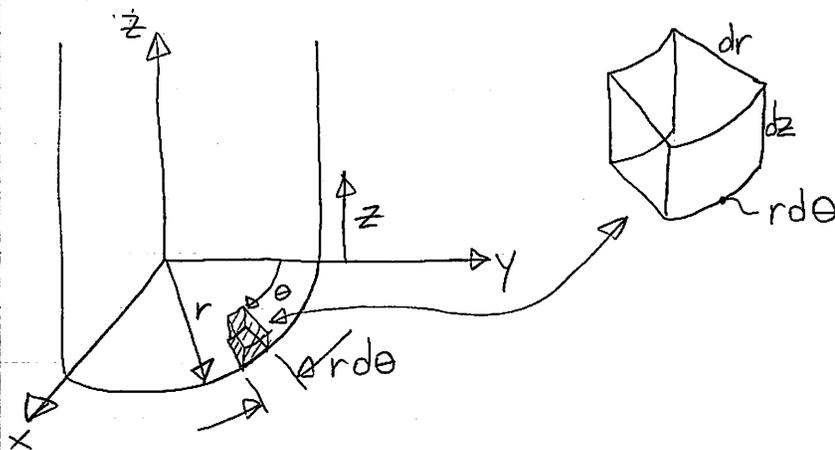
Recasting our heat equation

$$\frac{1}{ds_1 ds_2 ds_3} \left\{ \frac{\partial}{\partial s_1} \left(k ds_2 ds_3 \frac{\partial T}{\partial s_1} \right) ds_1 + \frac{\partial}{\partial s_2} \left(k ds_1 ds_3 \frac{\partial T}{\partial s_2} \right) ds_2 + \frac{\partial}{\partial s_3} \left(k ds_1 ds_2 \frac{\partial T}{\partial s_3} \right) ds_3 \right\} + \dot{Q}''' = \frac{\partial}{\partial t} (\rho c_p T)$$

Specific Cases:

① Cartesian, $\left. \begin{array}{l} ds_1 = dx \\ ds_2 = dy \\ ds_3 = dz \end{array} \right\}$ We'll get what we already solved

② Cylindrical



$$\begin{aligned} ds_1 &= dr \\ ds_2 &= r d\theta \\ ds_3 &= dz \end{aligned}$$

So our heat equation becomes

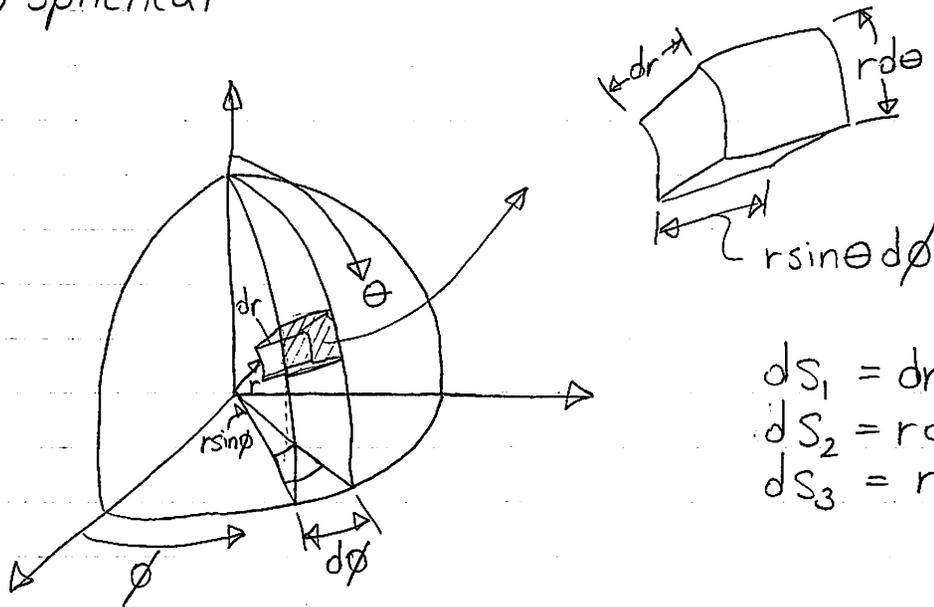
$$\frac{1}{r dr d\theta dz} \left\{ \frac{\partial}{\partial r} \left(k r d\theta dz \frac{\partial T}{\partial r} \right) dr + \frac{1}{r} \frac{\partial}{\partial \theta} \left(k dr dz \frac{\partial T}{\partial \theta} \right) r d\theta + \frac{\partial}{\partial z} \left(k r dr d\theta \frac{\partial T}{\partial z} \right) dz \right\} + \dot{Q}''' = \frac{\partial}{\partial t} (\rho c_p T)$$

Ask them to do spherical by themselves

$$\frac{1}{r} \frac{\partial}{\partial r} \left(k r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}$$

Bonus Solution - if we have time (Did not cover this in class)

③ Spherical



$$\begin{aligned} dS_1 &= dr \\ dS_2 &= r d\theta \\ dS_3 &= r \sin \theta d\phi \end{aligned}$$

Back substituting into our general formulation

$$\frac{1}{r^2 dr d\theta \sin \theta d\phi} \left\{ \frac{\partial}{\partial r} (k r d\theta r \sin \theta d\phi \frac{\partial T}{\partial r}) dr + \frac{1}{r} \frac{\partial}{\partial \theta} (k dr \cdot r \cdot \sin \theta d\phi \frac{\partial T}{\partial \theta}) r d\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (k dr \cdot r \cdot d\theta \frac{\partial T}{\partial \phi}) r \sin \theta d\phi \right\} + \dot{Q}''' = \frac{\partial}{\partial t} (\rho c_p T)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (k r^2 \frac{\partial T}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (k \sin \theta \frac{\partial T}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} (k \frac{\partial T}{\partial \phi}) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t}$$

Gas Kinetics

Some fundamental numbers to know are:

$A \Rightarrow$ Avogadro's # $\left[\frac{\# \text{ molecules}}{\text{kmol}} \right] = 6.022 \times 10^{26}$

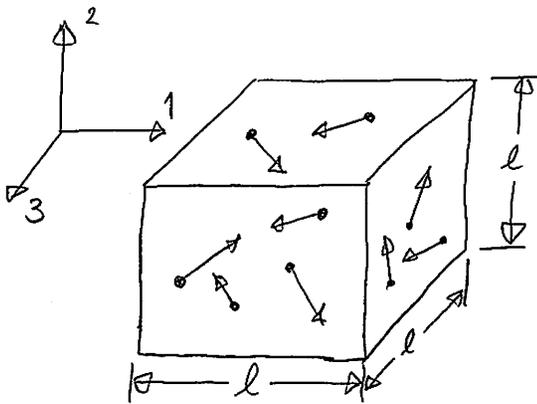
$R \Rightarrow$ Universal gas constant $\left[\frac{\text{J}}{\text{kmol} \cdot \text{K}} \right] = 8314.3$

$k_B \Rightarrow$ Boltzmann constant $\left[\frac{\text{J}}{\text{molecule} \cdot \text{K}} \right] = 1.38 \times 10^{-23} \Rightarrow$ Relates individual molecule energy to temperature

$M \Rightarrow$ Molecular weight $\left[\frac{\text{kg}}{\text{kmol}} \right]$

$$k_B = \frac{R}{A}$$

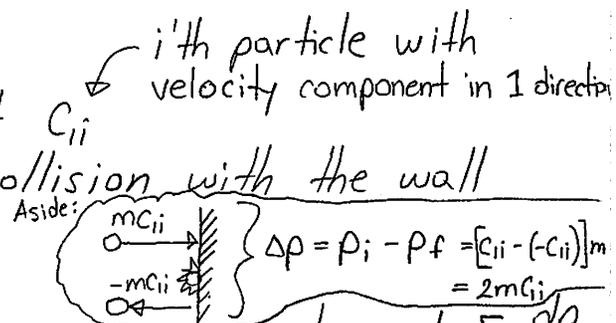
We can now develop a model to give meaning for pressure in a rarified gas (non-interacting gas molecules)



$$V = l^3$$

N molecules in our volume
 $l < \lambda_i$ (Rarified gas)
 $m \Rightarrow$ mass per particle

We have a particle traveling at C_{ii}
 The change in momentum from a collision with the wall normal to 1 is $2mC_{ii} = \Delta p$



Relating this to force: $F = \frac{\Delta p}{\Delta t}$, or as you know it $F = \frac{dp}{dt}$

Δt can be related to the frequency of particle collisions, f , with the wall,
 $f = \frac{C_{ii}}{2l} \Rightarrow \Delta t = \frac{1}{f} = \frac{2l}{C_{ii}}$

$$F_i = \frac{2mC_{ii}}{2l} = \frac{mC_{ii}^2}{l}$$

$$F_1 = \sum F_i = \frac{m}{l} \sum_{i=1}^N C_{ii}^2$$

$$P_1 = \frac{F_1}{l^2} = \frac{m}{l^3} \sum_{i=1}^N C_{ii}^2$$

(Pressure)

So for our three surfaces:

$$P_1 = \frac{m}{l^3} \sum_{i=1}^N C_{ii}^2; \quad P_2 = \frac{m}{l^3} \sum_{i=1}^N C_{2i}^2; \quad P_3 = \frac{m}{l^3} \sum_{i=1}^N C_{3i}^2$$

The total pressure in our chamber can be found by taking the average of our 3 directional pressures.

$$P = \frac{P_1 + P_2 + P_3}{3} = \frac{m}{l^3} \frac{1}{3} \cdot \sum_{i=1}^N \underbrace{C_{ii}^2 + C_{2i}^2 + C_{3i}^2}_{C_i^2}$$

$$P = \frac{m}{l^3} \frac{1}{3} \sum_{i=1}^N C_i^2$$

From statistical thermodynamics, we know that:

$$\underbrace{\frac{1}{2} m \bar{c}^2}_{\text{average molecule KE}} = \underbrace{\frac{3}{2} k_B T}_{\frac{1}{2} k_B T \text{ for every translational degree of freedom}}$$

We also know that: $m = \frac{M}{A}$ $\left[\frac{\text{kg}}{\text{molecule}} \right]$

$$\frac{1}{2} m \bar{c}^2 = \frac{3}{2} k_B T = \frac{1}{2} \frac{M}{A} \bar{c}^2$$

$$M \bar{c}^2 = 3 k_B A T \Rightarrow \text{we know that } k_B A = R$$

$$\boxed{M \bar{c}^2 = 3RT} \quad \textcircled{1}$$

Another more rigorous way to prove:

$$F_1 = \frac{m}{l} \sum_{i=1}^N C_{ii}^2$$

$$= \frac{M}{Al} \sum_{i=1}^N C_{ii}^2$$

$$= \frac{MN_{\text{mol}}}{l} \left\{ \frac{\sum_{i=1}^N C_{ii}^2}{N} \right\} \bar{c}_1^2$$

$$F_1 = \frac{MN_{\text{mol}}}{l} \bar{c}_1^2, \text{ we know that}$$

Due to random motion: $\bar{c}_1^2 = \frac{1}{3} \bar{c}^2$

Looking back at our pressure term

$$\begin{aligned}
 \rho &= \frac{m}{l^3} \cdot \frac{\sum c_i^2}{3} \\
 &= \frac{M}{Al^3} \cdot \frac{\sum c_i^2}{3} \Rightarrow A = \frac{N}{N_{\text{mol}}} \left[\frac{\# \text{ molecules}}{\# \text{ kmol}} \right] \\
 &= \frac{N_{\text{mol}} M}{N l^3} \frac{\sum c_i^2}{3} \Rightarrow \text{Since } \frac{\sum c_i^2}{N} = \overline{c^2} \text{ (average molecule speed)} \\
 &= \underbrace{\frac{N_{\text{mol}} M}{l^3}}_{\rho} \cdot \frac{1}{3} \overline{c^2} = \frac{1}{3} \rho \overline{c^2} \Rightarrow \boxed{\rho = \frac{1}{3} \rho \overline{c^2}} \quad (2)
 \end{aligned}$$

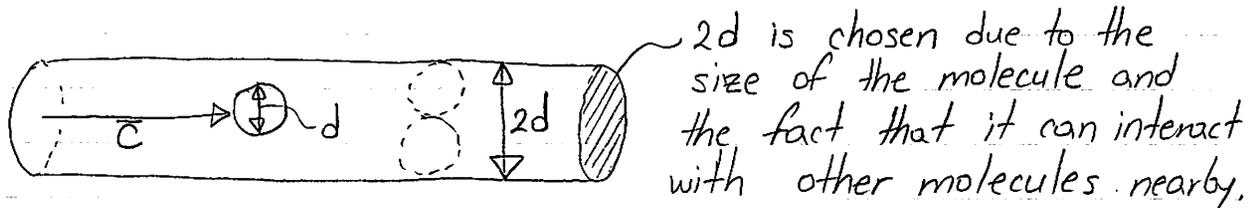
Now we can equate (1) and (2) & isolate for $\overline{c^2}$

$$\overline{c^2} = \frac{3P}{\rho} = \frac{3RT}{M}$$

$$P = \frac{\rho}{M} RT, \text{ note } \rho \Rightarrow \left[\frac{\text{kg}}{\text{m}^3} \right]; M \Rightarrow \left[\frac{\text{kg}}{\text{kmol}} \right]$$

$$\boxed{P = \rho_m RT} \Rightarrow \text{Ideal Gas Law} \Rightarrow \rho_m = \text{molecular density} \left[\frac{\text{kmol}}{\text{m}^3} \right] \\
 \text{or} \\
 \boxed{PV = N_{\text{mol}} RT} \\
 R = \text{universal gas const.} \\
 P = \text{pressure.}$$

Now if we look at the microscopic picture of the molecule



We could have chosen 3d, however there is a chance that the molecule doesn't interact at all then with molecules within the swept path.

$$\frac{\text{Swept Volume}}{\text{time}} = \pi d^2 \cdot c$$

$$\frac{\# \text{ collisions}}{\text{time}} = \pi d^2 n \cdot c, \quad n \Rightarrow \left[\frac{\# \text{ molecules}}{\text{volume}} \right]$$

$$\lambda = \text{mean free path} = \frac{\text{distance traveled / time}}{\# \text{ collisions / time}}$$

$$\lambda = \frac{\ell}{\pi d^2 n \ell} = \frac{1}{\pi d^2 n}$$

$$\boxed{\lambda = \frac{1}{\pi d^2 n}} \Rightarrow \text{Mean free path}$$

We can work with this a bit further to get it into a more useful form: (using our ideal gas law)

$$PV = N_{\text{mol}} RT \Rightarrow A = \frac{N}{N_{\text{mol}}} \Rightarrow N_{\text{mol}} = \frac{N}{A}$$

$$PV = \frac{N}{A} RT = \underbrace{\left(\frac{R}{A} \right)}_{k_B} NT$$

$$PV = k_B NT$$

$$\frac{N}{V} = n = \frac{P}{k_B T} \Rightarrow \text{Back substituting}$$

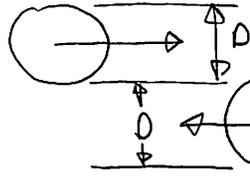
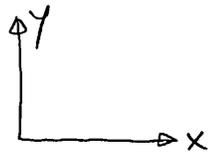
$$\boxed{\lambda = \frac{k_B T}{\pi d^2 P}}$$

If we look at our overall statistical distribution

$$\boxed{\lambda = \frac{k_B T}{\sqrt{2} \pi d^2 P}} \Rightarrow \begin{array}{l} \text{As } T \uparrow, \lambda \uparrow \\ \text{As } P \downarrow, \lambda \uparrow \end{array}$$

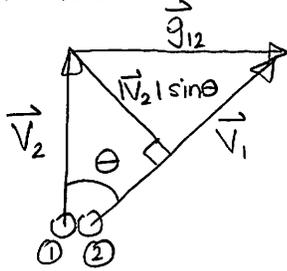
$\hookrightarrow \sqrt{2}$ comes from molecules having a velocity relative to one another.

Extra Derivation (Where the $\sqrt{2}$ in λ comes from)



This molecule is not stationary, it will have a speed as well.

We need to know the mean relative speed between molecules: (consider two arbitrary molecules)



$$|g_{12}| = \left[|v_2|^2 \sin^2 \theta + (|v_1| - |v_2| \cos \theta)^2 \right]^{1/2}$$

Rearranging, we obtain

$$|g_{12}|^2 = |v_2|^2 + |v_1|^2 - 2|v_1||v_2| \cos \theta$$

Taking \bar{g} to be the root mean square value of $|g_{12}|$, we have

$$\bar{g} = \langle |g_{12}|^2 \rangle^{1/2} = \langle |v_2|^2 + |v_1|^2 - 2|v_1||v_2| \cos \theta \rangle^{1/2}$$

In our model, we assumed $|v_1| = |v_2| = c$

$$\bar{g} = \langle |g_{12}|^2 \rangle^{1/2} = \langle c^2 + c^2 - 2c^2 \cos \theta \rangle^{1/2} = 2c^2 (1 - \langle \cos \theta \rangle)^{1/2}$$

If we average from $\theta = 0$ to $\theta = 2\pi \Rightarrow \langle \cos \theta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cos \theta d\theta$

$$\therefore \boxed{\bar{g} = (2c^2)^{1/2} = \sqrt{2}c} \Rightarrow \text{Use this in \# of collisions/time equation.}$$

Example #1 | If we look at air at 273K

$$M = 28.9 \text{ kg/kmol}$$

$$\rho_{\text{air}} = 1.288 \text{ kg/m}^3$$

$$\rho_{\text{liquid, air}} = 0.86 \times 10^3 \text{ kg/m}^3$$

$$A = 6.022 \times 10^{26} \text{ molecules/kmol}$$

$$R = 8314.3 \text{ J/kmol} \cdot \text{K}$$

$$k_B = 1.38 \times 10^{-23} \text{ J/K}$$

First we can calculate the mass per molecule (m)

$$m = \frac{M}{A} = 4.8 \times 10^{-26} \text{ kg/molecule}$$

$$n = \frac{\rho}{m} = 2.684 \times 10^{25} \text{ molecules/m}^3$$

$$\rho = 1.01325 \times 10^5 \text{ N/m}^2 \Rightarrow \text{Aside:}$$

Assuming that the volume of the air molecule can be approximated as d^3

$$\frac{1}{d^3} = n_{\text{liq}} = \frac{\rho_{\text{liq}}}{m}$$

$$d = \left(\frac{m}{\rho_{\text{liq}}} \right)^{1/3} = 3.83 \times 10^{-10} \text{ m}$$

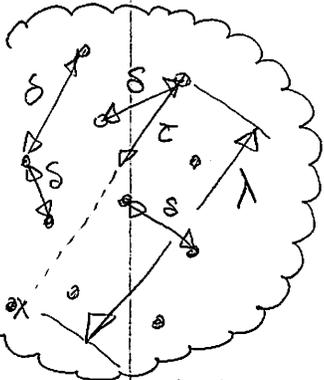
$$\lambda = \frac{1}{\sqrt{2} \pi d^2 n} = 5.717 \times 10^{-8} \text{ m} = \boxed{57.2 \text{ nm} = \lambda}$$

Distance between collision

$$\delta = \frac{1}{n^{1/3}} = 3.35 \times 10^{-9} \text{ m} = \boxed{3.35 \text{ nm}} \text{ (molecular spacing)}$$

Not same as λ .

Picture:



Note, we showed earlier that $T = \frac{M \bar{c}^2}{3 k_B A}$, so why doesn't your temperature increase when you're flying in a plane?

Aside: Comes from the classic Boltzmann dist.

$$f(E) = \frac{1}{Ae^{E/k_B T}} = \text{prob. that a particle has energy } E.$$

\hookrightarrow Constant $E.$
 $S = k_B \ln W$

Maxwellian Velocity Distribution

$$\phi(c_i) = \frac{1}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{1/2} e^{-\left(\frac{mc_i^2}{2k_B T} \right)}$$

$$0 < \phi(c_i) < 1$$

\hookrightarrow probability of a particle or molecule having a velocity c going in the i th direction (1D)

Like any other distribution:

$$\int_{-\infty}^{\infty} \phi(c_1) dc_1 = \int_{-\infty}^{\infty} \phi(c_2) dc_2 = \int_{-\infty}^{\infty} \phi(c_3) dc_3 = 1$$

We know that:

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx = \text{erf}(x), \quad \text{erf}(\infty) = 1, \quad \text{erf}(0) = 0$$

$$\int_0^{\infty} e^{-c^2} dc = \frac{\sqrt{\pi}}{2}$$

Now let's take a closer look:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{m}{2k_B T} \right)^{1/2} e^{-\left(\frac{mc_i^2}{2k_B T} \right)} dc_i \Rightarrow \text{Let } c \left(\frac{m}{2k_B T} \right)^{1/2} = \hat{c}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\hat{c}_i^2} d\hat{c}_i = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-\hat{c}_i^2} d\hat{c}_i = 1 \quad dc = \frac{1}{\left(\frac{m}{2k_B T} \right)^{1/2}} d\hat{c}$$

$$\int_0^{\infty} e^{-\hat{c}_i^2} d\hat{c}_i = \frac{\sqrt{\pi}}{2} = I_0$$

\hookrightarrow Multiplied by 2 and changed lower bound to 0 since distribution is symmetric and centered about 0.

Now if we look at higher power integrals:

$$I_m = \int_0^{\infty} c^n e^{-c^2} dc = \int_0^{\infty} \underbrace{c^{n-1}}_u \cdot \underbrace{c e^{-c^2}}_{dv} dc$$

Using integration by parts

$$\int u dv = uv - \int v du$$

$$\int dv = \int c e^{-c^2} dc \Rightarrow \text{Let } u = c^2 \Rightarrow \int \cancel{c} e^{-u} \frac{du}{\cancel{2c}} = \frac{1}{2} \int e^{-u} du$$

$du = 2c dc$ $\therefore v = -\frac{1}{2} e^{-c^2}$

$$\frac{dv}{dc} = -\frac{1}{2} (-2c) e^{-c^2} = c e^{-c^2} \Rightarrow dv = c e^{-c^2} dc \Rightarrow \text{WORKS!}$$

$$u = c^{n-1}$$

$$du = (n-1) c^{n-2} dc$$

Back substituting

$$\int u dv = uv - \int v du = (c^{n-1}) \left(-\frac{1}{2} e^{-c^2}\right) \Big|_0^\infty - \int_0^\infty -\frac{1}{2} e^{-c^2} (n-1) c^{n-2} dc$$

$$I_n = -\frac{c^{n-1}}{2} e^{-c^2} \Big|_0^\infty + \frac{(n-1)}{2} \int_0^\infty c^{n-2} e^{-c^2} dc$$

Need to use l'Hopital's rule $(\infty) \cdot (0)$
 Can prove to yourself that $\rightarrow 0$

$$I_n = \frac{n-1}{2} \int_0^\infty c^{n-2} e^{-c^2} dc = \frac{n-1}{2} I_{n-2}$$

Now we can have some fun:

$$I_0 = \frac{\sqrt{\pi}}{2} \quad I_1 = \frac{1}{2}$$

$$I_2 = \frac{\sqrt{\pi}}{4} \quad I_3 = \frac{1}{2}$$

$$I_4 = \frac{3}{8} \sqrt{\pi} \quad I_5 = 1$$

$$I_6 = \frac{15}{16} \sqrt{\pi} \quad \dots$$

↳ Recursive Solution

Example: $I_2 = \frac{2-1}{2} I_{2-2} = \frac{1}{2} I_0$
 $= \frac{1}{2} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}$

Note, we haven't solved for I_1 , however need to solve it to get I_3, I_5, I_7, \dots
 To solve for I_1 , let $u = c^2$ & integrate. Try it

~~END OF~~ LECTURE 2

Now we can look at each directional component of velocity

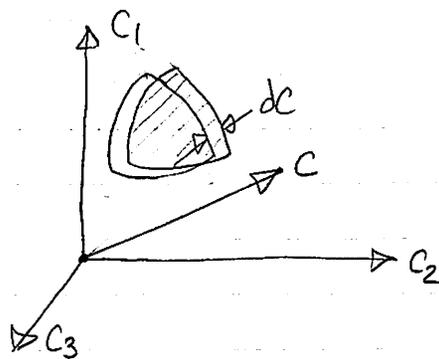
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(c_1) \phi(c_2) \phi(c_3) dc_1 dc_2 dc_3 = 1$$

This is true since the probability for three independent events is the product of the three individual probabilities.

$$\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{m(c_1^2 + c_2^2 + c_3^2)}{2k_B T}} dc_1 dc_2 dc_3 = 1$$

we know that $c^2 = c_1^2 + c_2^2 + c_3^2$

$$\left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{mc^2}{2k_B T}} dc_1 dc_2 dc_3$$



For a sphere: $4\pi c^2$ (surface area)
 $dV_c = 4\pi c^2 dc = dc_1 dc_2 dc_3$
 Thin shell approx.

Our integral becomes much simpler.

$$4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^{\infty} e^{-\frac{mc^2}{2k_B T}} c^2 dc = 1$$

So we previously defined $\hat{c} = c \left(\frac{m}{2k_B T}\right)^{1/2} \Rightarrow c = \frac{\hat{c}}{\left(\frac{m}{2k_B T}\right)^{1/2}}$

$$d\hat{c} = \left(\frac{m}{2k_B T}\right)^{1/2} dc$$

Back substituting:

$$= \frac{4\pi}{\pi^{3/2}} \left(\frac{m}{2k_B T}\right)^{3/2} \int_0^{\infty} e^{-\hat{c}^2} \hat{c}^2 \left(\frac{1}{\left(\frac{m}{2k_B T}\right)}\right) \frac{d\hat{c}}{\left(\frac{m}{2k_B T}\right)^{1/2}}$$

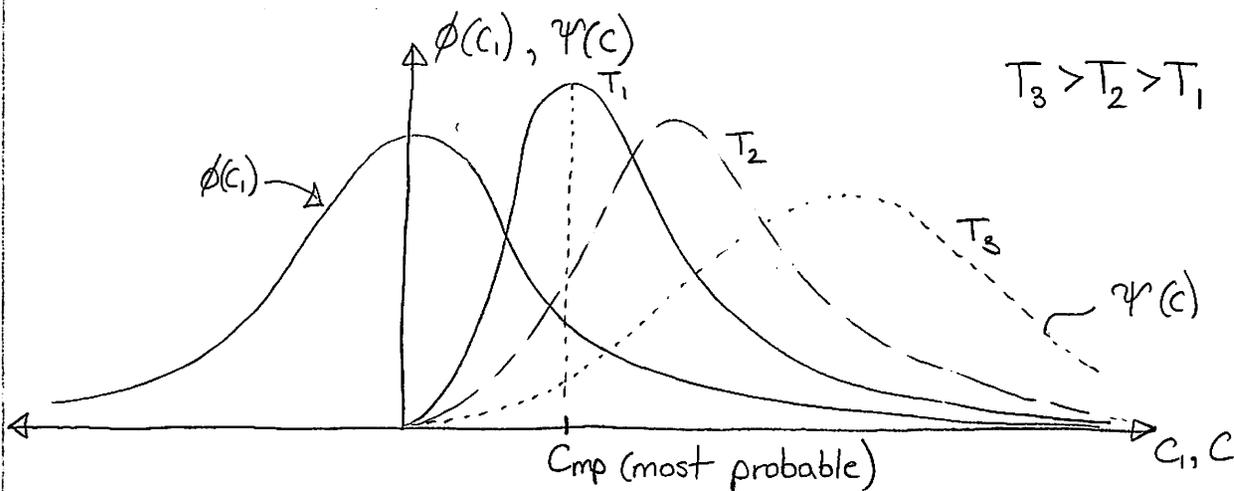
$$= \frac{4}{\sqrt{\pi}} \int_0^{\infty} \hat{c}^2 e^{-\hat{c}^2} d\hat{c} = 1 \Rightarrow \text{Works out}$$

$$I_2 = \frac{\sqrt{\pi}}{4}$$

From this, we see that the general speed distribution is:

$$\psi(c) = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_B T}\right)^{3/2} c^2 e^{-\frac{mc^2}{2k_B T}}$$

So what do our distributions look like?



So if we want to calculate the mean molecule speed (\bar{c})

$$\bar{c} = \int_0^{\infty} c \psi(c) dc$$

$$\bar{c} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{3/2} \int_0^{\infty} c^3 e^{-\frac{mc^2}{2k_B T}} dc$$

Doing our usual tricks $\hat{c} = c \left(\frac{m}{2k_B T} \right)^{1/2}$

$$\bar{c} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{3/2} \left(\frac{2k_B T}{m} \right)^2 \int_0^{\infty} \hat{c}^3 e^{-\hat{c}^2} d\hat{c}$$

Aside:

$$\int_0^{\infty} \hat{c}^3 e^{-\hat{c}^2} d\hat{c} = I_3$$

$$I_3 = \frac{n-1}{2} I_{n-2} = \frac{3-1}{2} I_1 = \frac{2}{2} \left(\frac{1}{2} \right)$$

We know this from our previous calculations as $I_3 = \frac{1}{2}$

$$\bar{c} = \frac{1}{2} \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{3/2} \left(\frac{2k_B T}{m} \right)^2 \frac{1}{2}$$

$$\boxed{\bar{c} = \sqrt{\frac{8k_B T}{\pi m}}}$$

\Rightarrow Average speed $\Rightarrow \bar{c} \sim \sqrt{T}$, as $T \uparrow, \bar{c} \uparrow$

So how about $\overline{c^2}$?

$$\overline{c^2} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{3/2} \int_0^{\infty} c^4 e^{-\frac{mc^2}{2k_B T}} dc \Rightarrow \hat{c} = c \left(\frac{m}{2k_B T} \right)^{1/2}$$

$$\overline{c^2} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{3/2} \left(\frac{2k_B T}{m} \right)^{5/2} \int \hat{c}^4 e^{-\hat{c}^2} d\hat{c}$$

$$\begin{aligned} \overline{c^2} &= \frac{4}{\sqrt{\pi}} \cdot \frac{3\sqrt{\pi}}{8} \left(\frac{m}{2k_B T} \right)^{3/2} \left(\frac{2k_B T}{m} \right)^{5/2} I_4 = \frac{n-1}{2} I_{n-2} = \frac{4-1}{2} I_{4-2} \\ &= \frac{3}{2} I_2 = \frac{3}{2} \frac{\sqrt{\pi}}{4} \\ &= \frac{3}{2} \cdot \frac{2k_B T}{m} = \frac{3k_B T}{m} \end{aligned}$$

$$\boxed{\overline{c^2} = \frac{3k_B T}{m}}$$

or

$$\boxed{C_{rms} = \sqrt{\frac{3k_B T}{m}}}$$

\Rightarrow Root mean square speed.

Looks familiar right?

We've proven that our average molecule kinetic energy assumption is indeed correct:

$$\frac{1}{2} m \overline{c^2} \approx \frac{3}{2} k_B T$$

How about the most probable velocity?

For this, we need to differentiate: $\frac{d}{dc} (\psi(c)) = 0$

$$\frac{d}{dc} \left(\frac{4}{\sqrt{\pi}} \cdot \left(\frac{m}{2k_B T} \right)^{3/2} c^2 e^{-\frac{mc^2}{2k_B T}} \right) = 0$$

$$\frac{4}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{3/2} \frac{d}{dc} (c^2 e^{-\frac{mc^2}{2k_B T}}) = 0$$

$$\frac{d}{dc} (c^2) \cdot e^{-\frac{mc^2}{2k_B T}} + c^2 \frac{d}{dc} (e^{-\frac{mc^2}{2k_B T}}) = 0$$

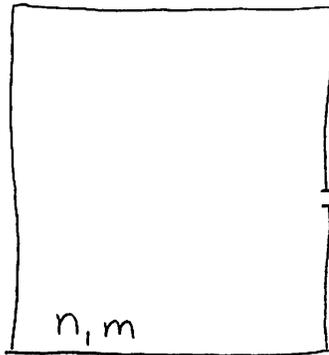
$$2c e^{-\frac{mc^2}{2k_B T}} + c^2 \left(-\frac{2mc}{2k_B T} \right) e^{-\frac{mc^2}{2k_B T}} = 0$$

$$1 - \frac{mc^2}{2k_B T} = 0 \Rightarrow$$

$$\boxed{C_{mp} = \sqrt{\frac{2k_B T}{m}}} \Rightarrow \text{most probable speed.}$$

To give you some insight, the speed of sound is $C_{sound} \approx 0.84 C_{mp}$ and $C_{sound} \approx 0.74 C_{mp} \Rightarrow$ Since sound waves propagate via molecular motion, makes sense that they are slightly slower than C_{mp} . (20)

Example #2 Reservoir full of air. What is the mass flux out of the orifice?



$d < D < \lambda$ where $n = \#$ of molecules/volume
 $d =$ size of molecule
 $D =$ orifice opening
 $\lambda =$ mean free path
 $m =$ mass per molecule
 $A =$ orifice area

$$\dot{m} = nmA \int_0^{\infty} c_1 \phi(c_1) \underbrace{\int_{-\infty}^{\infty} \phi(c_2) dc_2}_1 \underbrace{\int_{-\infty}^{\infty} \phi(c_3) dc_3}_1$$

\Rightarrow We don't care about the molecules y or z velocity
 Note, 0 is lower bound since we only want molecules that travel in the +x direction out of the reservoir.

$$\dot{m} = nmA \frac{1}{\sqrt{\pi}} \left(\frac{m}{2k_B T} \right)^{1/2} \int_0^{\infty} c_1 e^{-\frac{m c_1^2}{2k_B T}} dc_1$$

Using our old trick, and knowing that $n = \frac{p}{m}$

$$\dot{m} = pA \frac{1}{\sqrt{\pi}} \left(\frac{2k_B T}{m} \right)^{1/2} \underbrace{\int_0^{\infty} \hat{c}_1 e^{-\hat{c}_1^2} d\hat{c}_1}_{I_1 = \frac{1}{2}}$$

$$= \frac{pA}{2\sqrt{\pi}} \left(\frac{2k_B T}{m} \right)^{1/2}$$

\Rightarrow multiply by $\left(\frac{2}{2} \right)$

$$= \frac{pA}{4} \underbrace{\left(\frac{8k_B T}{\pi m} \right)^{1/2}}_{\bar{c}} = \boxed{\frac{pA\bar{c}}{4} = \dot{m}}$$

Another way to write it is:

$$\dot{m} = \frac{pA}{4} \left(\frac{8k_B T}{\pi m} \right)^{1/2} = \frac{\sqrt{8} pA}{4\sqrt{\pi}} \sqrt{\frac{k_B T}{m}} = \frac{2^{3/2} pA}{2^2 \pi^{1/2}} \sqrt{\frac{k_B T}{m}}$$

We also know that: $R_{\text{gas}} = \frac{k_B}{m}$ ^{Aside} \Rightarrow

$$\dot{m} = \frac{pA}{\sqrt{2\pi}} \sqrt{R_{\text{gas}} T} \approx 0.4 pA \sqrt{RT}$$

$$R_{\text{gas}} = \frac{R}{M}$$

$$k_B = \frac{R}{A} = \frac{M R_{\text{gas}}}{A}$$

but $M = A \cdot m$

$$k_B = \frac{A m R_{\text{gas}}}{A} = R_{\text{gas}} m$$

Avogadro
NOT!
Area.

For comparison, what if $D > \lambda$, so we can use continuum mechanics to solve for \dot{m} . Now we have a choked flow condition. From compressible fluid mechanics:

$$\dot{m} = pA \sqrt{R_{\text{gas}} T} \cdot \underbrace{\left[\gamma \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}} \right]^{1/2}}_{0.684}$$

$$\dot{m} = 0.684 pA \sqrt{RT} \Rightarrow \text{Using continuum mechanics}$$

We overestimate the result by 50% using continuum mechanics when compared to rarified gas theory.

So what is the energy of the escaping gas molecules? Solve for $\overline{c^2}$ and c_{esc} . You will see that

$$\overline{c_{\text{esc}}^2} = 4 \frac{R_{\text{gas}} T}{m} = \frac{4k_B T}{m}$$

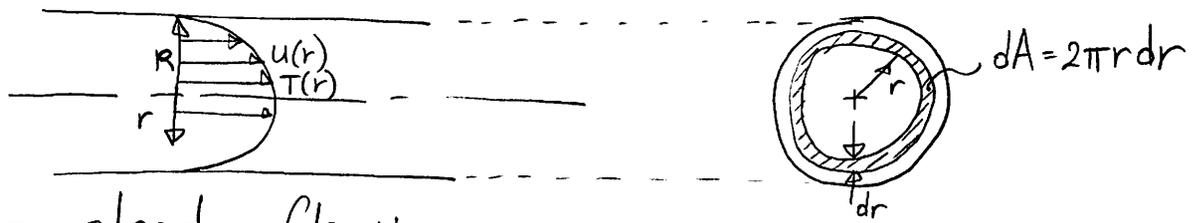
$$\frac{4k_B T}{m} > \frac{3k_B T}{m} = \langle E \rangle_{\text{Reservoir}}$$

\hookrightarrow Escaping molecules have on average a larger energy than the rest of the molecules in the reservoir. Makes sense, they travel faster and scatter off the walls more frequently. (22)

Extra Derivation (Pinhole problem)

Solution to energy of escaping gas molecules from a pinhole:

The way to solve this problem is analogous to calculating bulk flow temperature of a fluid in a pipe.



For a steady flow:

$$\bar{u} = \frac{1}{\rho \pi R^2} \int_0^R \rho u(r) 2\pi r dr$$

$$\bar{T} = \frac{1}{\rho c_p \bar{u} \pi R^2} \int_0^R \rho c_p T(r) u(r) 2\pi r dr$$

So we now have: Total convected energy in the flow.

$$\bar{u} = \frac{1}{\pi R^2} \int_0^R u 2\pi r dr$$

$$\bar{T} = \frac{1}{\pi R^2 \bar{u}} \int_0^R (uT) 2\pi r dr$$

⇒ We will need to use an analogous approach.

In our case, we need $\overline{C_{esc}^2}$. We can write:

$$\overline{C_{esc}^2} = \frac{1}{\overline{C_{esc}}} \left(\frac{m}{2k_B T} \right)^{3/2} \frac{1}{\pi^{3/2}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty C_1 (c^2) e^{-\frac{mc^2}{2k_B T}} dC_1 dC_2 dC_3$$

Need to divide by $\overline{C_{esc}}$ due to the C_1 in the integral.

↳ energy advected in the 1th component

This term is like the \bar{T} calculation. We are computing the average energy, $\overline{C^2}$, advected in the C_1 direction

We can also express $\overline{C_{esc}}$ as: $\overline{C_{esc}} = \frac{\dot{m}}{\rho A}$

Expanding our integral and back substituting $\overline{C_{esc}}$:

$$\overline{C_{esc}^2} = \frac{\rho A}{\dot{m}} \left(\frac{m}{2k_B T} \right)^{3/2} \frac{1}{\pi^{3/2}} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty C_1 (C_1^2 + C_2^2 + C_3^2) e^{-\frac{mC_1^2}{2k_B T}} e^{-\frac{mC_2^2}{2k_B T}} e^{-\frac{mC_3^2}{2k_B T}} \cdot dC_1 dC_2 dC_3$$

Now we can expand the integral

$$\overline{C_{esc}^2} = \underbrace{\frac{\rho A}{\dot{m}} \left(\frac{m}{2k_B T} \right)^{3/2} \frac{1}{\pi^{3/2}}}_{X} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty C_1^3 e^{-\frac{mC_1^2}{2k_B T}} e^{-\frac{mC_2^2}{2k_B T}} e^{-\frac{mC_3^2}{2k_B T}} dC_1 dC_2 dC_3$$

$$+ \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty C_1 C_2^2 e^{-\frac{mC_1^2}{2k_B T}} e^{-\frac{mC_2^2}{2k_B T}} e^{-\frac{mC_3^2}{2k_B T}} dC_1 dC_2 dC_3$$

$$+ \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty C_1 C_3^2 e^{-\frac{mC_1^2}{2k_B T}} e^{-\frac{mC_2^2}{2k_B T}} e^{-\frac{mC_3^2}{2k_B T}} dC_1 dC_2 dC_3 \Bigg]$$

$$\begin{aligned} \overline{C_{esc}^2} = X & \left[\int_0^\infty C_1^3 e^{-\frac{mC_1^2}{2k_B T}} dC_1 \int_{-\infty}^\infty e^{-\frac{mC_2^2}{2k_B T}} dC_2 \int_{-\infty}^\infty e^{-\frac{mC_3^2}{2k_B T}} dC_3 \right. \\ & + \int_0^\infty C_1 e^{-\frac{mC_1^2}{2k_B T}} dC_1 \int_{-\infty}^\infty C_2^2 e^{-\frac{mC_2^2}{2k_B T}} dC_2 \int_{-\infty}^\infty e^{-\frac{mC_3^2}{2k_B T}} dC_3 \\ & \left. + \int_0^\infty C_1 e^{-\frac{mC_1^2}{2k_B T}} dC_1 \int_{-\infty}^\infty e^{-\frac{mC_2^2}{2k_B T}} dC_2 \int_{-\infty}^\infty C_3^2 e^{-\frac{mC_3^2}{2k_B T}} dC_3 \right] \end{aligned}$$

Now we can normalize to make our integral simpler

$$\text{Let: } c_1 = \frac{\hat{c}_1}{\left(\frac{m}{2k_B T}\right)^{1/2}}, \quad c_2 = \frac{\hat{c}_2}{\left(\frac{m}{2k_B T}\right)^{1/2}}, \quad c_3 = \frac{\hat{c}_3}{\left(\frac{m}{2k_B T}\right)^{1/2}}$$

$$dc_1 = \frac{d\hat{c}_1}{\left(\frac{m}{2k_B T}\right)^{1/2}}, \quad dc_2 = \frac{d\hat{c}_2}{\left(\frac{m}{2k_B T}\right)^{1/2}}, \quad dc_3 = \frac{d\hat{c}_3}{\left(\frac{m}{2k_B T}\right)^{1/2}}$$

Back substituting

$$\overline{c_{esc}^2} = \underbrace{\prod \left(\frac{2k_B T}{m}\right)^{3/2}}_{\text{from } \hat{c}} \underbrace{\left(\frac{2k_B T}{m}\right)^{3/2}}_{\text{from } d\hat{c}} \cdot \underbrace{\int_0^\infty \hat{c}_1 e^{-\hat{c}_1^2} d\hat{c}_1}_{I_3} \underbrace{\int_{-\infty}^\infty e^{-\hat{c}_2^2} d\hat{c}_2}_{2I_0} \underbrace{\int_{-\infty}^\infty e^{-\hat{c}_3^2} d\hat{c}_3}_{2I_0}$$

$$+ \underbrace{\int_0^\infty \hat{c}_1 e^{\hat{c}_1^2} d\hat{c}_1}_{I_1} \underbrace{\int_{-\infty}^\infty \hat{c}_2 e^{-\hat{c}_2^2} d\hat{c}_2}_{2I_2} \underbrace{\int_{-\infty}^\infty e^{-\hat{c}_3^2} d\hat{c}_3}_{2I_0}$$

$$+ \underbrace{\int_0^\infty \hat{c}_1 e^{-\hat{c}_1^2} d\hat{c}_1}_{I_1} \underbrace{\int_{-\infty}^\infty e^{-\hat{c}_2^2} d\hat{c}_2}_{2I_0} \underbrace{\int_{-\infty}^\infty \hat{c}_3^2 e^{-\hat{c}_3^2} d\hat{c}_3}_{2I_2}$$

Putting everything together,

$$\overline{c_{esc}^2} = \frac{\rho A}{\dot{m}} \left(\frac{m}{2k_B T}\right)^{3/2} \frac{1}{\pi^{3/2}} \left(\frac{2k_B T}{m}\right)^{3/2} \left[I_3 (2I_0)^2 + I_1 (2I_2)(2I_0) + 4I_0 I_1 I_2 \right]$$

$$= \frac{\rho A}{\dot{m}} \frac{1}{\pi^{3/2}} \left(\frac{2k_B T}{m}\right)^{3/2} \left[\frac{1}{2} \left(2 \frac{\sqrt{\pi}}{2}\right)^2 + \frac{1}{2} \left(2 \frac{\sqrt{\pi}}{4}\right) \left(2 \frac{\sqrt{\pi}}{2}\right) + \frac{1}{2} \left(2 \frac{\sqrt{\pi}}{2}\right) \left(2 \frac{\sqrt{\pi}}{4}\right) \right]$$

$$= \frac{\rho A}{\dot{m}} \frac{1}{\pi^{3/2}} \left(\frac{2k_B T}{m}\right)^{3/2} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = \frac{\rho A}{\dot{m}} \left(\frac{2k_B T}{m}\right)^{3/2} \frac{1}{\sqrt{\pi}}$$

But we already solved that

$$\dot{m} = \frac{pA}{4} \left(\frac{8k_B T}{\pi m} \right)^{1/2}$$

Back substituting:

$$\begin{aligned} \overline{z}_{\text{Cesc}}^2 &= \frac{pA}{\frac{pA}{4} \left(\frac{8k_B T}{\pi m} \right)^{1/2}} \cdot \left(\frac{2k_B T}{m} \right)^{3/2} \frac{1}{\sqrt{\pi}} \\ &= 4 \left(\frac{\pi m}{8k_B T} \right)^{1/2} \left(\frac{2k_B T}{m} \right)^{3/2} \frac{1}{\sqrt{\pi}} \\ &= 4 \frac{k_B T}{m} \end{aligned}$$

$$\therefore \boxed{\overline{z}_{\text{Cesc}}^2 = \frac{4k_B T}{m}} > \frac{3k_B T}{m} = \langle E \rangle_{\text{Reservoir}}$$

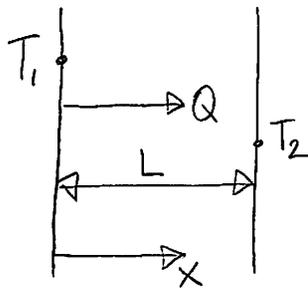
Result seems counter intuitive, but actually makes sense. Molecules with a larger energy move at a higher speed, and scatter off the reservoir walls more frequently. This makes them more probable to hit the pinhole and escape, hence the larger average energy.

Conduction

$\vec{v} = 0$, $1-D$, Steady State, $\dot{Q}''' = 0$, $k = \text{constant}$

We know that $\nabla^2 T = 0$ (heat equation)

① Slab



$Q = \text{heat transfer rate [W]}$

Using our heat equation

$$\int \frac{\partial^2 T}{\partial x^2} = \int 0$$

$$\int \frac{\partial T}{\partial x} = \int C_1$$

$$T(x) = C_1 x + C_2 \Rightarrow \text{B.C.'s} \Rightarrow \begin{aligned} T(x=0) &= T_1 \\ T(x=L) &= T_2 \end{aligned}$$

$$T(x=0) = T_1 = C_1(0) + C_2 \Rightarrow C_2 = T_1$$

$$T(x=L) = T_2 = C_1 L + T_1 \Rightarrow C_1 = \frac{T_2 - T_1}{L}$$

$$\boxed{T(x) = \frac{T_2 - T_1}{L} x + T_1} \Rightarrow \text{Temperature profile in the solid. Linear and decreasing from hot to cold}$$

So if we define a heat transfer resistance; R , $A = \text{area}$

Aside:
Flow of Current:
 $I = \frac{\Delta V}{R}$

$$Q = \frac{\Delta T}{R} \Rightarrow Q = -kA \frac{\partial T}{\partial x} \Rightarrow \text{from above } \frac{\partial T}{\partial x} = \frac{T_2 - T_1}{L}$$

$$\frac{\partial T}{\partial x} = -\frac{(T_1 - T_2)}{L} = -\frac{\Delta T}{L} \Rightarrow \text{Back substituting}$$

$$Q = \frac{kA}{L} \Delta T$$

$$= \frac{kA}{L} \Delta T$$

$$\frac{1}{R} \Rightarrow$$

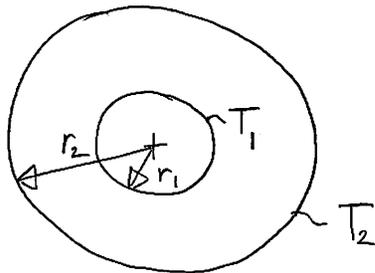
$$R_{\text{SLAB}} = \frac{L}{kA}$$

\Rightarrow Thermal Resistance of a solid slab.

*(Read the interesting paper about Fourier)

L = slab thickness
 A = heat transfer area
 k = thermal conductivity

② Circular cylinder



We know our heat equation in radial coordinates is:

$$\frac{1}{r} \cdot \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = 0$$

$$\int \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = \int 0$$

$$r \frac{\partial T}{\partial r} = C_1$$

$$\int \frac{\partial T}{\partial r} = \int \frac{C_1}{r}$$

$$T(r) = C_1 \ln r + C_2$$

Applying our boundary conditions:

$$T_1 = C_1 \ln r_1 + C_2 \quad \text{①}$$

$$T_2 = C_1 \ln r_2 + C_2 \quad \text{②}$$

Subtract ① & ② to get

$$C_1 = \frac{T_1 - T_2}{\ln r_1 - \ln r_2} = \frac{T_1 - T_2}{\ln(r_1/r_2)}$$

Back substitute into ①

$$C_2 = T_1 - \frac{(T_1 - T_2)}{\ln(r_1/r_2)} \ln(r_1)$$

Back substituting into our initial equation $T(r)$

$$T = \frac{T_1 - T_2}{\ln(r_1/r_2)} \ln r + T_1 - \frac{(T_1 - T_2) \ln r_1}{\ln(r_1/r_2)}$$

Finally:

$$\frac{T - T_1}{T_1 - T_2} = \frac{\ln(r/r_1)}{\ln(r_1/r_2)}$$

or

$$\boxed{\frac{T - T_1}{T_2 - T_1} = \frac{\ln(r/r_1)}{\ln(r_2/r_1)}}$$

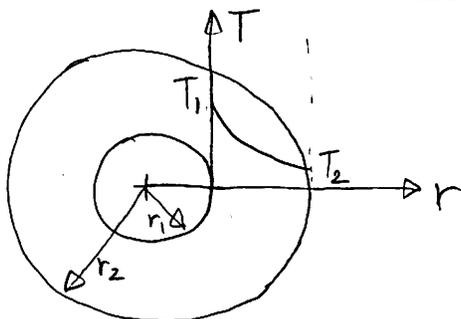
↳ Radial temperature profile.

Calculating the thermal resistance:

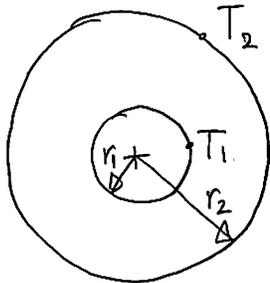
$$Q = \frac{\Delta T}{R} = -k A \left. \frac{\partial T}{\partial r} \right|_r = -k (2\pi r L) \frac{\overbrace{T_2 - T_1}^{\Delta T}}{\ln(r_2/r_1)} \cdot \frac{1}{r}$$

$$\boxed{R_{cyl} = \left| \frac{\ln(r_2/r_1)}{2\pi k L} \right|}$$

⇒ Cylindrical thermal resistance
Note the absolute value on the resistance. Doesn't matter if $r_2 > r_1$, or $r_1 > r_2$ ⇒ resistance is the same.



③ Spherical System



Using our spherical heat equation

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = 0$$

$$\int \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) = \int 0$$

$$r^2 \frac{\partial T}{\partial r} = C_1 \Rightarrow \int \frac{\partial T}{\partial r} = \int \frac{C_1}{r^2}$$

$$T = -\frac{C_1}{r} + C_2$$

Our boundary conditions are:

$$T_1 = -\frac{C_1}{r_1} + C_2 \quad (1)$$

$$T_2 = -\frac{C_1}{r_2} + C_2 \quad (2)$$

① - ②

$$C_1 = \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

$$C_2 = T_2 + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \cdot \frac{1}{r_2}$$

$$T = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \cdot \frac{1}{r} + \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \cdot \frac{1}{r_2} + T_2$$

$$\boxed{\frac{T - T_2}{T_1 - T_2} = \frac{\left(\frac{1}{r} - \frac{1}{r_2}\right)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}}$$

\Rightarrow Spherical temperature profile

Calculating our thermal resistance:

$$Q = -kA \frac{\partial T}{\partial r} \quad ; \quad \frac{\partial T}{\partial r} = \frac{T_1 - T_2}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \left(-\frac{1}{r^2}\right)$$

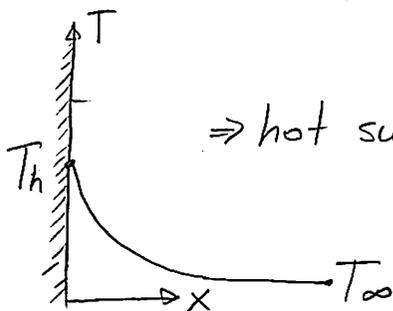
$$= -k(4\pi r^2) \frac{T_1 - T_2}{\frac{1}{r_1} - \frac{1}{r_2}} \left(-\frac{1}{r^2}\right)$$

$$Q = \frac{T_1 - T_2}{\left\{ \frac{\frac{1}{r_1} - \frac{1}{r_2}}{4\pi k} \right\}} = \frac{\Delta T}{R_{\text{sph}}}$$

$$\boxed{R_{\text{sph}} = \frac{\left| \frac{1}{r_1} - \frac{1}{r_2} \right|}{4\pi k}} \Rightarrow \text{Spherical thermal resistance.}$$

We can do the same thermal resistance analogy for convection heat transfer:

$$Q = hA \Delta T \quad ; \quad h = \text{heat transfer coefficient [W/m}^2 \cdot \text{K]} \\ A = \text{area}$$



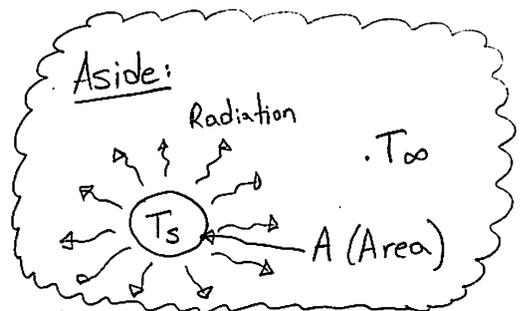
\Rightarrow hot surface exchanging heat with a fluid

Newton developed this concept in the 1700's. Known as Newton's law of cooling.

$$Q = hA \Delta T = \frac{\Delta T}{(1/hA)} \Rightarrow \boxed{R_{\text{Th, conv}} = \frac{1}{hA}}$$

\hookrightarrow Thermal resistance associated with convection

Radiation thermal resistance:



$$Q_{rad} = \epsilon \sigma A (T_s^4 - T_{\infty}^4) \Rightarrow \text{Stefan Boltzmann Law (Black Body)}$$

Emisivity ϵ Stefan-Boltzmann Constant (5.67×10^{-8})

Factoring: $(T_s^2 + T_{\infty}^2)(T_s + T_{\infty})(T_s - T_{\infty})$

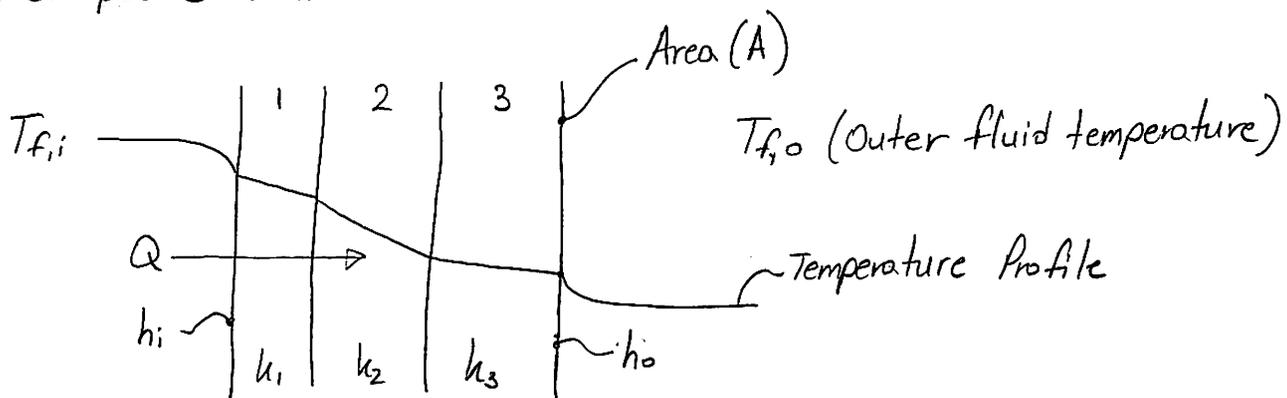
$$Q_{rad} = \underbrace{\epsilon \sigma A (T_s^2 + T_{\infty}^2)(T_s + T_{\infty})}_{h_{rad} \cdot A} \underbrace{(T_s - T_{\infty})}_{\Delta T}$$

$$h_{rad} = \epsilon \sigma (T_s^2 + T_{\infty}^2)(T_s + T_{\infty}) \Rightarrow \text{Radiation heat transfer coefficient.}$$

$$R_{\text{radiation}} = \frac{1}{h_{rad} A} \Rightarrow \text{Thermal resistance associated with radiation. (nonlinear)}$$

Composite problems

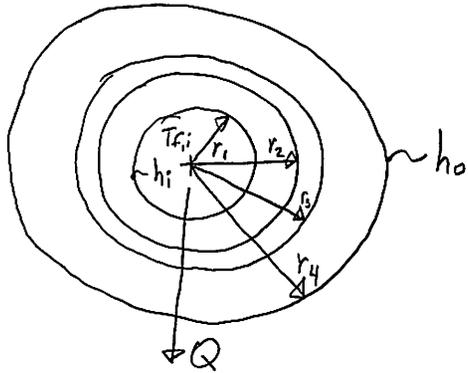
① Composite wall:



$$Q = \frac{A (T_{f,i} - T_{f,o})}{\frac{1}{h_i} + \frac{1}{h_o} + \sum_{j=1}^n \frac{L_j}{k_j}}$$

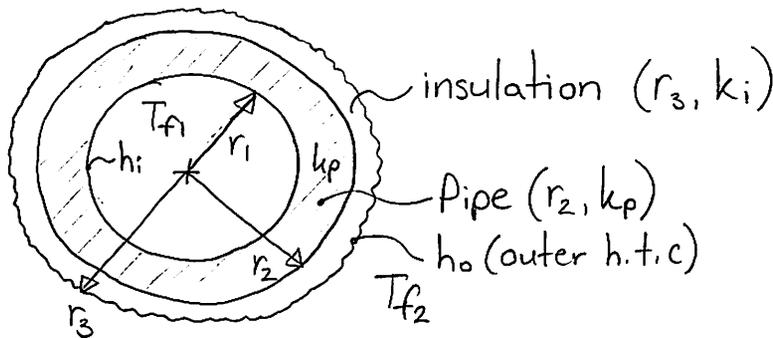
L_j = thickness of the j'th wall
 k_j = thermal conductivity of the j'th wall
 h_i = inner wall heat transfer coeff.
 h_o = outer wall h.t.c.

② Composite cylinder



$$Q = \frac{2\pi L (T_{f,i} - T_{f,o})}{\frac{1}{h_i r_1} + \frac{1}{h_o r_{n+1}} + \sum_{j=1}^n \frac{\ln(r_{j+1}/r_j)}{k_j}}$$

Critical Thickness of Insulation:



k_i = insulation thermal c.
 k_p = pipe thermal cond.
 T_{f1} = inner fluid temp.
 T_{f2} = outer fluid temp.
 h_o = outer h.t.c.
 $r_1 < r_2 < r_3$

Note, this is an optimization problem since by adding insulation:

1) The outer area, for which h_o convects heat away increases $\Rightarrow Q_{loss} \uparrow$

2) The resistance of the insulation increases $\Rightarrow Q_{loss} \downarrow$

$$R_{TOT} = \underbrace{\frac{1}{2\pi r_1 h_i}}_{\text{inner convection}} + \underbrace{\frac{\ln(r_2/r_1)}{2\pi k_p L}}_{\text{pipe cond.}} + \underbrace{\frac{\ln(r_3/r_2)}{2\pi k_i L}}_{\text{insulation cond.}} + \underbrace{\frac{1}{2\pi r_3 h_o L}}_{\text{outer convection}}$$

We need to maximize R_{TOT} with respect to r_3

$$\frac{\partial R_{TOT}}{\partial r_3} = 0$$

$$\frac{1}{2\pi k_i L} \frac{\partial}{\partial r_3} \ln(r_3/r_2) + \frac{1}{2\pi h_o L} \frac{\partial}{\partial r_3} \left(\frac{1}{r_3} \right) = 0$$

$$\frac{1}{k_i} \frac{\partial}{\partial r_3} \ln\left(\frac{r_3}{r_2}\right) + \frac{1}{h_o} \frac{\partial}{\partial r_3} \left(\frac{1}{r_3}\right) = 0$$

$$\frac{1}{k_i} \cdot \frac{k_i}{r_3 \cdot r_2} + \frac{1}{h_o} \left(-\frac{1}{r_3^2}\right) = 0$$

$$\frac{1}{k_i} - \frac{1}{h_o r_3} = 0$$

$$\boxed{r_{3,\text{crit}} = \frac{k_i}{h_o}}$$

\Rightarrow Critical thickness of insulation for a pipe.

Now we need to check to make sure this is a minimum for Q_{TOT}

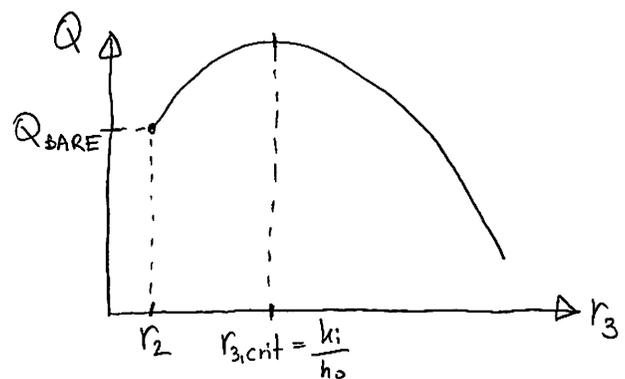
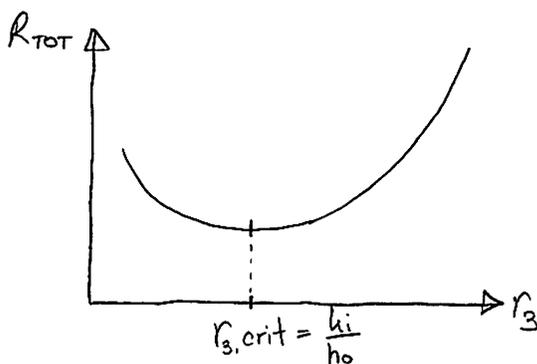
$$\begin{aligned} \frac{\partial^2 R_{\text{TOT}}}{\partial r_3^2} &= \left[\frac{\partial}{\partial r} \left(\frac{1}{k_i r_3}\right) - \frac{\partial}{\partial r} \left(\frac{1}{h_o r_3^2}\right) \right] \frac{1}{2\pi L} \\ &= \left(-\frac{1}{k_i r_3^2} + \frac{2}{h_o r_3^3} \right) \frac{1}{2\pi L} \end{aligned}$$

Substitute in $r_{3,\text{crit}} = \frac{k_i}{h_o}$ into equation ①

$$= -\frac{1}{k_i \left(\frac{k_i^2}{h_o^2}\right)} + \frac{2}{h_o \left(\frac{k_i^3}{h_o^3}\right)} = \left(-\frac{h_o^2}{k_i^3} + \frac{2h_o^2}{k_i^3} \right) \cdot \frac{1}{2\pi L}$$

$$= +\frac{h_o}{2\pi k_i^3} > 0 \quad (\text{Always})$$

So our $R_{\text{TOT}}(r_{3,\text{cr}})$ is a global minimum.
No optimum thickness exists, only a critical insulation thickness.



If we do a sample calculation

$$\left. \begin{array}{l} k_i = 0.1 \text{ W/m}\cdot\text{K} \\ h_o = 5 \text{ W/m}^2\cdot\text{K} \end{array} \right\} r_{3, \text{crit}} = \frac{0.1 \text{ W/m}\cdot\text{K}}{5 \text{ W/m}^2\cdot\text{K}} = 0.02 \text{ m} = 2 \text{ cm}$$

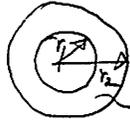
↑
this is a radius!

Note, here $r_{3, \text{crit}} < r_2$ meaning it's OK to insulate.

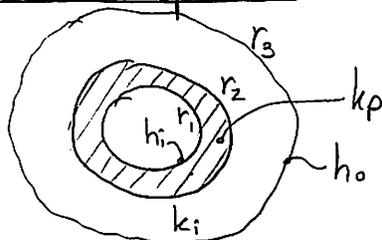
In general: $k_i \sim 0.1 \text{ W/m}\cdot\text{K}$ (common insulating materials)
 $h_{o, \text{minimum}} \sim 5 \text{ W/m}^2\cdot\text{K}$ (natural convection)

$r_{\text{crit}} \sim 1 \text{ cm} \Rightarrow$ We usually operate well above this.

∴ We can insulate hot water and steam pipes without worrying about increasing external heat transfer losses.

How about electrical wires? \Rightarrow  $\Rightarrow r_1 \sim 1 \text{ mm}$
 Good to insulate to increase losses & cool the wire.

For a sphere:



$$r_1 < r_2 < r_3$$

$$R_{\text{TOT}} = \frac{1}{4\pi r_1^2 h_i} + \frac{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}{4\pi k_p} + \frac{\left(\frac{1}{r_2} - \frac{1}{r_3}\right)}{4\pi k_i} + \frac{1}{4\pi r_3^2 h_o}$$

$$\frac{\partial R_{\text{TOT}}}{\partial r_3} = \frac{1}{4\pi k_i} \left(\frac{1}{r_3^2}\right) + \frac{1}{4\pi h_o} \left(-\frac{2}{r_3^3}\right) = 0$$

$$\frac{1}{r_{3, \text{cr}}^2 k_i} - \frac{2}{h_o r_{3, \text{cr}}^3} = 0 \Rightarrow$$

$$\boxed{r_{3, \text{cr}} = \frac{2k_i}{h_o}}$$

↓ Critical insulation thickness for a sphere.

Heat Generation (Slab)Steady State, 1D, constant properties, Uniform \dot{Q}'''

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{Q}''' = \rho c_p \frac{\partial T}{\partial t} \quad \text{s.s.}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{Q}'''}{k} = 0 \quad (1)$$

$$T(x=L) = T_w$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0$$

Integrating (1)

$$\frac{\partial T}{\partial x} = -\frac{\dot{Q}'''}{k}x + C_1$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 = -\frac{\dot{Q}'''}{k}(0) + C_1 = 0 \Rightarrow C_1 = 0$$

$$T(x) = -\frac{\dot{Q}'''x^2}{2k} + C_2$$

$$T(x=L) = T_w = -\frac{\dot{Q}'''L^2}{2k} + C_2 \Rightarrow C_2 = T_w + \frac{\dot{Q}'''L^2}{2k}$$

$$T(x) = T_w + \frac{\dot{Q}'''L^2}{2k} \left(1 - \left(\frac{x}{L} \right)^2 \right)$$

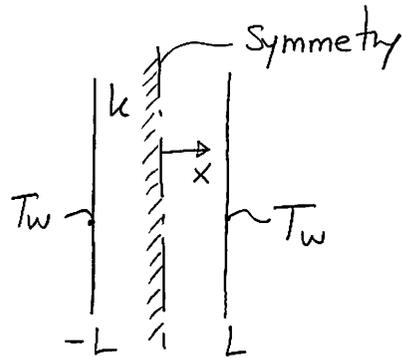
Where is T_{\max} ?

$$\frac{\partial T}{\partial x} = -\frac{\dot{Q}'''x}{k} = 0 \Rightarrow x=0 \Rightarrow \frac{\partial^2 T}{\partial x^2} = -\frac{\dot{Q}'''}{k} < 0 \quad (T \text{ is a max at } x=0)$$

$$T_{\max} = T_w + \frac{\dot{Q}'''L^2}{2k}$$

$$q'' = q(L) = -k \left. \frac{\partial T}{\partial x} \right|_{x=L} = +k \frac{\dot{Q}'''L}{k} = \dot{Q}'''L$$

$$q''_{\text{out}} = \dot{Q}'''L$$



In general, for heat generation we have a formulation:

$$-\frac{k}{r^n} \frac{\partial}{\partial r} \left(r^n \frac{\partial T}{\partial r} \right) = \dot{Q}''' \quad \begin{cases} n=0 \text{ (slab)} \\ n=1 \text{ (cylinder)} \\ n=2 \text{ (sphere)} \end{cases}$$

This only works if $\dot{Q}''' = \text{constant}$

$$-k \frac{\partial}{\partial r} \left(r^n \frac{\partial T}{\partial r} \right) = \dot{Q}''' r^n$$

$$-k r^n \frac{\partial T}{\partial r} = \dot{Q}''' \int r^n dr = \frac{\dot{Q}'''}{n+1} r^{n+1} + C_1$$

$$q'' = -k \frac{\partial T}{\partial r} = \frac{\dot{Q}''' r}{n+1} + \frac{C_1}{r^n}$$

Choosing $q''(0) = 0 \rightarrow C_1 = 0$ for $n=0$

For $n=1, 2$, $C_1 = 0$ if $|q''| < \infty$ (since $\frac{C_1}{r^1}$ or $\frac{C_1}{r^2}$ at $r \rightarrow 0$ blows up)

$$\frac{\partial T}{\partial r} = -\frac{\dot{Q}''' r}{(n+1)k}$$

$$T = -\frac{\dot{Q}'''}{(n+1)k} \int r dr = C_2 - \frac{\dot{Q}''' r^2}{2(n+1)k}$$

$$\text{Letting } T(a) = T_a \Rightarrow T_a = C_2 - \frac{\dot{Q}''' a^2}{2(n+1)k} \Rightarrow C_2 = T_a + \frac{\dot{Q}''' a^2}{2(n+1)k}$$

So our generalized result is:

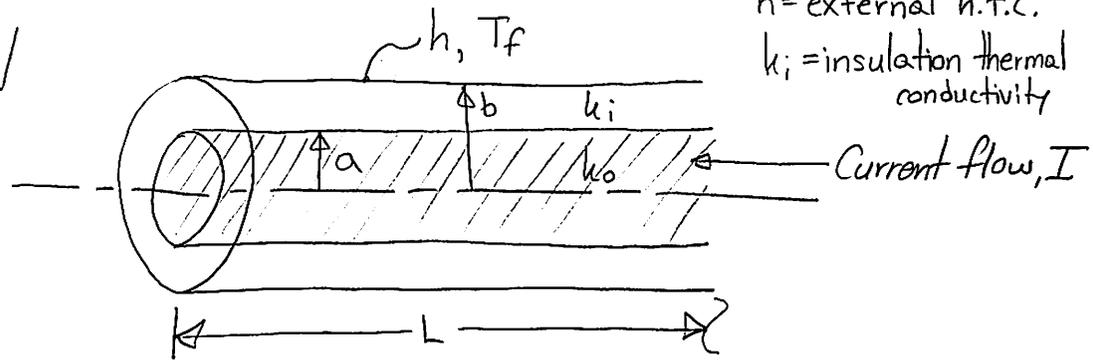
$$T(r) = T_a + \frac{\dot{Q}''' a^2}{2(n+1)k} \left(1 - \frac{r^2}{a^2} \right) \Rightarrow \text{Temperature profile}$$

$$q'' = \frac{\dot{Q}''' r}{n+1} \Rightarrow \text{Heat flux}$$

$$T_{\max} = T(r=0) = T_a + \frac{\dot{Q}''' a^2}{2(n+1)k}$$

\Rightarrow Maximum temperature where T_a is the boundary temperature on the outside.

Example #3/



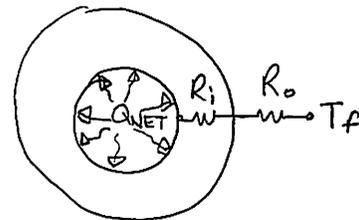
k_o = wire thermal cond.
 h = external h.t.c.
 k_i = insulation thermal conductivity

We have the wire electrical resistance, R_e , and the current, I , what is T_{max} ?

$$\text{Energy loss (Joule Heating)} = I^2 R_e$$

$$\dot{Q}''' = \frac{I^2 R_e}{\pi a^2 L}$$

$$Q_{net} = \pi a^2 L \dot{Q}''' = \frac{\Delta T}{R_{TOT}}$$



$$\pi a^2 L \dot{Q}''' = \frac{T_a - T_f}{\frac{\ln(b/a)}{2\pi L k_i} + \frac{1}{2\pi b L h}}$$

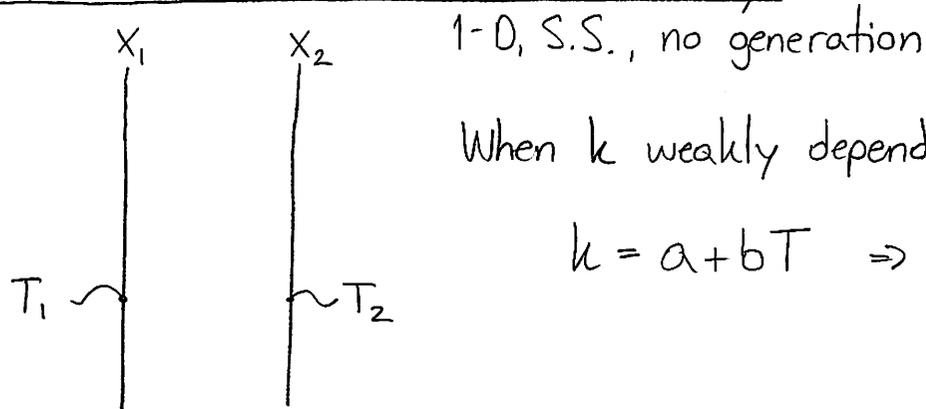
$$T_a = T_f + \frac{a^2 \dot{Q}'''}{2} \left[\frac{\ln(b/a)}{k_i} + \frac{1}{bh} \right] \quad (1)$$

We just learned that for a cylinder ($n=1$)

$$T_{max} = T_a + \frac{\dot{Q}''' a^2}{4k} \quad (2)$$

Substitute (1) into (2)

$$T_{max} = T_f + \frac{a^2 \dot{Q}'''}{2} \left[\frac{1}{2k_o} + \frac{\ln(b/a)}{k_i} + \frac{1}{bh} \right]$$

Wall with Variable Thermal ConductivityWhen k weakly depends on T ,

$$k = a + bT \Rightarrow \text{units } a \Rightarrow \frac{\text{W}}{\text{m}\cdot\text{K}}$$

$$b \Rightarrow \frac{\text{W}}{\text{m}\cdot\text{K}^2}$$

No need to use heat equation, even simpler

$$q_x = -k \frac{\partial T}{\partial x} \Rightarrow \text{Integrate}$$

$$\int_{x_1}^{x_2} q_x dx = - \int_{T_1}^{T_2} (a + bT) dT$$

We know that the heat flux is independent of location, (from the conservation of energy)

$$q_x(x_2 - x_1) = -a(T_2 - T_1) - \frac{b(T_2^2 - T_1^2)}{2} \Rightarrow \text{Factor}$$

$$q_x \Delta x = - \left(a + b \frac{T_2 + T_1}{2} \right) (T_2 - T_1)$$

$$\text{So: } q_x \Delta x = -(a + b\bar{T}) \Delta T$$

Rearranging, an interesting interpretation evolves

$$q_x = - \underbrace{(a + b\bar{T})}_{\bar{k}} \frac{\Delta T}{\Delta x}$$

$$\boxed{q_x = -\bar{k} \frac{\Delta T}{\Delta x}}$$

For the temperature distribution:

$$q_w = -k \frac{\partial T}{\partial x} = -(a+bT) \frac{\partial T}{\partial x}$$

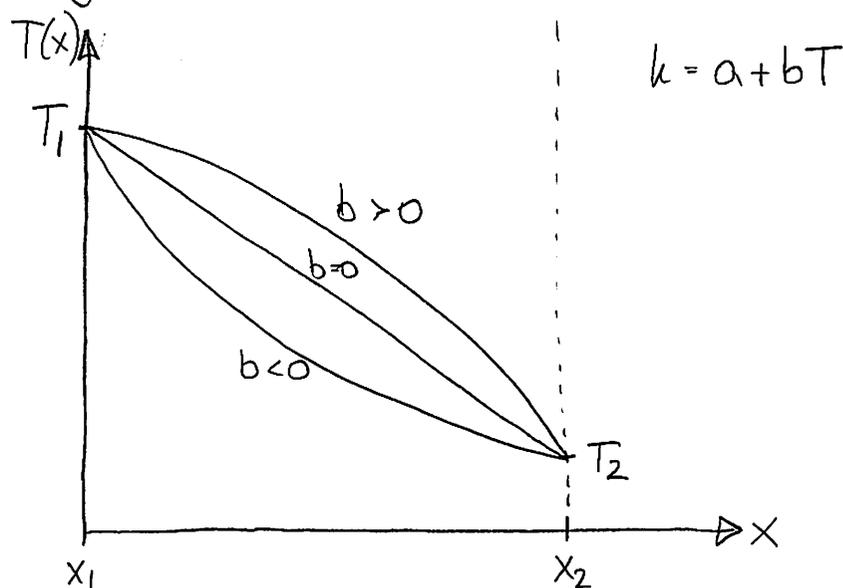
$$\int_{x_1}^x q_w dx = - \int_{T_1}^T (a+bT) dT \quad (\text{Note bounds of integration are different})$$

$$q_w (x-x_1) = -a(T(x)-T_1) - \frac{b(T(x)^2 - T_1^2)}{2} \Rightarrow \text{Quadratic equat.}$$

Solving our quadratic equation above:

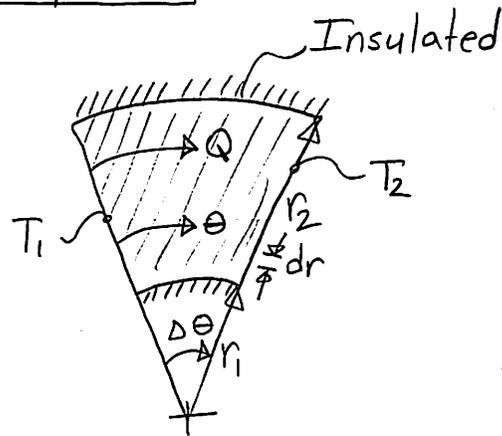
$$T(x) = -\frac{a}{b} \pm \left\{ \left(\frac{a}{b}\right)^2 - \frac{2}{b} q_w (x-x_1) - T_1^2 - 2\frac{a}{b} T_1 \right\}^{1/2}$$

Plotting our result:



Very usefull solution since many materials have a thermal conductivity temperature dependence that can be well approximated with a linear temperature dependent profile.

Example #4



$$T = T(\theta), \quad l = \text{thickness into page}$$

Note, more complex since heat flux is also a function of r since heat flow through the region closer to the center happens easier than the outside (lower length and thermal resistance)

Using our general formulation: $u_1 = \theta, \quad ds_1 = r d\theta$ } For radial coordinates
 $d(A(u_1)) = l dr$

$$dQ = -k d(A(u_1)) \frac{\partial T}{\partial s_1} = -k d(A(u_1)) \frac{1}{r} \cdot \frac{\partial T}{\partial u_1}$$

$$= -k l dr \frac{\partial T}{r \partial \theta} = -k l \cdot \frac{1}{r} dr \frac{\partial T}{\partial \theta}$$

$$\int dQ = \int -k l \frac{\partial T}{\partial \theta} \cdot \frac{dr}{r}$$

$$Q = -k l \frac{\partial T}{\partial \theta} \int_{r_1}^{r_2} \frac{dr}{r} = -k l \frac{\partial T}{\partial \theta} \cdot \ln(r_2/r_1)$$

$$Q = -k l \ln(r_2/r_1) \frac{\partial T}{\partial \theta} = \frac{k l \ln(r_2/r_1)}{\underbrace{\Delta \theta}_{R^{-1}}} \cdot \Delta T$$

$$\boxed{R = \frac{\Delta \theta}{k l \ln(r_2/r_1)}} \Rightarrow \text{Thermal resistance of the arc}$$

Let's check the bounds of our solution to see if it makes sense: What if $r_1 \approx r_2 = r_2 - \delta$

$$\ln\left(\frac{r_2}{r_1}\right) = \ln\left(\frac{r_2}{r_2 - \delta}\right) = \ln\left(\frac{1}{1 - \frac{\delta}{r_2}}\right) = \ln\left[\frac{1 + \frac{\delta}{r_2}}{(1 - \frac{\delta}{r_2})(1 + \frac{\delta}{r_2})}\right]$$

$$= \ln \left[\frac{1 + \frac{\delta}{r_2}}{1 + \frac{\delta}{r_2} - \frac{\delta}{r_2} - \frac{\delta^2}{r_2^2}} \right], \text{ since } \frac{\delta}{r_2} \ll 1$$

$$= \ln \left(1 + \frac{\delta}{r_2} \right) \quad \frac{\delta^2}{r_2^2} \ll 1, \approx 0$$

Using series expansion:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots - \frac{(-x)^{N+1}}{N+1} + \dots$$

$$x = \frac{\delta}{r_2}$$

$$\ln \left(1 + \frac{\delta}{r_2} \right) = \frac{\delta}{r_2} - \frac{\delta^2}{2r_2^2} + \frac{\delta^3}{3r_2^3} + \dots$$

$$\ln \left(1 + \frac{\delta}{r_2} \right) \approx \frac{\delta}{r_2}$$

So plugging back into our resistance calculation:

$$R = \frac{\Delta\theta}{k l \ln(r_2/r_1)} = \frac{\Delta\theta r_2}{k l \delta} \Rightarrow \underbrace{\delta}_{L} = A \text{ (Area)}$$

$$\underbrace{\Delta\theta r_2}_{L} = L \text{ (Length)}$$

$$\frac{L}{kA} \Rightarrow \text{Works like a charm!}$$

General Orthogonal Coordinate System (Pure Conduction)

Defining u_1, u_2, u_3 (3 coordinate dimensions)

$$\left. \begin{aligned} ds_1 &= h_1 du_1 \\ ds_2 &= h_2 du_2 \\ ds_3 &= h_3 du_3 \end{aligned} \right\} \text{For example} \Rightarrow \text{Cartesian} \quad \begin{aligned} h_1 &= 1, \quad du_1 = dx \\ h_2 &= 1, \quad du_2 = dy \\ h_3 &= 1, \quad du_3 = dz \end{aligned}$$

We can generalize the following:

$$\boxed{q'' = -k \frac{1}{h_i} \frac{\partial T}{\partial u_i}}$$

$$Q = \iint q'' ds_2 ds_3 = \iint q'' h_2 h_3 du_2 du_3$$

$$Q = -k \frac{2T}{2u_1} \iint \frac{h_2 h_3}{h_1} du_2 du_3$$

$$R = \frac{1}{k} \int_{(u_1)_a}^{(u_1)_b} \frac{du_1}{\iint \frac{h_2 h_3}{h_1} du_2 du_3}$$

⇒ The most usefull & general form you'll learn. Very powerfull.

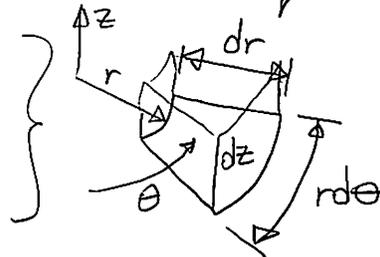
Let's check with two quick examples ⇒ radial & spherical

① Radial:

$$u_1 = r$$

$$u_2 = \theta$$

$$u_3 = z$$



$$ds_1 = (1) dr \Rightarrow h_1 = 1$$

$$ds_2 = r d\theta \Rightarrow h_2 = r$$

$$ds_3 = (1) dz \Rightarrow h_3 = 1$$

Back substituting into our formulation

$$Q = -k \frac{2T}{2r} \int_0^L \int_0^{2\pi} \frac{r(z)}{r} d\theta dz = -k \frac{2T}{2r} \int_0^L dz \int_0^{2\pi} r d\theta$$

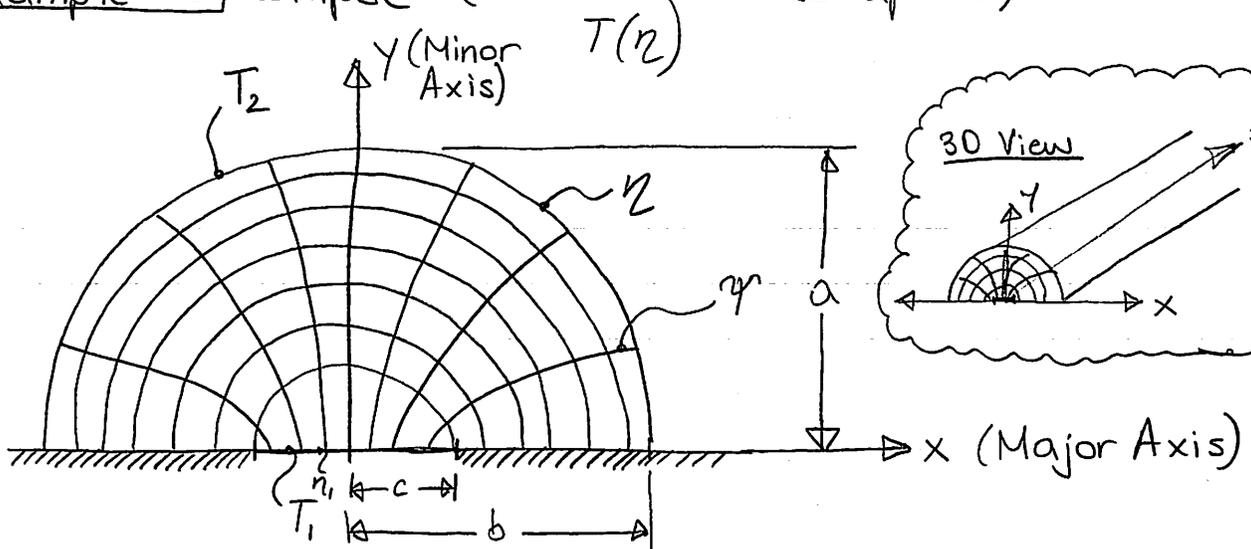
$$= -k \underbrace{(2\pi r)}_A L \frac{2T}{2r} = -k A \frac{2T}{2r}$$

$$R = \frac{1}{k} \int_{r_1}^{r_2} \frac{dr}{\int_0^L \int_0^{2\pi} \frac{r(z)}{r} d\theta dz}$$

$$= \frac{1}{k} \int_{r_1}^{r_2} \frac{dr}{r \int_0^L dz \int_0^{2\pi} d\theta} = \frac{1}{2\pi k L} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\ln(r_2/r_1)}{2\pi k L} = R_{cyl}$$

We see that the formalism works. Try it for spherical.

Example #5 | Ellipse (Foci located $2c$ apart)



$$\begin{aligned}
 U_1 &= \eta & h_1 &= \text{unknown yet (we will solve for it)} \\
 U_2 &= \psi & h_2 &= \text{unknown yet} \\
 U_3 &= z, & h_3 &= 1
 \end{aligned}$$

First we need to solve for h_1 & h_2 . We can use some nice rules & identities for ellipses:

Rule #1:

Coordinate Transformation:

$$b^2 - a^2 = c^2 \Rightarrow \begin{aligned} x &= c \cosh \eta \cos \psi & b &= c \cosh \eta_1 \\ y &= c \cosh \eta \sin \psi & a &= c \sinh \eta_1 \end{aligned}$$

$$(b^2 - a^2) = c^2 (\cosh^2 \eta_1 - \sinh^2 \eta_1)$$

$$1 \Rightarrow \text{so } b^2 - a^2 = c^2 \text{ (we are OK!)}$$

Some identities we'll need are

$$\cosh \eta = \frac{e^\eta + e^{-\eta}}{2}$$

$$\sinh \eta = \frac{e^\eta - e^{-\eta}}{2}$$

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow \left. \begin{aligned} \left| \frac{d\vec{r}}{d\eta} \right| &= h_1 = c (\sinh^2 \eta \cos^2 \psi + \cosh^2 \eta \sin^2 \psi)^{1/2} \\ \left| \frac{d\vec{r}}{d\psi} \right| &= h_2 = c (\cosh^2 \eta \sin^2 \psi + \sinh^2 \eta \cos^2 \psi)^{1/2} \end{aligned} \right\} h_1 = h_2$$

(Coordinate Transformation)

This makes our lives a lot easier:

$$R = \frac{1}{k} \int_0^{n_1} \frac{dn}{\int_0^\pi \int_0^L d\varphi dz} = \frac{n_1}{\pi k L}$$

We know that: $\frac{a+b}{c} = \cosh n_1 + \sinh n_1$
 $= \frac{e^{n_1} + e^{-n_1}}{2} + \frac{e^{n_1} - e^{-n_1}}{2} = e^{n_1}$

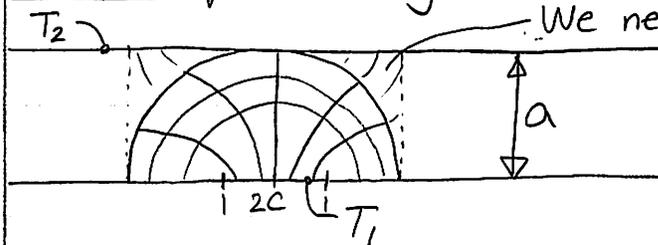
$$n_1 = \ln\left(\frac{a+b}{c}\right)$$

So our solution becomes:

$Q = \frac{\Delta T \pi k L}{\ln\left(\frac{a+b}{c}\right)}$	$R = \frac{\ln\left(\frac{a+b}{c}\right)}{\pi k L}$
--	---

Note, this solution is extremely usefull since it can be used to solve the following cases:

Case 1: Spot Welding

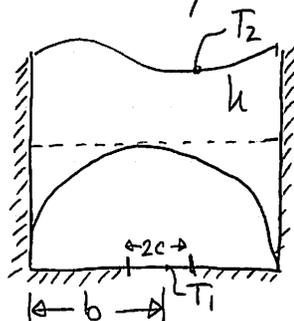


We neglect this resistance. But because spreading resistance here is high, safe approximation.

$$b^2 = \sqrt{a^2 + c^2}$$

Using the above resistance puts an upper bound on our heat transfer Q by assuming cylindrical spreading.

Case 2:

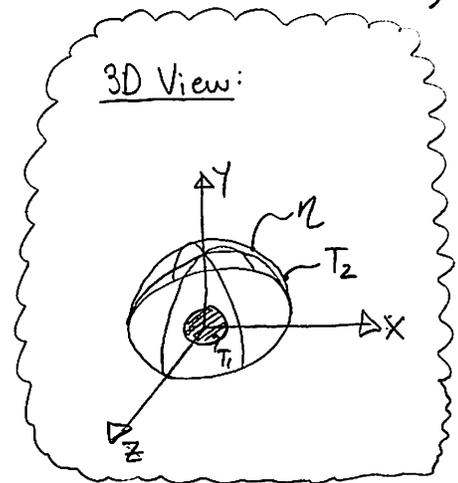
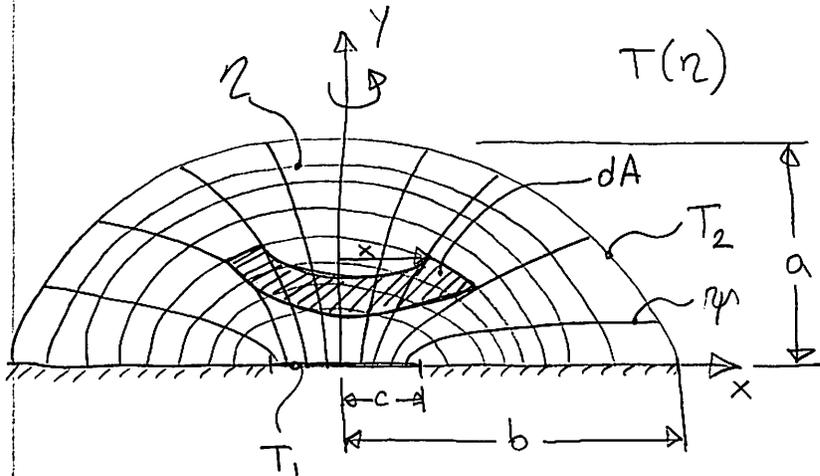


$$\Rightarrow R_1 = \frac{L}{kA}$$

$$\Rightarrow R_c = \frac{\ln\left(\frac{a+b}{c}\right)}{\pi k L} \Rightarrow a = \sqrt{b^2 - c^2}$$

Broken down into 2 resistances.
 Good approximation.

Revolved Ellipsoid b (Major axis), a (Minor axis), $2c$ (Distance between foci)



$$q'' = -k \frac{1}{h_1} \cdot \frac{\partial T}{\partial \eta}$$

$$dA = 2\pi x \cdot \underbrace{h_2 \cdot d\psi}_{ds_2}$$

$$Q = \iint q'' dA = 2\pi k \frac{\partial T}{\partial \eta} \iint \frac{x h_2}{h_1} d\psi$$

$$Q = -2\pi k c \cosh \eta \frac{\partial T}{\partial \eta} \underbrace{\int_0^{\pi/2} \cos \psi d\psi}_1$$

$$= -2\pi k c \cosh \eta \frac{\partial T}{\partial \eta}$$

$$\Delta T = Q \underbrace{\frac{1}{2\pi k c} \int_0^{\eta_1} \frac{d\eta}{\cosh \eta}}_R$$

$$R = \frac{1}{2\pi k c} \int_0^{\eta_1} \frac{2 d\eta}{e^\eta + e^{-\eta}} = \frac{1}{\pi k c} \int_0^{\eta_1} \frac{e^\eta d\eta}{e^{2\eta} + 1}, \text{ let } \lambda = e^\eta$$

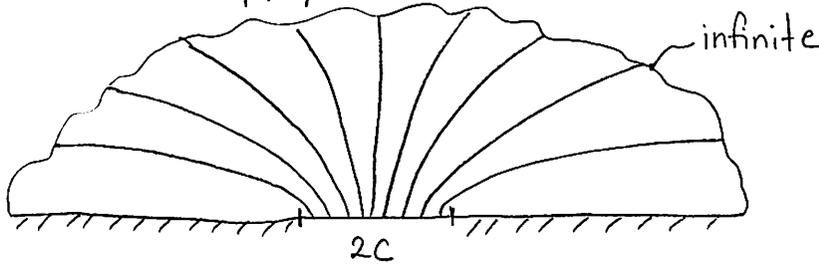
$$R = \frac{1}{\pi k c} \int_0^{e^{\eta_1}} \frac{d\lambda}{\lambda^2 + 1}$$

$$R = \frac{1}{\pi k c} \left[\tan^{-1}(e^{\eta_1}) - \frac{\pi}{4} \right] \Rightarrow \text{we know that } \eta_1 = \ln\left(\frac{a+b}{c}\right)$$

$$\boxed{R = \frac{1}{\pi k c} \left[\tan^{-1}\left(\frac{a+b}{c}\right) - \frac{\pi}{4} \right]}$$

$$e^{\eta_1} = \frac{a+b}{c}$$

So if we apply our solution to a semi-infinite body:

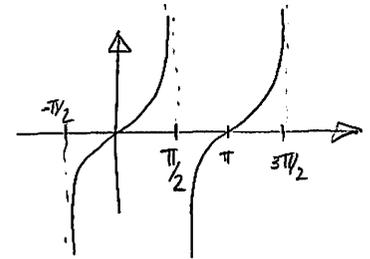


$$a \rightarrow \infty$$

$$b \rightarrow \infty$$

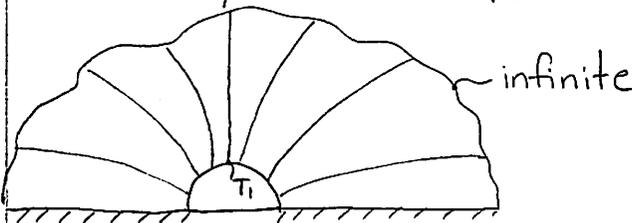
$$R = \frac{1}{\pi k c} \left[\tan^{-1} \left(\frac{a+b}{c} \right) - \frac{\pi}{4} \right]$$

$$\tan^{-1}(\infty) = \frac{\pi}{2}$$



$$R = \frac{1}{\pi k c} \left[\frac{\pi}{2} - \frac{\pi}{4} \right] = \frac{1}{4 k c}$$

When NASA was working with this problem in the 50's and 60's, they used an approximation using spherical resistance.



$$R_{\text{approx}} \approx \frac{1}{2\pi k} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

note, divide by 2
not 4 since half-sphere.

$$r_2 \rightarrow \infty$$

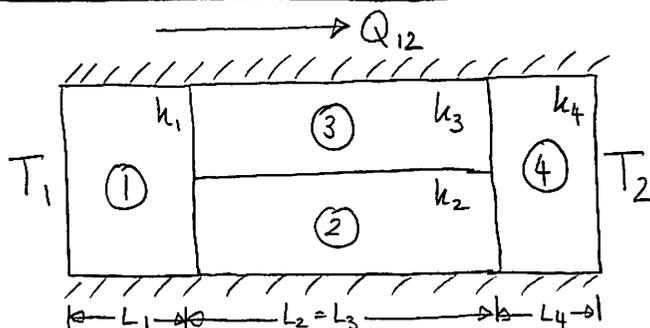
$$r_1 \rightarrow c$$

$$R_{\text{approx}} = \frac{1}{2\pi k c} \quad (\approx 50\% \text{ off})$$

$$R_{\text{REAL}} = \frac{1}{4 k c} < R_{\text{approx}}$$

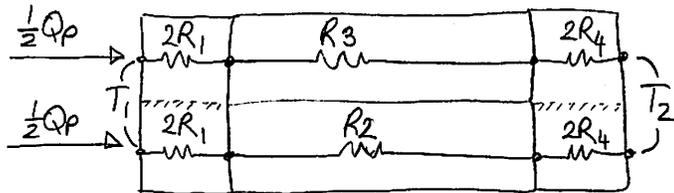
END OF LECTURE 5

Composite Wall Problems



There are two ways to analyze this problem (both giving different answers)

① Parallel Approach



$$R_L = \frac{L_1}{k_1 A_2} + \frac{L_2}{k_2 A_2} + \frac{L_4}{k_4 A_2}$$

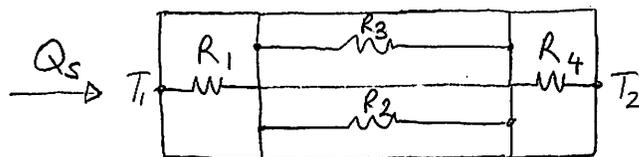
$$R_u = \frac{L_1}{k_1 A_3} + \frac{L_2}{k_3 A_3} + \frac{L_4}{k_4 A_3}$$

$$\left. \begin{matrix} R_L \\ R_u \end{matrix} \right\} \frac{1}{R} = \frac{1}{R_L} + \frac{1}{R_u}$$

$$\frac{1}{R_p} = \left(\frac{A_2}{L_1/k_1 + L_2/k_2 + L_4/k_4} + \frac{A_3}{L_1/k_1 + L_3/k_3 + L_4/k_4} \right)$$

$$Q_p = \frac{\Delta T}{R_p} = \left(\frac{A_2}{L_1/k_1 + L_2/k_2 + L_4/k_4} + \frac{A_3}{L_1/k_1 + L_3/k_3 + L_4/k_4} \right) \Delta T$$

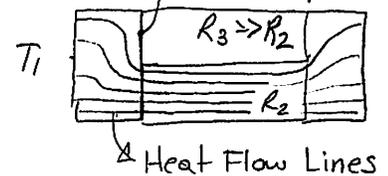
② Series Approach



$$R_s = \frac{L_1}{k_1 A_1} + \left(\frac{k_2 A_2}{L_2} + \frac{k_3 A_3}{L_2} \right)^{-1} + \frac{L_4}{k_4 A_1}$$

$$Q_s = \frac{\Delta T}{R_s} = \left(\frac{L_1}{k_1 A_1} + \frac{L_2}{k_2 A_2 + k_3 A_3} + \frac{L_4}{k_4 A_1} \right)^{-1} \Delta T$$

Aside: Think of this way:
Real Life: T will vary here



It is always true that:

$$Q_p \leq Q_{REAL} \leq Q_s$$

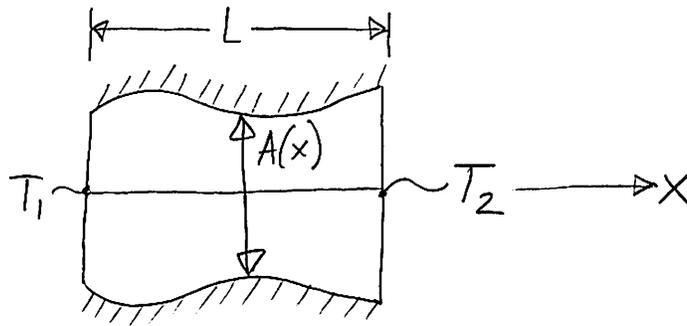
↳ Lower Bound estimate

↳ Upper Bound estimate.

In general:

- 1) Adding adiabatic surfaces: $R \uparrow$, $Q \downarrow$
- 2) Allowing infinite conductance: $R \downarrow$, $Q \uparrow$

Example #6



$$dR = \frac{dx}{kA(x)} \rightarrow R \geq \frac{1}{k} \int_0^L \frac{dx}{A(x)}$$

$$Q \leq \frac{\Delta T}{\frac{1}{k} \int_0^L \frac{dx}{A(x)}}$$

Assuming it's a 1D problem. In reality, 2D and added resistance will come into play due to longer path.

The Nieve approach: $\bar{A} = \frac{1}{L} \int_0^L A(x) dx \rightarrow R \approx \frac{L^2}{k \int_0^L A(x) dx}$

$$Q \approx \frac{k \Delta T}{L^2} \int_0^L A(x) dx$$

But, it's important to note that:

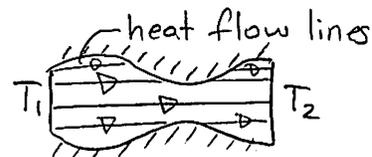
$$\underbrace{\frac{1}{L} \int_0^L A(x) dx}_{\text{Arithmetic Mean}} \geq \underbrace{\left(\frac{1}{L} \int_0^L \frac{dx}{A(x)} \right)^{-1}}_{\text{Harmonic Mean}}$$

$$R_{\text{real}} \geq \frac{1}{k} \int_0^L \frac{dx}{A(x)} \geq \frac{L}{k} \left(\frac{1}{L} \int_0^L A(x) dx \right)^{-1}$$

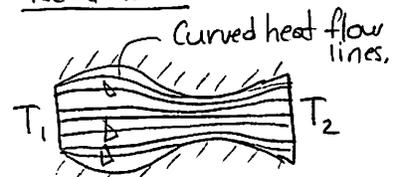
$$Q_{\text{real}} \leq \frac{\Delta T}{\frac{1}{k} \int_0^L \frac{dx}{A(x)}} \leq \frac{k \Delta T}{L^2} \int_0^L A(x) dx$$

Aside:

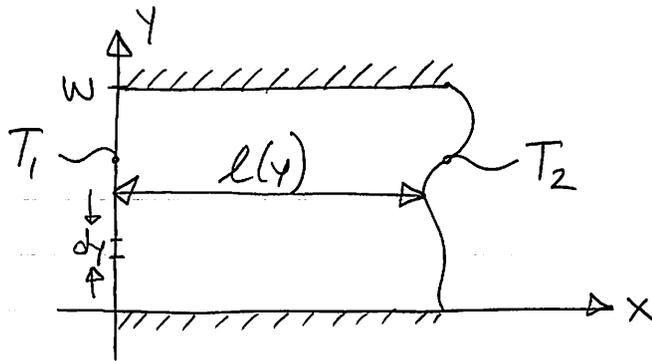
1-D solution



Real Life:



Example #7



Q' = heat transfer rate per unit length into board or paper.

$$dQ' \approx k dy \frac{\Delta T}{l(y)}$$

$$Q' \approx k \Delta T \int_0^w \frac{dy}{l(y)}$$

$$R' \approx \frac{1}{k} \left(\int_0^w \frac{dy}{l(y)} \right)^{-1}$$

The Nieve' Approach:

$$\bar{l} \approx \frac{1}{w} \int_0^w l(y) dy$$

$$\bar{R} \approx \frac{1}{kw} \int_0^w \frac{dy}{l(y)}$$

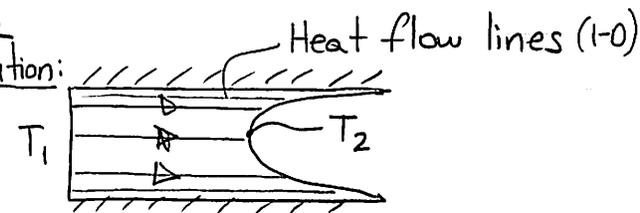
$$Q' \approx kw \Delta T \left(\int_0^w \frac{dy}{l(y)} \right)^{-1}$$

Real heat transfer.

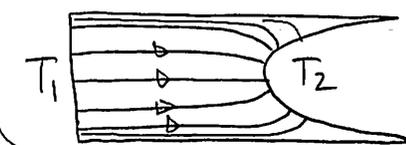
$Q' \approx (\dots)$ since in reality, heat can spread laterally and take the path of least resistance. This is not considered here.

i.e.

Solution:



Real Life:



From this we can say that:

$$R'_{\text{real}} \leq \frac{1}{k} \left(\int_0^w \frac{dy}{l(y)} \right)^{-1} \leq \frac{1}{kw} \int_0^w \frac{dy}{l(y)}$$

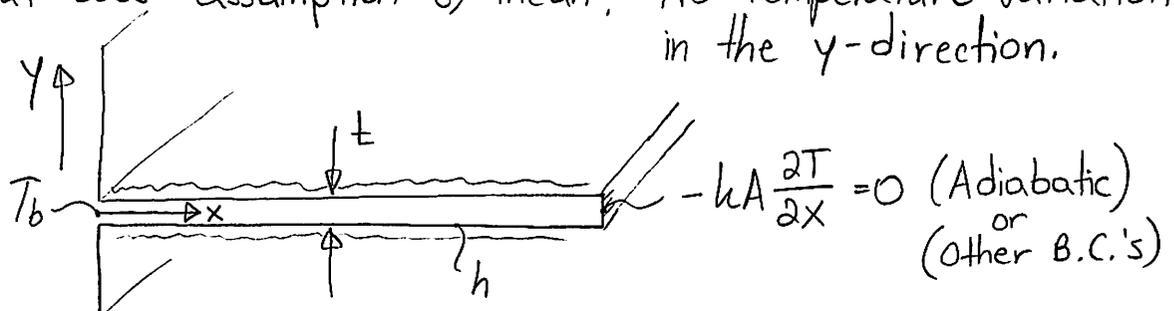
$$Q'_{\text{real}} \geq kw \Delta T \left(\int_0^w \frac{dy}{l(y)} \right)^{-1} \geq kw \Delta T \int_0^w \frac{dy}{l(y)}$$

\Rightarrow We've already learned that Arithmetic mean \geq Harmonic mean.

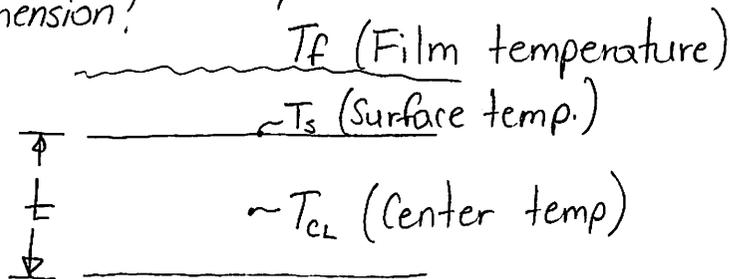
Quasi-1D Conduction: Fins

- Assumptions; 1) $\rho, A = \text{constant}$ (Perimeter and Area of Fin)
 2) No heat generation
 3) The heat transfer is 1-D

What does assumption 3) mean? No temperature variation in the y -direction.



So how do we quantify the temperature variation in the y -dimension?



Doing a heat balance across the fin, we obtain:

$$\underbrace{-k \frac{\partial T}{\partial y} (L dx)}_{\text{Conduction}} = \underbrace{h (L dx) (T_s - T_f)}_{\text{Convection}}$$

$$\frac{\partial T}{\partial y} = \frac{T_{CL} - T_s}{t/2} = h (T_s - T_f)$$

$$\frac{T_{CL} - T_s}{T_s - T_f} = -\frac{1}{2} \frac{ht}{k} \approx 0.05$$

$$\frac{1}{2} \left(\frac{ht}{k} \right) \leq \frac{1}{20} \Rightarrow \boxed{\frac{ht}{k} \leq \frac{1}{10} = Bi_t} \Rightarrow \text{for fin heat transfer in the thickness direct. Biot Number. } (48)$$

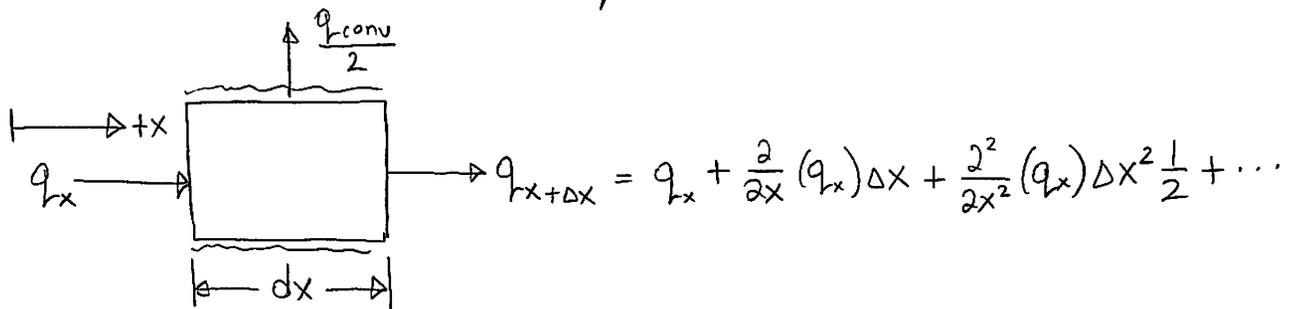
- Interpretation:
- 1) Temp. changes across the fin thickness are small compared to those external to the fin. (isothermal in y , at a particular x)
 - 2) Internal resistance to conduction across the thickness of the fin is small compared to resistance due to external convection. i.e. $R_{cond} \ll R_{conv}$

We can double check this right away:

$$\frac{R_{cond}}{R_{conv}} = \frac{\frac{t}{kA}}{\frac{1}{hA}} = \frac{ht}{k} \ll 1$$

$\underbrace{\hspace{1.5cm}}_{Bi_t} \ll 1$

Now we can model the temperature distribution:



$$q_{conv} = hP dx (T - T_f) ; \quad P = \text{perimeter or surface area/unit length}$$

We also know that: $q_x = -kA \frac{\partial T}{\partial x}$

Energy balance:

$$q_x - \left\{ q_x + \frac{\partial q_x}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 + \dots \right\} = hP \Delta x (T - T_f)$$

$$-\frac{\partial q_x}{\partial x} \cdot dx = hP dx (T - T_f)$$

$$\frac{d}{dx} \left(kA \frac{\partial T}{\partial x} \right) - hP (T - T_f) = 0$$

Defining $\theta = T - T_f$, $d\theta = dT$

$$kA \frac{\partial^2 \theta}{\partial x^2} - hP\theta = 0 \Rightarrow \frac{\partial^2 \theta}{\partial x^2} - \frac{hP}{kA} \theta = 0$$

$$\frac{\partial^2 \theta}{\partial x^2} - m^2 \theta = 0$$

$$\text{Let } \frac{hP}{kA} = m^2$$

Works if:

$$\Rightarrow a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

Linear, second order ODE

Our characteristic equation is: (ODE concept) Euler did it!

$$\lambda^2 - m^2 = 0$$

$$\lambda^2 = m^2$$

$$\lambda = \pm m$$

So our general solution is:

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Aside: Proof of solution:

$$\theta = C_1 e^{mx} + C_2 e^{-mx}$$

$$\theta' = mC_1 e^{mx} - mC_2 e^{-mx}$$

$$\theta'' = m^2 C_1 e^{mx} + m^2 C_2 e^{-mx}$$

$$\theta'' - m^2 \theta = m^2 (C_1 e^{mx} + C_2 e^{-mx}) - m^2 (C_1 e^{mx} + C_2 e^{-mx}) = 0 \quad \text{QED}$$

We have four directions to go to:

① Infinite Fin case ($x \rightarrow \infty$)

Our B.C.'s become:

$$\theta(x=0) = T_b - T_f = \theta_b$$

$$\theta(x=L \rightarrow \infty) = T_f - T_f = 0$$

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

$$\theta(\infty) = 0 = C_1 e^{m(\infty)} + C_2 e^{-m(\infty)}$$

$C_1 = 0$ for solution to be valid

$$\theta(x=0) = \theta_b$$

$$\theta_b = T_b - T_f = C_2 e^{-m(0)} \Rightarrow C_2 = \theta_b$$

$$\theta(x) = \theta_b e^{-mx}$$

$$\frac{T - T_f}{T_b - T_f} = e^{-\frac{hP}{kA} x}$$

What about heat transfer from the base to the fin?

$$Q_{\text{net}} = Q(0) = -kA \frac{\partial \theta}{\partial x} \Big|_{x=0} = +kA \theta_b m e^{-m(0)}$$

$$Q_{net} = +kA\theta_b \sqrt{\frac{hP'}{kA}} = \sqrt{kAhP'}\theta_b$$

Note: $\frac{Q_{net}}{Q_0} = \frac{\sqrt{kAhP'}\theta_b}{hA\theta_b} = \sqrt{\frac{kP'}{hA}} = \sqrt{\frac{k}{ht}} = Bi_t^{-1/2} \gg 1$

Also note here that: $\frac{P}{A} = \frac{1}{t}$. In general $L_c = \frac{V_{body}}{A_{surface}} = \frac{AK}{PK} = \frac{A}{P} = t$

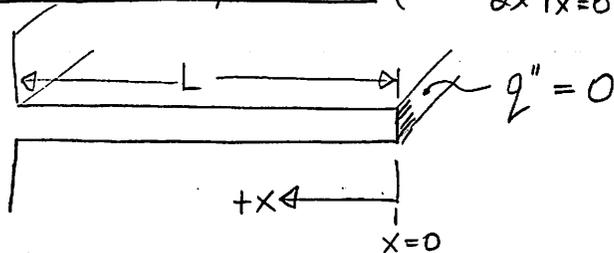
Sanity check: $Q_{net} = \int_0^{\infty} hP\theta dx = \int_0^{\infty} hP\theta_b e^{-mx} dx$

\uparrow Characteristic Length. $= t$

$$= hP\theta_b \frac{e^{-mx}}{m} \Big|_0^{\infty} = \frac{hP\theta_b}{m} = \sqrt{kAhP'}\theta_b$$

\Downarrow Works!

② Insulated Tip case ($-k \frac{\partial T}{\partial x} \Big|_{x=0} = 0$)



B.C.'s:

$$-k \frac{\partial \theta}{\partial x} \Big|_{x=0} = 0$$

$$\theta(x=L) = \theta_b$$

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

We can rewrite this as:

$$\theta(x) = C_1 \cosh(mx) + C_2 \sinh(mx)$$

Substitute in our B.C.'s

$$\frac{\partial \theta}{\partial x} = C_1 m \sinh(mx) + C_2 m \cosh(mx)$$

$$\frac{\partial \theta}{\partial x} \Big|_{x=0} = C_1 m \sinh(0) + C_2 m \cosh(0) \Rightarrow C_2 m = 0 \Rightarrow C_2 = 0$$

$$\theta(x) = C_1 \cosh(mx) \Rightarrow \text{Apply our second B.C.}$$

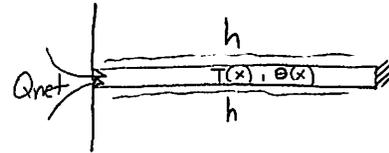
$$\theta(x=L) = \theta_b = C_1 \cosh(mL)$$

$$C_1 = \frac{\theta_b}{\cosh(mL)}$$

Back substituting

$$\Theta(x) = \frac{\Theta_b \cosh(mx)}{\cosh(mL)}$$

$$\boxed{\frac{T - T_f}{T_b - T_f} = \frac{\cosh(mx)}{\cosh(mL)}}$$



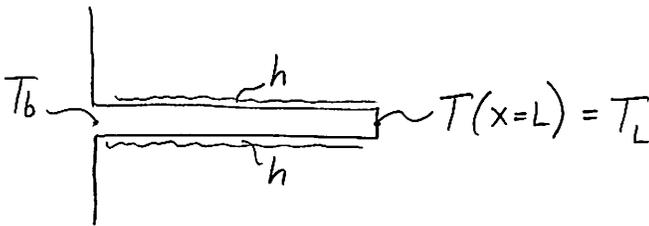
How about heat transfer?

$$Q_{net} = -kA \left. \frac{\partial \Theta}{\partial x} \right|_{x=L} = +kA \Theta_b m \frac{\sinh(mL)}{\cosh(mL)} = kA \Theta_b \sqrt{\frac{hP}{kA}} \frac{\sinh(mL)}{\cosh(mL)}$$

$$\boxed{Q_{net} = \sqrt{kAhP} (T_b - T_f) \frac{\sinh(mL)}{\cosh(mL)}} \Rightarrow \text{Heat transfer from the insulated tip fin.}$$

$\underbrace{\hspace{10em}}_{\tanh(mL)}$

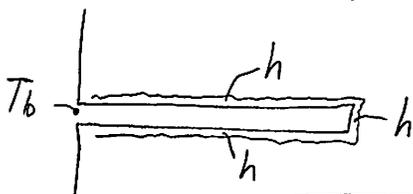
③ Prescribed Tip Temperature ($T(x=L) = T_L$)



I won't go through the math, (you can do this) but the solution is:

$$\boxed{\frac{T - T_f}{T_b - T_f} = \frac{T_L - T_f}{T_b - T_f} \frac{\sinh(mx) + \sinh[m(L-x)]}{\sinh(mL)}}$$

④ Convection on Tip ($h(T(x=L) - T_f) = -k \left. \frac{\partial T}{\partial x} \right|_{x=L}$)



$$\boxed{\frac{T - T_f}{T_b - T_f} = \frac{\cosh(m(L-x)) + \frac{h}{mk} \sinh(m(L-x))}{\cosh(mL) + \frac{h}{mk} \sinh(mL)}}$$

Fin Efficiency (Insulated tip)

$$\text{Fin Efficiency} = \eta_{\text{fin}} = \frac{\text{Actual Heat Transfer}}{\text{Heat Transfer if } \theta = \theta_b \text{ everywhere}}$$

$$= \frac{q_{\text{actual}}}{q_{\text{ideal}}}$$

We know: $q_{\text{ideal}} = h(PL)(T_b - T_f)$

$$q_{\text{actual}} = \int_0^L hP(T - T_f) dx = \underbrace{\sqrt{kAPh} \theta_b \tanh(mL)}_{\text{Insulated Tip } Q_{\text{net}}}$$

$$\eta_{\text{fin}} = \frac{\sqrt{kAPh} \theta_b \tanh(mL)}{hPL \theta_b} = \underbrace{\sqrt{\frac{kA}{hP}}}_{\frac{L}{m}} \cdot \frac{\tanh(mL)}{L}$$

$$\boxed{\eta_{\text{fin}} = \frac{\tanh(mL)}{mL}} = \frac{q_{\text{actual}}}{q_{\text{ideal}}}$$

For other configurations, see Mills pg. 104-105

Fin Resistance (R_{fin}) \Rightarrow For insulated tip case

$$R_{\text{fin}} = \frac{\theta_b}{q_{\text{actual}}} = \frac{T_b - T_f}{q_{\text{actual}}} = \frac{\theta_b}{\sqrt{kAPh} \tanh(mL) \theta_b} = \frac{1}{h A_{\text{fin}} \eta_{\text{fin}}}$$

$$\boxed{R_{\text{fin}} = \frac{1}{h A_{\text{fin}} \eta_{\text{fin}}}}$$

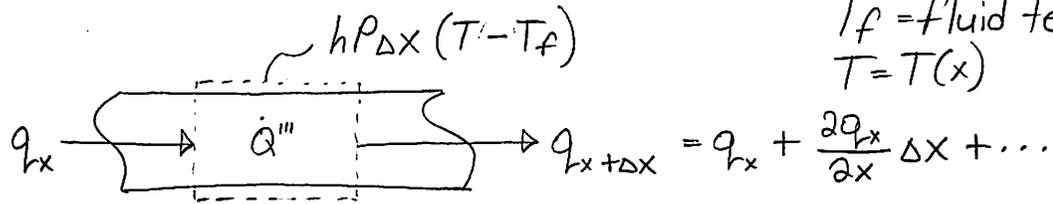
A_{fin} = outside area of fin (PL)
 η_{fin} = fin efficiency
 h = heat transfer coeff.

Fin Effectiveness (ϵ_{fin})

$$\epsilon_f = \frac{\text{actual heat transfer}}{\text{heat transfer if no fin}}$$

$$= \frac{q_{\text{ideal}} \cdot \eta_f}{q_{\text{ideal}} \left(\frac{A}{PL}\right)} = \eta_f \left(\frac{PL}{A}\right) \Rightarrow \boxed{\epsilon_f = \eta_f \frac{PL}{A}}$$

P = perimeter
 L = fin length
 A = c.s.a. of fin.

Fins with Heat Generation (\dot{Q}''')

Writing out our energy balance

$$Aq_x - A\left(q_x + \frac{\partial q_x}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 + \text{H.O.T.}\right) + \dot{Q}''' A \Delta x - hP \Delta x (T - T_f) = 0$$

We know from Fourier's Law: $q_x = -k \frac{\partial T}{\partial x}$

$$+kA \left(\frac{\partial^2 T}{\partial x^2} + \frac{1}{2} \frac{\partial^3 T}{\partial x^3} \Delta x + \text{H.O.T.} \right) + \dot{Q}''' A - hP (T - T_f) = 0$$

Letting $\Delta x \rightarrow 0$, and $\theta = T - T_f$
 $d\theta = dT$

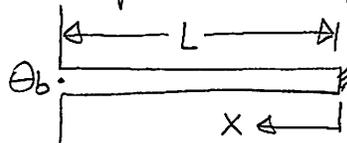
We obtain:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\dot{Q}'''}{k} - m^2 \theta = 0, \quad m^2 = \frac{hP}{kA}$$

Assuming the insulated tip condition, our B.C.'s become

$$\frac{\partial \theta}{\partial x} \Big|_{x=0} = 0$$

$$\theta(x=L) = \theta_b$$



To solve, we can do the particular and homogeneous solution:
Particular:

$$\frac{\partial^2 \theta_p}{\partial x^2} + \frac{\dot{Q}'''}{k} - m^2 \theta_p = 0$$

Set this to zero (quick solution for the particular)

$$\theta_p = \frac{\dot{Q}'''}{km^2} = \frac{\dot{Q}''' kA}{khP} = \frac{\dot{Q}''' A}{hP} \Rightarrow \boxed{\theta_p = \frac{\dot{Q}''' A}{hP}}$$

Now we move on to the homogeneous part.

Homogeneous: $\frac{\partial^2 \Theta_h}{\partial x^2} - m^2 \Theta_h = 0$

We've already solved this problem before. Nothing but

$$\Theta_h = C_1 \cosh(mx) + C_2 \sinh(mx)$$

Now we combine $\Theta_p + \Theta_h = \Theta(x)$ and apply our B.C.'s

$$\Theta(x) = \Theta_p(x) + \Theta_h(x) = \frac{\dot{Q}''' A}{hP} + C_1 \cosh(mx) + C_2 \sinh(mx)$$

$$\left. \frac{\partial \Theta}{\partial x} \right|_{x=0} = 0 = -C_1 m \sinh(m(0)) + C_2 m \cosh(m(0)) \Rightarrow C_2 = 0$$

$$\Theta(x=L) = \Theta_b = \frac{\dot{Q}''' A}{hP} + C_1 \cosh(mL)$$

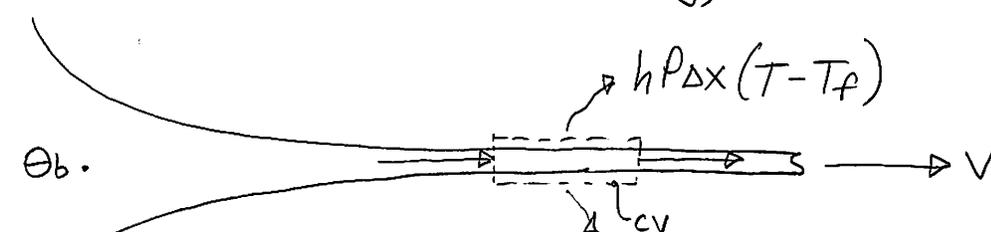
$$C_1 = \frac{\Theta_b - \frac{\dot{Q}''' A}{hP}}{\cosh(mL)}$$

$$\therefore \Theta(x) = \frac{\dot{Q}''' A}{hP} + \frac{\Theta_b - \frac{\dot{Q}''' A}{hP}}{\cosh(mL)} \cdot \cosh(mx)$$

$$\Theta(x) = \Theta_b \frac{\cosh(mx)}{\cosh(mL)} + \frac{\dot{Q}''' A}{hP} \left(1 - \frac{\cosh(mx)}{\cosh(mL)} \right)$$

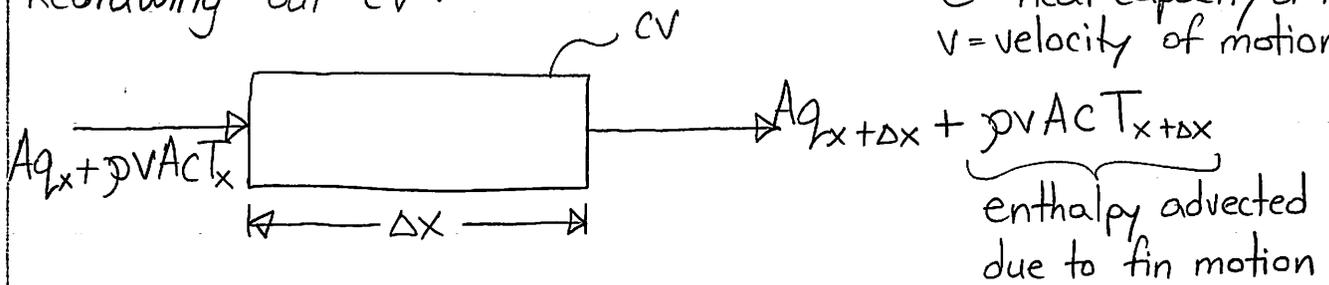
↳ Temperature profile in the fin with heat generat.

Moving Fins (For example, metal extrusion and solidification
⇒ continuous casting)



Our inlet and outlet terms become a bit more complicated.

Redrawing our CV:



Energy balance on our control volume:

$$Aq_x + \rho v A c T_x - (Aq_{x+\Delta x} + \rho v A c T_{x+\Delta x}) - h P \Delta x (T - T_f) = 0$$

Substituting in Fourier's Law and doing a Taylor series expans.

$$\begin{aligned}
 & Aq_x + \rho v A c T_x - \left(Aq_x + A \frac{\partial q_x}{\partial x} \Delta x + A \frac{1}{2} \frac{\partial^2 q_x}{\partial x^2} \Delta x^2 + \text{H.O.T.} + \rho v A c T_x \right. \\
 & \quad \left. + \rho v A c \frac{\partial T}{\partial x} \Delta x + \rho v A c \frac{\partial^2 T}{\partial x^2} \frac{\Delta x^2}{2} + \text{H.O.T.} \right) - h P \Delta x (T - T_f) = 0 \\
 & k A \frac{\partial^2 T}{\partial x^2} \Delta x + \frac{1}{2} k A \frac{\partial^3 T}{\partial x^3} \Delta x^2 + \text{H.O.T.} - \rho v A c \frac{\partial T}{\partial x} \Delta x - \rho v A c \frac{\partial^2 T}{\partial x^2} \frac{\Delta x^2}{2} \\
 & \quad + \text{H.O.T.} - h P \Delta x (T - T_f) = 0
 \end{aligned}$$

Let $\Delta x \rightarrow 0$, all H.O.T. terms drop out, we are left with:

$$k A \frac{\partial^2 T}{\partial x^2} - \rho v A c \frac{\partial T}{\partial x} - h P (T - T_f) = 0$$

$$\frac{\partial^2 T}{\partial x^2} - \frac{\rho v c}{k} \frac{\partial T}{\partial x} - \frac{h P}{k A} (T - T_f) = 0$$

m^2

$$\text{Let } \theta = T - T_f$$

$$d\theta = dT$$

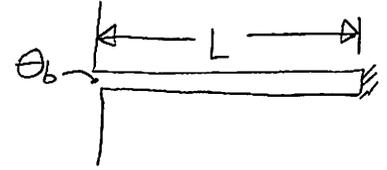
$$\frac{\partial^2 \theta}{\partial x^2} - \frac{\rho v c}{k} \frac{\partial \theta}{\partial x} - m^2 \theta = 0$$

Our B.C.'s are (for infinite fin.) $\Rightarrow \begin{cases} (1) \theta(x=0) = \theta_b \\ (2) \theta(x \rightarrow \infty) = 0 \end{cases}$

Note, if we non dimensionalize, it becomes more powerfull

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{\rho v c}{k} \frac{\partial \theta}{\partial x} - m^2 \theta = 0$$

Let $\theta^* = \frac{\theta}{\theta_b}$, $x^* = \frac{x}{L}$



$$d\theta = \theta_b d\theta^*, \quad dx = L dx^* \Rightarrow \text{Back substituting}$$

$$\frac{\partial^2 (\theta_b \theta^*)}{L^2 \partial x^{*2}} - \frac{\rho v c}{k} \frac{\partial (\theta_b \theta^*)}{L \partial x^*} - m^2 (\theta_b \theta^*) = 0$$

$$\frac{1}{L^2} \frac{\partial^2 \theta^*}{\partial x^{*2}} - \frac{\rho v c}{k L} \frac{\partial \theta^*}{\partial x^*} - m^2 \theta^* = 0$$

Multiply through by L^2 on both sides

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} - \underbrace{\frac{L \rho v c}{k}}_{\alpha} \frac{\partial \theta^*}{\partial x^*} - (mL)^2 \theta^* = 0$$

This is interesting: $\alpha = \frac{k}{\rho c}$

This becomes: $\frac{LV}{\alpha} \rightarrow$ Thermal advection
 $\alpha \rightarrow$ Thermal diffusion

Note, a side definition: Convection = motion of fluid in response to heat. (Diffusion + Advection)

Advection = motion of fluid, mass, momentum or heat due to the motion of the fluid itself.

Back to our problem:

$$Pe = \text{Peclet Number} = \frac{\text{advective transport rate}}{\text{diffusive transport rate}} = \frac{LV}{\alpha}$$

Lets first test the limits of our equation

$$\frac{LV}{\alpha} = \frac{\rho c L V}{k} = \frac{\rho c V (\Delta T)}{k \frac{\Delta T}{L}} = \frac{\rho c V \Delta T}{k \frac{\Delta T}{L}} \Rightarrow \frac{q''_{adv}}{q''_{cond}} \quad (57)$$

If advection dominates, $\frac{LV}{\alpha} \gg 1$
 Our diffusion term drops out.
 0 (negligible)

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{\rho V C}{k} \frac{\partial \theta}{\partial x} - m^2 \theta = 0$$

$$\frac{V}{\alpha} \frac{\partial \theta}{\partial x} + m^2 \theta = 0 \quad (\text{Rearranging})$$

$$\frac{1}{\theta} \frac{\partial \theta}{\partial x} = -\frac{m^2 \alpha}{V} = -\frac{h P k}{k A V \rho C} = -\frac{h P}{\rho C V A}$$

$$\int \frac{d\theta}{\theta} = -\frac{h P}{\rho C V A} \int dx$$

$$\ln(\theta) = -\frac{h P}{\rho C V A} x + C_1 \quad \begin{array}{l} \text{Constant of integration, not} \\ \text{heat capacity } C. \end{array}$$

$$e^{\ln(\theta)} = e^{-\frac{h P}{\rho C V A} x + C_1} = C_1' e^{-\frac{h P}{\rho C V A} x}$$

$$\theta(x) = C_1' e^{-\frac{h P x}{\rho C V A}}$$

We know that $\theta(x=0) = \theta_b = C_1' e^{\cancel{0}} \Rightarrow C_1' = \theta_b$

Let's check our second b.c.

$$\theta(x \rightarrow \infty) = 0 \Rightarrow \theta_b e^{-\infty} \rightarrow 0 \text{ OK!}$$

$$\therefore \boxed{\theta(x) = \theta_b e^{-\frac{h P x}{\rho C V A}}} \Rightarrow \text{When advection dominates diffusion.}$$

But what if conduction is non-negligible? (i.e. $Pe \sim 1$)

Now we have to solve the full differential equation

Start with the characteristic equation:

$$\text{Let } \theta = e^{\lambda x}$$

$$\frac{\partial \theta}{\partial x} = \lambda e^{\lambda x}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \lambda^2 e^{\lambda x}$$

} Substitute into our ODE

$$\lambda^2 e^{\lambda x} - \frac{v}{\alpha} \lambda e^{\lambda x} - m^2 e^{\lambda x} = 0$$

$$\lambda^2 - \frac{v}{\alpha} \lambda - m^2 = 0 \quad (\text{Quadratic equation})$$

$$\lambda = \frac{v}{2\alpha} - \sqrt{\frac{v^2}{4\alpha^2} + m^2} \Rightarrow \text{Only use negative root since we need } \Theta(x \rightarrow \infty) = 0$$

$$\lambda = -\frac{v}{2\alpha} \left[\sqrt{1 + \frac{4m^2\alpha^2}{v^2}} - 1 \right] \Rightarrow \Theta = \Theta_b e^{\lambda x}$$

To check our solution, if $\left(\frac{\alpha}{v}\right)^2 \ll 1$ (Advection Dominates)

Then $\frac{4m^2\alpha^2}{v^2} \ll 1$, we can do a Taylor series expansion

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{4} + \frac{3x^3}{8} - \dots$$

$$\begin{aligned} f(x) &= (1+x)^{1/2} \\ \frac{\partial f}{\partial x} &= \frac{1}{2}(1+x)^{-1/2} \\ \frac{\partial^2 f}{\partial x^2} &= -\frac{1}{4}(1+x)^{-3/2} \\ &\vdots \end{aligned}$$

H.O.T. can be neglected

So our solution becomes: if $x \ll 1$.

$$\begin{aligned} \lambda &= -\frac{v}{2\alpha} \left[\sqrt{1 + \frac{4m^2\alpha^2}{v^2}} - 1 \right] \\ &= -\frac{v}{2\alpha} \left(1 + \frac{2m^2\alpha^2}{v^2} - 1 \right) = -\frac{m^2\alpha}{v} \end{aligned}$$

$$\Theta = \Theta_b e^{\left(-\frac{hP\alpha}{kAV} x\right)}$$

$$\Theta = \Theta_b e^{-\frac{hPx}{PcAV}} \Rightarrow \text{Matches our previous solution}$$

So what if we move with our element and map out our temperature with respect to time.



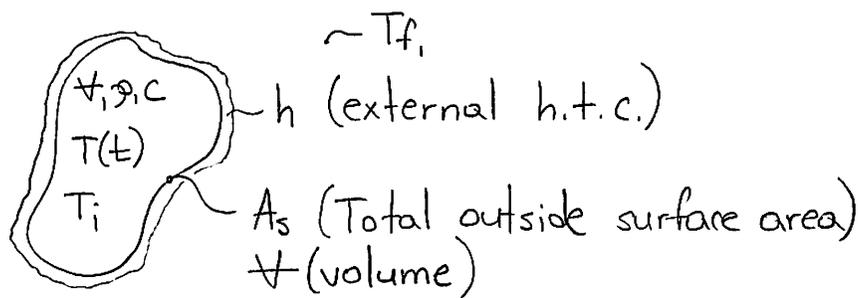
$$\theta = \theta_b e^{-\frac{hPx}{\rho cAV}} \Rightarrow \text{Note, } t = \frac{x}{v}$$

$$\theta = \theta_b e^{-\frac{hP}{\rho cA} t} \Rightarrow \text{Multiply by } \frac{\Delta X}{\Delta X} \text{ in the exponent}$$

$$\theta = \theta_b e^{-\frac{h(\rho_{\Delta x})}{\rho c(\Delta x)} t} \Rightarrow \rho_{\Delta x} = A_s \text{ (Total outside surface area)} \\ A_{\Delta x} = \forall \text{ (Total fin volume in element)}$$

$$\boxed{\theta = \theta_b e^{-\frac{hA_s}{\rho c\forall} t}} \Rightarrow \text{Looks familiar (Lumped capacitance model)}$$

Lumped Capacitance Why do we care? Forensic analysis of dead bodies.



Energy ballance: $\dot{E}_{\text{stored}} + \dot{E}_{\text{in}} - \dot{E}_{\text{out}} + \dot{E}_{\text{gen}} = 0$

$$\frac{\partial}{\partial t} (\forall \rho c T) = h A_s (T - T_f)$$

$$\forall \rho c \frac{\partial T}{\partial t} - h A_s (T - T_f) = 0$$

$$\frac{\partial T}{\partial t} - \frac{h A_s}{\forall \rho c} (T - T_f) = 0 \Rightarrow \text{Let } \theta = T - T_f$$

$$\frac{\partial \theta}{\partial t} - \frac{h A_s}{\forall \rho c} \theta = 0 \Rightarrow \text{Let } \lambda = \frac{h A_s}{\forall \rho c}$$

$$\frac{\partial \theta}{\partial t} - \lambda \theta = 0 \Rightarrow \int \frac{\partial \theta}{\theta} = \int \lambda dt$$

$$\ln \theta + \ln C_1 = \lambda t + C_2 \Rightarrow \text{Combine constants}$$

$$\ln \theta = \lambda t + C_3$$

$$e^{\ln \theta} = e^{\lambda t + C_3} \Rightarrow \theta = C_4 e^{\lambda t}$$

Our B.C. is $\theta(t=0) = \theta_i = T_i - T_f$

$$\theta(t=0) = C_4 e^{-\lambda \cdot 0} = \theta_i \Rightarrow C_4 = \theta_i$$

$$\therefore \boxed{\theta = \theta_i e^{-\lambda t} = \theta_i e^{-\frac{hA_s t}{\rho c}}} \Rightarrow \text{Lumped capacitance model.}$$

Note, only valid if the whole body is isothermal; $\frac{hL}{k} < 0.1$

Non-constant Fluid Temperature ($T(t)$ = body temperature)
Our previous equation would become: ($T_f(t)$ = fluid temp.)

$$\left[\frac{\partial T}{\partial t} + \lambda T = \lambda T_f(t) \right] \cdot e^{\lambda t} \quad (\text{Multiply by } e^{\lambda t}, \text{ integrating factor method})$$

$$\underbrace{e^{\lambda t} \frac{\partial T}{\partial t} + \lambda e^{\lambda t} T}_{\frac{\partial}{\partial t} (e^{\lambda t} T)} = \lambda e^{\lambda t} T_f(t)$$

$$\frac{\partial}{\partial t} (e^{\lambda t} T) = \lambda e^{\lambda t} T_f(t) \quad (\text{Integrate both sides})$$

$$\int_0^t \frac{\partial}{\partial t} (e^{\lambda t} T) dt = \lambda \int_0^t e^{\lambda t} T_f(t) dt$$

$$\boxed{e^{\lambda t} T \Big|_0^t = \lambda \int_0^t e^{\lambda t} T_f(t) dt} \Rightarrow \text{General solution, } \lambda = \frac{hA_s}{\rho c}$$

Example | Let $T_f = \beta \cdot t$ where $\beta = \text{constant}$

I.C. $\Rightarrow t=0, T_f = T = 0$

$$e^{\lambda t} T \Big|_0^t = \lambda \int_0^t e^{\lambda t} (\beta t) dt$$

$$e^{\lambda t} T(t) - e^{\lambda \cdot 0} T(0) = e^{\lambda t} T(t) = \lambda \beta \int_0^t t e^{\lambda t} dt$$

So how do we solve this?

$$\text{Let } s = \lambda t, ds = \lambda dt$$

$$e^{\lambda t} T(t) = \cancel{\lambda} B \int_0^{\frac{s}{\lambda}} \frac{s}{\cancel{\lambda}} e^s \frac{ds}{\lambda}$$

$$= \frac{B}{\lambda} \int_0^{\frac{s}{\lambda}} s e^s ds = \frac{B}{\lambda} (s e^s - e^s) \Big|_0^{\frac{s}{\lambda}}$$

$$e^{\lambda t} T(t) = \frac{B}{\lambda} e^{\lambda t} (\lambda t - 1) \Big|_0^t$$

$$e^{\lambda t} T(t) = \frac{B}{\lambda} [e^{\lambda t} (\lambda t - 1) - e^0 (0 - 1)]$$

$$= \frac{B}{\lambda} [e^{\lambda t} (\lambda t - 1) + 1]$$

Dividing through by $e^{\lambda t}$

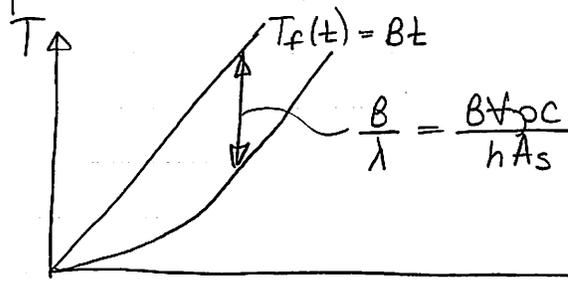
$$T(t) = \frac{B}{\lambda} [(\lambda t - 1) + e^{-\lambda t}]$$

$$= \underbrace{Bt}_{T_f(t)} - \frac{B}{\lambda} (1 - e^{-\lambda t})$$

$$T(t) = T_f(t) - \frac{B}{\lambda} (1 - e^{-\lambda t}) ; \lambda = \frac{hAs}{V\rho c}$$

Lag term. This is why its difficult to measure temperature changes in a fluid since measurement lags if B is high.

If we plot our result:



For a thermocouple:

$$\frac{B}{\lambda} = \frac{B V \rho c}{h A s} = \frac{B R^2 \rho c}{h R^2}$$

$$= \frac{B R \rho c}{h} \Rightarrow R \sim 0.001$$

$$h \sim 2500$$

$$\rho \sim 1000$$

$$c \sim 150$$

$$B \sim 2 \quad (62)$$

$$\frac{B}{\lambda} = \frac{2}{25} \text{ } ^\circ\text{C}$$

↳ For Water w/ TC.

Example | Time of death.

Person is found dead at 5 PM in a room with $T_f = 20^\circ\text{C}$. The body temperature is $T = 25^\circ\text{C}$ when found.

$$h = 8 \text{ W/m}^2 \cdot \text{K}$$

Geometry: $D \approx 30 \text{ cm}$, $L = 1.7 \text{ m}$

Estimate the time of death.

A healthy persons temperature is $T_i = 37^\circ\text{C}$

Assume radiation is negligible: Check:

$$\begin{aligned} h_{\text{rad}} &= \epsilon \sigma (T_s^2 + T_\infty^2)(T_s + T_\infty) \\ &= (1.0)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)((37+273)^2 + (20+273)^2)(37+273 + (20+273)) \\ h_{\text{rad}} &= 6.26 \text{ W/m}^2 \cdot \text{K} \end{aligned}$$

Add another $\bar{h}_{\text{rad}} \approx 5 \text{ W/m}^2 \cdot \text{K}$ to our convection h.t.c.

$$h_{\text{TOT}} = h_c + h_{\text{rad}} \approx 13 \text{ W/m}^2 \cdot \text{K}$$

Now lets assume a lumped capacitance model.

$$L_c = \frac{V}{A_s} = \frac{\pi r_o^2 L}{2\pi r_o L + 2\pi r_o^2} = 0.069 \text{ m}$$

$$\text{Check the Biot number: } Bi_{Lc} = \frac{hL_c}{k}$$

Assume $k = k_{\text{water}} = 0.6 \text{ W/m} \cdot \text{K}$ (since body is made up of 72% water by mass)

$$Bi_{Lc} \approx \frac{(13 \text{ W/m}^2 \cdot \text{K})(0.069 \text{ m})}{0.6 \text{ W/m} \cdot \text{K}} = 1.5 > 0.1$$

So lumped capacitance is not applicable but we use it to get a rough estimate:

$$\lambda = \frac{hA_s}{\rho CV} = \frac{h}{\rho CL_c} = 4.53 \times 10^{-5} \text{ s}^{-1} \Rightarrow \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-\lambda t} \Rightarrow t = 7.5 \text{ hr}$$

Two Dimensional Conduction (x, t)

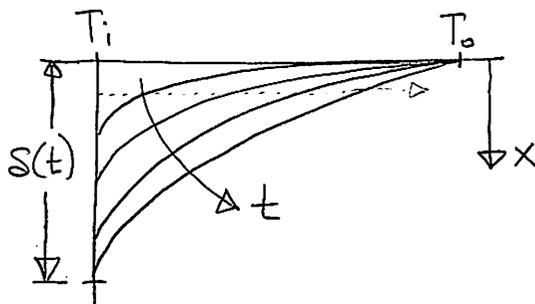
$$T(x, t), \dot{Q}''' = 0, v = 0, k = \text{constant}, \alpha = \text{constant}$$

We derived previously that:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{Q}'''}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Here we need 2 boundary conditions and 1 initial condition



T_o = hot body temperature
 T_i = initial temperature

So our governing PDE is: $\theta = \frac{T - T_o}{T_i - T_o}$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t}$$

$$x = 0, \theta = 0$$

$$x \rightarrow \infty, \theta = 1$$

$$t = 0, \theta = 1$$

Separation of variables won't work since the medium is infinite.

The only way to solve this is to look for a similarity variable, that relates x, t . One that works and is convenient is:

$$\eta = \frac{x}{\sqrt{t}}, \text{ so } \theta(\eta), f(t) = \text{function of time}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{t}} \frac{\partial \theta}{\partial \eta}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \theta}{\partial x} \right) = \frac{1}{\sqrt{t}} \frac{\partial}{\partial x} \left(\frac{\partial \theta}{\partial \eta} \right) = \frac{1}{\sqrt{t}} \frac{\partial}{\partial \eta} \left(\frac{\partial \theta}{\partial \eta} \right) \frac{\partial \eta}{\partial x} = \frac{1}{t} \frac{\partial^2 \theta}{\partial \eta^2}$$

Note, if you use Buckingham PI theorem you will get $\pi_1 = \theta, \pi_2 = \frac{x}{\sqrt{\alpha t}}$.

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = - \frac{\partial \theta}{\partial \eta} \cdot x \cdot \frac{f'}{f^2} = - \eta \frac{\partial \theta}{\partial \eta} \frac{f'}{f}$$

Now if we back substitute into our PDE

$$\frac{1}{f^2} \frac{\partial^2 \theta}{\partial \eta^2} + \frac{f'}{\alpha f} \eta \cdot \frac{\partial \theta}{\partial \eta} = 0$$

$$\frac{\partial^2 \theta}{\partial \eta^2} + \underbrace{\left(\frac{f \cdot f'}{\alpha} \right)}_{\text{Let this term equal to a constant}} \eta \frac{\partial \theta}{\partial \eta} = 0$$

$$\frac{ff'}{\alpha} = 2 \quad (\text{can choose any arbitrary number})$$

$$f \frac{\partial f}{\partial t} = 2\alpha \Rightarrow \int f \partial f = \int 2\alpha \partial t$$

$$\frac{f^2}{2} = 2\alpha t$$

$$\boxed{f = 2\sqrt{\alpha t}}$$

$$\frac{ff'}{\alpha} \cdot \eta = 2\eta$$

$$\boxed{\eta = \frac{x}{2\sqrt{\alpha t}}} \Rightarrow \frac{\partial^2 \theta}{\partial \eta^2} + 2\eta \frac{\partial \theta}{\partial \eta} = 0 \Rightarrow \text{ODE}$$

Our new B.C.'s become: $\left. \begin{array}{l} \eta = 0, \theta = 0 \\ \eta \rightarrow \infty, \theta = 1 \end{array} \right\}$ Turned 2BC's & IC into 2BC's only

Rewriting our ODE

$$\frac{\theta''}{\theta'} = -2\eta$$

Rewrite as: $\frac{\partial}{\partial \eta} (\ln \theta')$

$$\frac{\partial}{\partial \eta} \ln(\theta') = -2\eta$$

$$\int \frac{\partial}{\partial \eta} \ln(\theta') = \int -2\eta \partial \eta$$

$$\ln(\theta') = -\frac{2\eta^2}{2} + C = -\eta^2 + C \quad (\text{Take } \ln(\dots) \text{ of both sides})$$

$$\theta' = C e^{-\eta^2}$$

Aside

$$\Rightarrow \frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \cdot \frac{d}{dx} (f(x))$$

Integrating again,

$$\theta = c \int_0^{\eta} e^{-\eta^2} d\eta + C_2$$

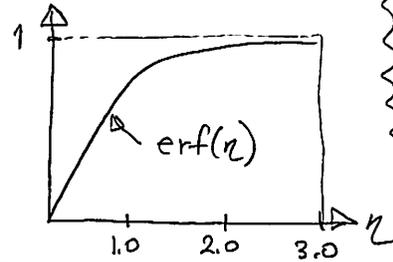
We know $\theta(\eta=0) = 0 \Rightarrow C_2 = 0$
 $\theta(\eta \rightarrow \infty) = 1$

$$1 = c \int_0^{\infty} e^{-\eta^2} d\eta \quad (\text{Gaussian integral})$$

$$I_0 = \frac{\sqrt{\pi}}{2} \Rightarrow c = \frac{2}{\sqrt{\pi}}$$

$$\therefore \theta = \frac{T - T_0}{T_i - T_0} = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Aside: Error function



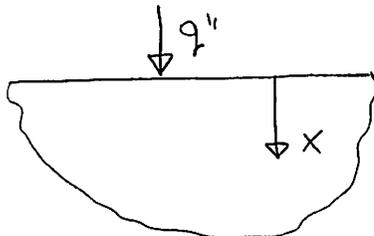
This integral cannot be solved analytically. Must be done numerically.

For the error function, see Table B4, pg 924 in Mills. Also: $1 - \text{erf}(\eta) = \text{erfc}(\eta)$

error function, tabulated results

complimentary error function.

So what about heat transfer during transient conduction?



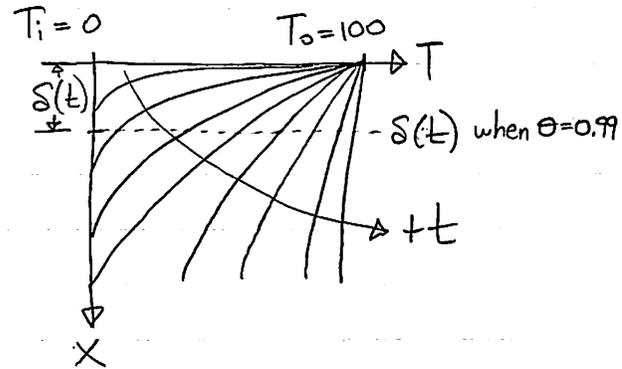
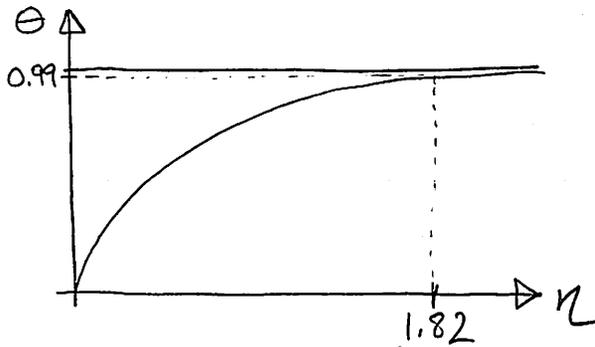
$$q''_{x=0} = -k \frac{\partial T}{\partial x} \Big|_{x=0} \Rightarrow \text{let } \theta = \frac{T - T_0}{T_i - T_0}, \quad d\theta = \frac{1}{T_i - T_0} dT$$

$$= -k (T_i - T_0) \frac{\partial \theta}{\partial x} \Big|_{x=0} = k \underbrace{(T_0 - T_i)}_{\Delta T} \frac{\partial \eta}{\partial x} \underbrace{\left(\frac{\partial \theta}{\partial \eta} \right)}_{\frac{1}{\sqrt{\pi}} \frac{2}{2\sqrt{\alpha t}}} \Big|_{x=0}$$

$$\boxed{q'' = \frac{k \Delta T}{\sqrt{\pi \alpha t}}}, \quad \boxed{\alpha = \frac{k}{\rho c}} \Rightarrow \text{Thermal diffusivity}$$

Note, to get total energy transferred, you need to integrate q'' .

Lets analyze our solution: $\Theta = \frac{T - T_0}{T_i - T_0}$



$$\frac{S}{2\sqrt{\alpha t}} = 1.82 \Rightarrow \boxed{S(t) = 3.04\sqrt{\alpha t}} \Rightarrow \text{Thermal penetration depth}$$

Lets look at some other cases:

① Specified surface heat flux, $q'' = \text{constant}$

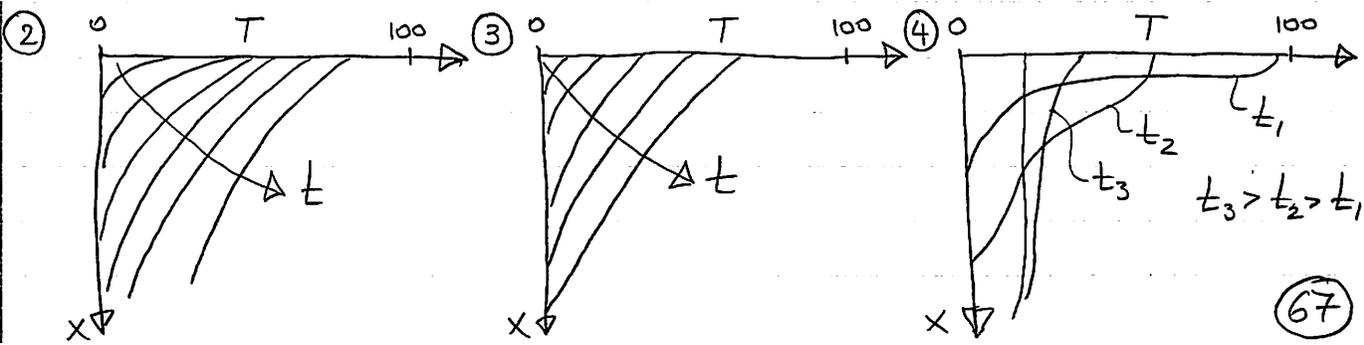
$$T(x,t) - T_i = \frac{q''}{k} \left[\frac{\sqrt{4\alpha t}}{\pi} e^{-\frac{x^2}{4\alpha t}} - x \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \right]$$

② Convection on the surface, $q'' = h(T_\infty - T(0,t))$

$$\frac{T(x,t) - T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - e^{\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

③ Energy pulse at surface, $e_s = \text{constant} \left[\frac{\text{J}}{\text{m}^2}\right] @ t=0$
Like a laser pulse, no losses, all heat goes into the solid

$$T(x,t) - T_i = \frac{e_s}{k\sqrt{\pi t/\alpha}} e^{-\frac{x^2}{4\alpha t}}$$



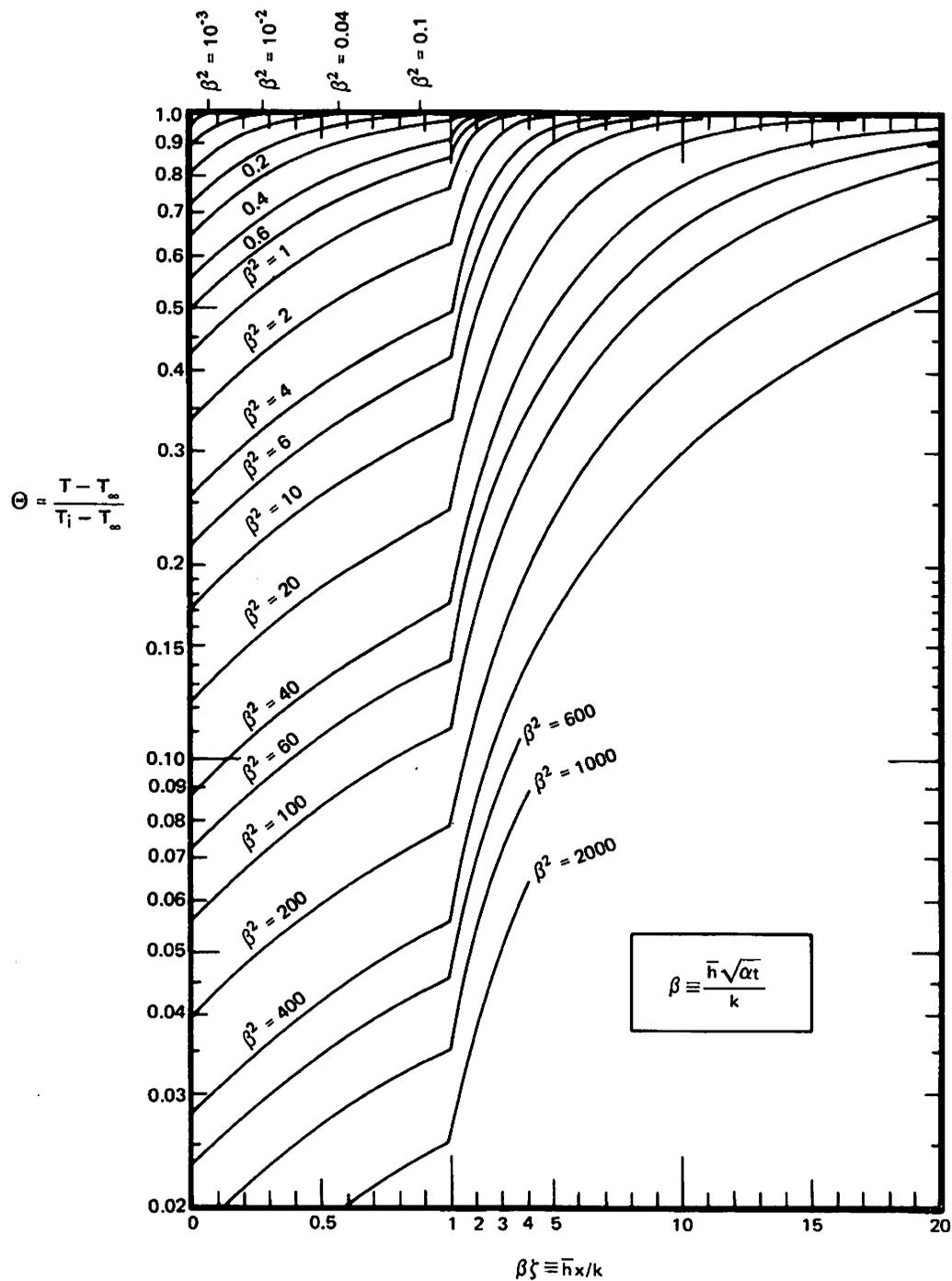
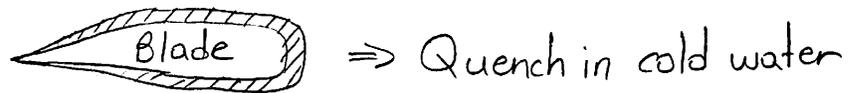


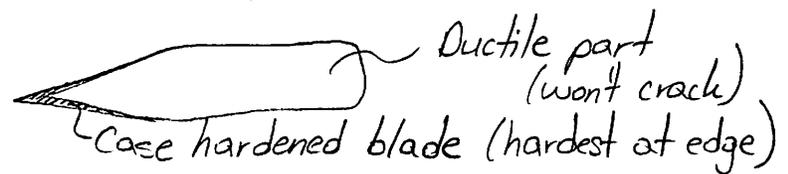
Figure 5.16 The cooling of a semi-infinite region by an environment at T_∞ , through a heat transfer coefficient, \bar{h} .

From: J.H. Lienhard IV and J.H. Lienhard V. *A Heat Transfer Textbook*, Phlogiston Press, Cambridge, MA, 3rd edition, 2008. Used by permission.

Application: Samurai in Japan used to make their swords with transient heat conduction in mind. They would coat the blade with clay sheath & quench

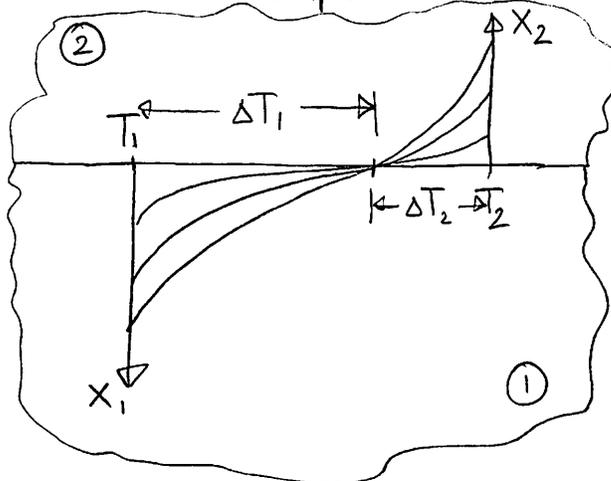


⇒ Quench in cold water



Contact of Two Semi-Infinite Solids

Two bodies at T_1 & T_2 brought into contact, they instantly achieve a temperature that is constant at their interface



Note, $T_{interface} \neq f(t) = \text{constant}$

$T_{interface}$ will develop very quickly after the two bodies touch, & remain the same.

Applying an energy balance, $\Delta T = T_1 - T_2$

$$q'' = \frac{k \Delta T}{\sqrt{\pi \alpha t}} = \frac{(k \rho c)^{1/2}}{\sqrt{\pi}} \cdot \frac{\Delta T}{\sqrt{t}}$$

$$q''_1 = q''_2 \Rightarrow \frac{(k_1 \rho_1 c_1)^{1/2}}{\sqrt{\pi}} \cdot \frac{\Delta T_1}{\sqrt{t}} = \frac{(k_2 \rho_2 c_2)^{1/2}}{\sqrt{\pi}} \cdot \frac{\Delta T_2}{\sqrt{t}}$$

$$(k_1 \rho_1 c_1)^{1/2} \Delta T_1 = (k_2 \rho_2 c_2)^{1/2} \Delta T_2 \quad (1)$$

$$\Delta T_1 + \Delta T_2 = \Delta T \quad (2)$$

Back substituting (2) into (1)

$$\Delta T_2 = \frac{(k\rho c)_1^{1/2}}{(k\rho c)_2^{1/2}} \cdot \Delta T_1$$

$$\Delta T = \left(1 + \sqrt{\frac{(k\rho c)_1}{(k\rho c)_2}}\right) \Delta T_1$$

$$\Delta T_1 = \frac{\Delta T}{1 + \frac{(k\rho c)_1^{1/2}}{(k\rho c)_2^{1/2}}} = \frac{(k\rho c)_2^{1/2} \cdot \Delta T}{(k\rho c)_2^{1/2} + (k\rho c)_1^{1/2}} \neq f(t), \infty T_{\text{interface}} = \text{const.}$$

$$q''_{x=0} = \frac{(k\rho c)_1^{1/2} (k\rho c)_2^{1/2}}{(k\rho c)_1^{1/2} + (k\rho c)_2^{1/2}} \cdot \frac{\Delta T}{\sqrt{\pi t}} \Rightarrow \text{Heat flux between the two contacting bodies}$$

Note, this is fundamentally why when you touch certain objects in a room, they feel "colder" than others, even though they are at the same temperature.

Example | Touching a brass & wooden doorknob.

Brass: $k_1 = 109 \text{ W/m}\cdot\text{K}$
 $\rho_1 = 8730 \text{ kg/m}^3$
 $c_1 = 380 \text{ J/kg}\cdot\text{K}$

Wood: $k_2 = 0.17 \text{ W/m}\cdot\text{K}$ (Oak)
 $\rho_2 = 750 \text{ kg/m}^3$
 $c_2 = 1700 \text{ J/kg}\cdot\text{K}$

$$\underbrace{(k_1 \rho_1 c_1)^{1/2}} = 19016$$

$$\underbrace{(k_2 \rho_2 c_2)^{1/2}} = 466$$

Lets say $\Delta T = T_{\text{body}} - T_{\text{room}} \approx 20^\circ\text{C}$
 Now we need human flesh properties.

Human Hand: $k_3 \approx 0.6 \text{ W/m}\cdot\text{K}$
 $\rho_3 \approx 1000 \text{ kg/m}^3$
 $c_3 \approx 4190 \text{ J/kg}\cdot\text{K}$

$$\left. \begin{array}{l} k_3 \approx 0.6 \text{ W/m}\cdot\text{K} \\ \rho_3 \approx 1000 \text{ kg/m}^3 \\ c_3 \approx 4190 \text{ J/kg}\cdot\text{K} \end{array} \right\} (k_3 \rho_3 c_3)^{1/2} = 1586$$

Case 1: Brass Doorknob

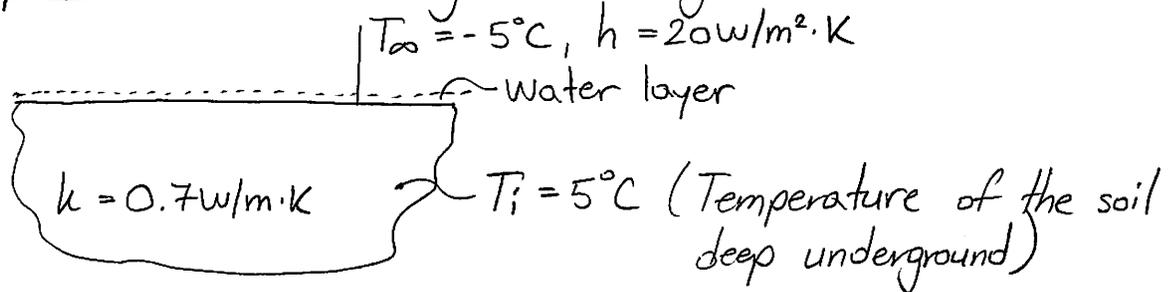
$$q''_{x=1} = \frac{(19016)(1586)}{(19016) + (1586)} \cdot \frac{\Delta T}{\sqrt{\pi t}} = 1463 \frac{\Delta T}{\sqrt{\pi t}}$$

Case 2: Wooden Doorknob

$$q_{3 \rightarrow 2}'' = \frac{(466)(1586)}{466 + 1586} \frac{\Delta T}{\sqrt{\pi t}} = 360 \frac{\Delta T}{\sqrt{\pi t}}$$

We see that: $\frac{q_{3 \rightarrow 1}''}{q_{3 \rightarrow 2}''} \approx 4.1 \Rightarrow$ The brass doorknob will feel about 4 times "colder"

Your body senses heat by monitoring heat flux, not temperature, so the feeling of relative coldness depends on how fast you lose heat.

Example | Water freezing on the ground

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \frac{0^\circ\text{C} - (-5^\circ\text{C})}{5^\circ\text{C} - (-5^\circ\text{C})} = 0.5 \quad , \text{ note } T = 0^\circ\text{C} \text{ because of water freezing temp.}$$

$$x = 0 \text{ (top of soil)}$$

Using our chart: 5.16 from Lienhard & Lienhard, 2008

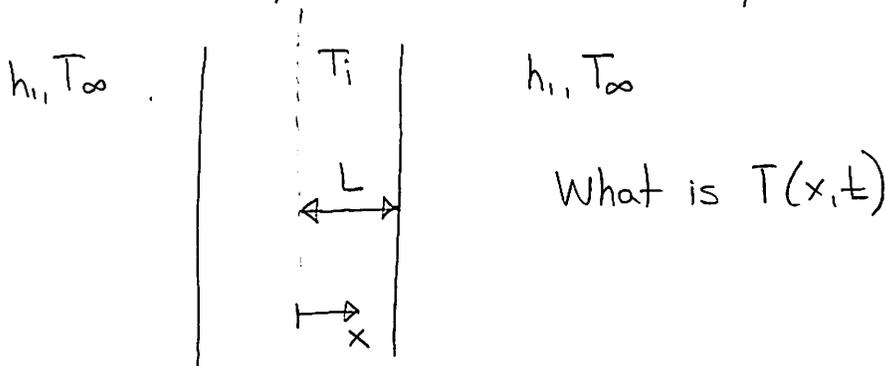
$$\beta^2 = 0.6 = \left(\frac{h}{k}\right)^2 \alpha t \Rightarrow \boxed{t = 24 \text{ minutes}}$$

Therefore, ice will freeze in ≈ 24 minutes on the ground.

Transient Heat Conduction in Finite Bodies

Previously, we covered lumped capacitance: $T = f(t) \neq f(x)$
 and transient heat conduction in infinite media: $0 \leq x \leq \infty$
 What about finite bodies: $0 \leq x \leq L$?

Plane wall problem: Wall initially at T_i placed in medium at T_∞
 & h_i on outside.



Writing out our heat equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$T(x, t=0) = T_i \quad (\text{initial condition})$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \quad (\text{1'st B.C.})$$

$$-k \left. \frac{\partial T}{\partial x} \right|_{x=L} = h(T - T_\infty) \quad (\text{2'nd B.C.})$$

To make things simpler, it's helpful to non-dimensionalize

$$\text{Let } \bar{x} = \frac{x}{L}, \quad \partial \bar{x} = \frac{1}{L} \partial x \quad (\text{Dimensionless distance from center})$$

$$\Theta = \frac{T - T_\infty}{T_i - T_\infty}, \quad \partial \Theta = \frac{1}{T_i - T_\infty} \partial T \quad (\text{Dimensionless temperature})$$

What about dimensionless time τ ?

$$\frac{\partial T}{\partial x} = \frac{(T_i - T_\infty) \partial \Theta}{L \partial \bar{x}}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{(T_i - T_\infty) \partial^2 \Theta}{L^2 \partial \bar{x}^2}$$

$$\frac{\partial T}{\partial t} = (T_i - T_\infty) \frac{\partial \Theta}{\partial t}$$

Back substituting

$$\frac{(T_i - T_\infty)}{L^2} \frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{(T_i - T_\infty)}{2t} \frac{\partial \theta}{\partial t} \cdot \frac{1}{\alpha}$$

$$\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{L^2}{\alpha} \frac{\partial \theta}{\partial t}$$

So this helps us solve for τ : $\int 2t = \int \frac{L^2}{\alpha} d\tau$

$$\tau = \frac{\alpha t}{L^2} \Rightarrow \text{Fourier number}$$

Aside: Fourier # simplified

$$F_0 = \frac{\alpha t}{L^2} = \frac{k t}{\rho c L^2} \cdot \frac{\Delta T}{\Delta T}$$

$$= \frac{(k \frac{\Delta T}{L})}{(\frac{\rho c \Delta T L}{t})}$$

$$= \frac{q''_{\text{cond}}}{q''_{\text{stored}}}$$

$$\tau = F_0 = \frac{\alpha t}{L^2} = \frac{\text{diffusive heat conduction rate}}{\text{heat storage rate (transient)}} = \text{dimensionless time in h.t. problems.}$$

Now if we non-dimensionalize our second boundary condition

$$-k \frac{\partial T}{\partial x} \Big|_{x=L} = h(T - T_\infty)$$

$$-k \frac{(T_i - T_\infty)}{L} \frac{\partial \theta}{\partial \bar{x}} \Big|_{\bar{x}=1} = h \theta (T_i - T_\infty)$$

We can rewrite this as:

$$\frac{\partial \theta(1, \tau)}{\partial \bar{x}} = - \underbrace{\frac{hL}{k}}_{-Bi_L} \theta(1, \tau)$$

$-Bi_L \Rightarrow$ Biot number = dimensionless h.t.c.
 $= \frac{\text{conduction resistance}}{\text{convection resistance}}$

Our other B.C. & I.C. are:

$$\frac{\partial \theta(0, \tau)}{\partial \bar{x}} = 0$$

$$\theta(\bar{x}, 0) = 1$$

Note we've taken the problem from: $T(x,t) = f(x, L, t, \alpha, h, T_\infty, T_i)$

to: $\theta = f(\bar{x}, Bi, F_0) \Rightarrow$ Very Powerful!

To solve, we need to use separation of variables:

$$\Theta(\bar{x}, \tau) = F(\bar{x}) \cdot G(\tau)$$

Back substituting & dividing by $F \cdot G$

$$\underbrace{\frac{1}{F} \frac{\partial^2 F}{\partial \bar{x}^2}}_{f(\bar{x}) \text{ only}} = \underbrace{\frac{1}{G} \frac{\partial G}{\partial \tau}}_{f(\tau) \text{ only}} = \text{Constant} \quad (\text{The only valid solution})$$

$\neq 0$ since that says no time dependence. $G \neq f(\tau)$.
(\bar{x} , & τ can be varied independently)

We will assume a solution of the form $-\lambda^2$

$$\frac{\partial^2 F}{\partial \bar{x}^2} + \lambda^2 F = 0 \quad (1)$$

$$\frac{\partial G}{\partial \tau} + \lambda^2 G = 0 \quad (2)$$

Solving (1) first: $\lambda'^2 + \lambda^2 = 0$
 $\lambda' = \pm \sqrt{-1} \lambda = \lambda i$

So our solution becomes:

$$F = C_1 e^{+i\lambda\bar{x}} + C_2 e^{-i\lambda\bar{x}} \Rightarrow \text{since } \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Can rewrite our solution as:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$F = C_1 \cos(\lambda\bar{x}) + C_2 \sin(\lambda\bar{x})$$

For G :

$$\lambda' + \lambda^2 = 0 \Rightarrow \lambda' = -\lambda^2$$

$$G = C_3 e^{-\lambda^2 \tau}$$

Combining our solution, we obtain:

$$\Theta = F \cdot G = C_3 e^{-\lambda^2 \tau} (C_1 \cos(\lambda\bar{x}) + C_2 \sin(\lambda\bar{x})) = e^{-\lambda^2 \tau} [A \cos(\lambda\bar{x}) + B \sin(\lambda\bar{x})] \quad (73)$$

Applying our B.C.'s:

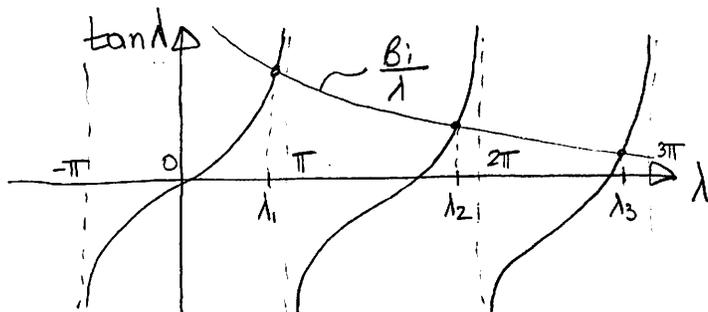
$$\frac{\partial \theta(0, \tau)}{\partial \bar{x}} = 0 \Rightarrow -e^{-\lambda^2 \tau} (A \lambda \sin(0) + B \lambda \cos(0)) = 0 \rightarrow B = 0$$

$$\theta = A e^{-\lambda^2 \tau} \cos(\lambda \bar{x})$$

$$\frac{\partial \theta(1, \tau)}{\partial \bar{x}} = -Bi \theta(1, \tau) \Rightarrow -A e^{-\lambda^2 \tau} \lambda \sin \lambda = -Bi A e^{-\lambda^2 \tau} \cos \lambda$$

$$\lambda \tan \lambda = Bi$$

We know $\tan \lambda$ is a periodic function with period π .
The solution can lie anywhere between 0 & π , π & 2π , ...



\Rightarrow Multiple roots exist

So we have multiple solutions

$$\lambda_n \tan \lambda_n = Bi \quad (\text{Eigenfunction with eigenvalues } \lambda_n)$$

There exist an infinite number of solutions of the form $A e^{-\lambda^2 \tau} \cos(\lambda \bar{x})$. The final solution is a linear combination of all of them:

$$\theta = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} \cos(\lambda_n \bar{x})$$

Where the constants A_n are determined from our I.C.

$$\theta(\bar{x}, 0) = 1 \rightarrow 1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n \bar{x})$$

Using orthogonality \Rightarrow multiply both sides by $\cos(\lambda_m \bar{x})$ & integrate

$$\int_0^1 \cos(\lambda_m \bar{x}) d\bar{x} = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 \cos(\lambda_n \bar{x}) \cos(\lambda_m \bar{x}) d\bar{x}}_{=0 \text{ if } n \neq m}$$

So our solution becomes

$$\int_0^1 \cos(\lambda_n \bar{x}) d\bar{x} = A_n \int_0^1 \cos^2(\lambda_n \bar{x}) d\bar{x}$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin(2\lambda_n)} \Rightarrow \text{Constants in the } \theta \text{ solution series}$$

So now what? We can do the same for a cylinder & sphere.

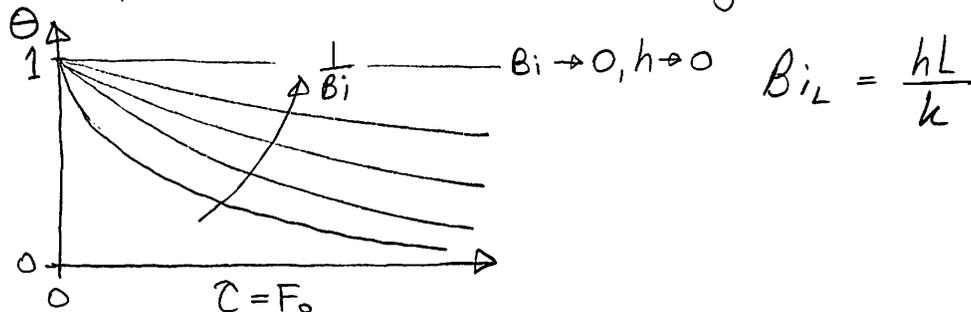
Since solution involves an infinite series, not very usefull to solve analytically. However, good approximation is made with the first few terms since rest decay rapidly due to the $e^{-\lambda_n^2 \tau}$ term.

END OF LECTURE 8

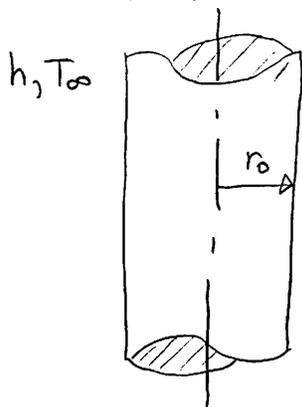
Tabular Solutions

We know our solution is : $\theta = f(\bar{x}, Bi, Fo)$

We can plot our results as: (for a given \bar{x})



Similar if we had a cylinder :

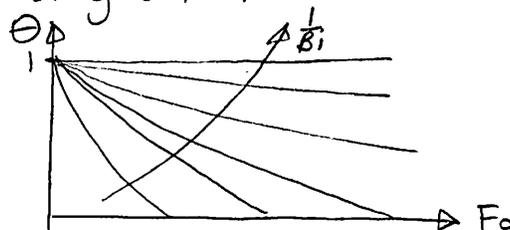


h, T_∞

$$\bar{r} = \frac{r}{r_0}, \quad \theta = f(\bar{r}, Bi_{r_0}, Fo)$$

$$Bi = \frac{hr_0}{k}$$

For a given \bar{r} :



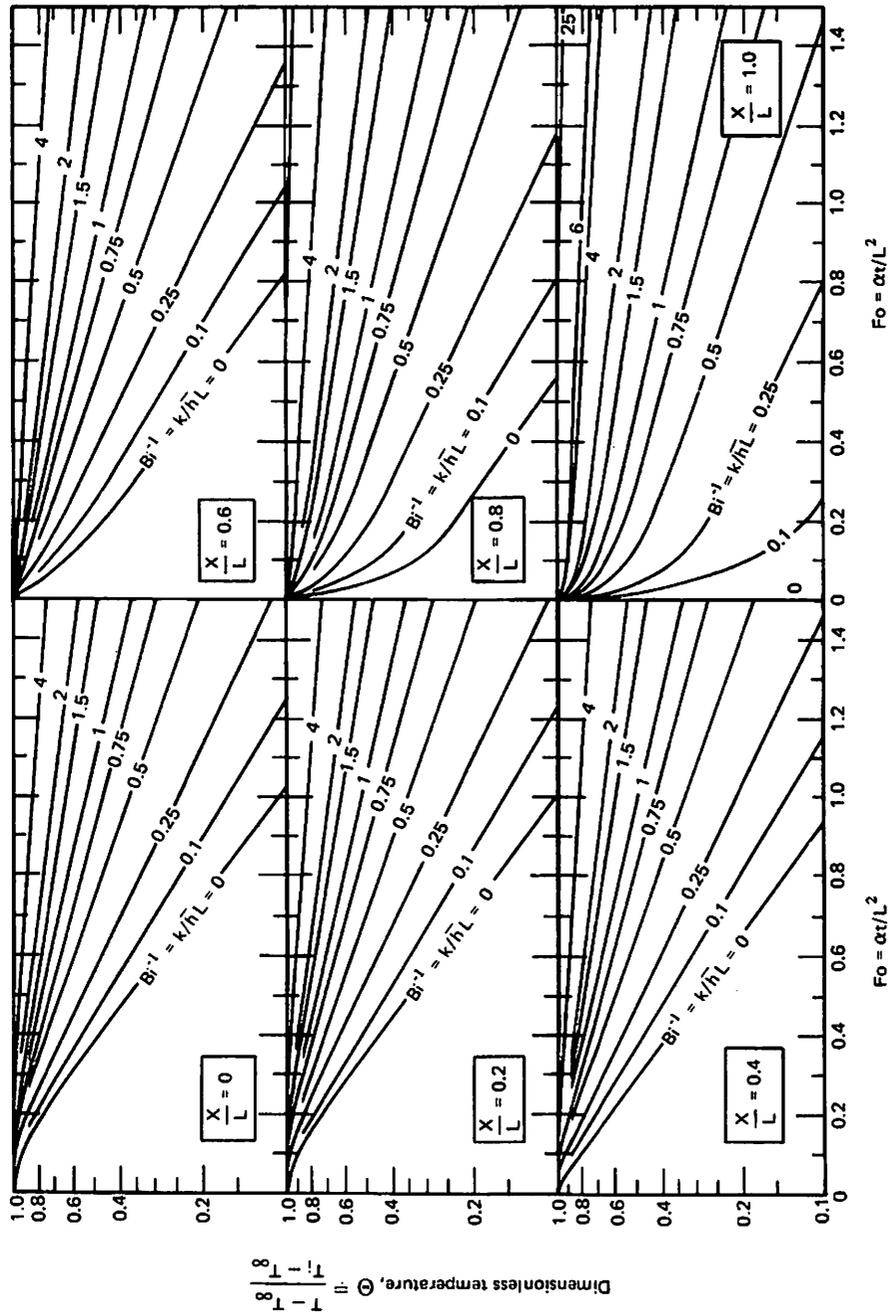


Figure 5.7 The transient temperature distribution in a slab at six positions: $x/L = 0$ is the center, $x/L = 1$ is one outside boundary.

From: J.H. Lienhard IV and J.H. Lienhard V. *A Heat Transfer Textbook*, Phlogiston Press, Cambridge, MA, 3rd edition, 2008. Used by permission.

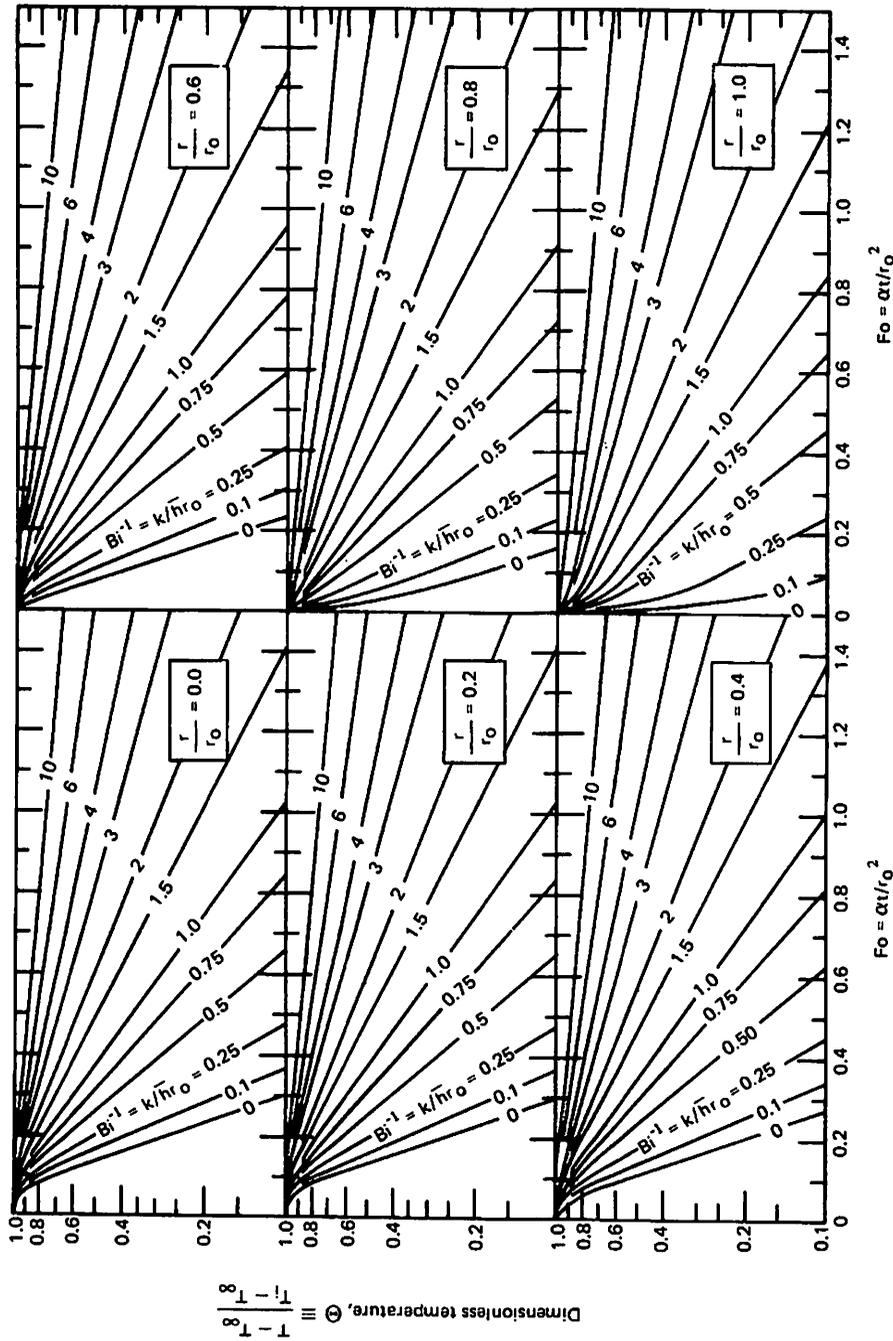
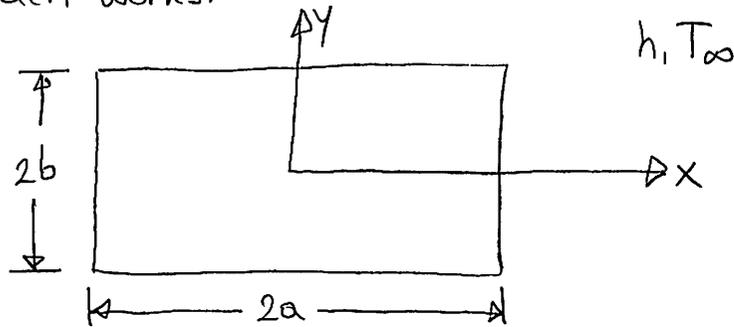


Figure 5.9 The transient temperature distribution in a sphere of radius r_0 at six positions: $r/r_0 = 0$ is the center; $r/r_0 = 1$ is the outside boundary.

From: J.H. Lienhard IV and J.H. Lienhard V. *A Heat Transfer Textbook*, Phlogiston Press, Cambridge, MA, 3rd edition, 2008. Used by permission.

2-D Finite conduction (or 3-D)

Note, if we had a 2-D slab, the same separation of variables approach works:



$$\Theta = f(\bar{x}, \bar{y}, Bi_x, Bi_y, Fo_x, Fo_y)$$

$$\Theta = \Theta_{x,t} \cdot \Theta_{y,t}$$

$$\Theta_{x,t} = f(\bar{x}, Fo_x, Bi_x)$$

$$\bar{x} = \frac{x}{a}$$

$$Fo_x = \frac{\alpha t}{a^2}$$

$$Bi_x = \frac{ha}{k}$$

$$\Theta_{y,t} = f(\bar{y}, Fo_y, Bi_y)$$

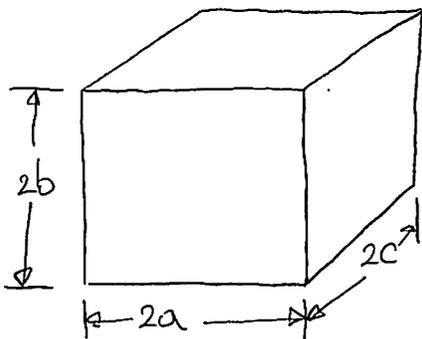
$$\bar{y} = \frac{y}{b}$$

$$Fo_y = \frac{\alpha t}{b^2}$$

$$Bi_y = \frac{hb}{k}$$

Use charts to solve for each direction & then $\Theta = \Theta_{x,t} \cdot \Theta_{y,t}$

For a 3-D body, it would be the same:



$$\Theta = \Theta_{x,t} \cdot \Theta_{y,t} \cdot \Theta_{z,t}$$

Use tabulated results.

Steady Multi-Dimensional Heat Transfer (The Shape factor)

A heat conduction shape factor, S , may be defined for steady problems involving isothermal surfaces as follows:

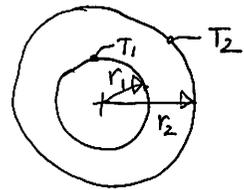
$$Q = Sk\Delta T$$

↳ Obtained analytically or numerically (for complex problems)

$$R = \frac{1}{kS}$$

For example, if we look at a cylinder:

$$S_{cyl} = \frac{1}{kR_{cyl}} \Rightarrow R_{cyl} = \frac{|\ln(r_2/r_1)|}{2\pi kL}$$



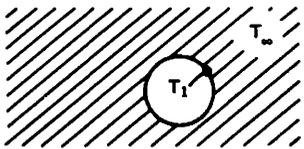
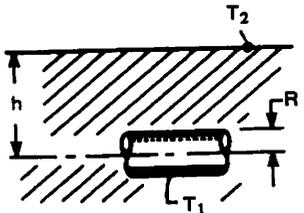
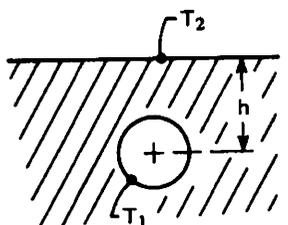
$$S_{cyl} = \frac{2\pi L}{\ln(r_2/r_1)}$$

In general, values are tabulated and solutions are in the form:

$$Q = Sk\Delta T, \quad R_{\pm} = \frac{1}{kS}$$

See Mills Table 3.2, page 162-163

Table 5.4 Conduction shape factors: $Q = S k \Delta T$, $R_t = 1/(kS)$.

Situation	Shape factor, S	Dimensions	Source
1. Conduction through a slab	A/L	meter	Example 2.2
2. Conduction through wall of a long thick cylinder	$\frac{2\pi}{\ln(r_o/r_i)}$	none	Example 5.9
3. Conduction through a thick-walled hollow sphere	$\frac{4\pi(r_o r_i)}{r_o - r_i}$	meter	Example 5.10
4. The boundary of a spherical hole of radius R conducting into an infinite medium 	$4\pi R$	meter	Problems 5.19 and 2.15
5. Cylinder of radius R and length L , transferring heat to a parallel isothermal plane; $h \ll L$ 	$\frac{2\pi L}{\cosh^{-1}(h/R)}$	meter	[5.16]
6. Same as item 5, but with $L \rightarrow \infty$ (two-dimensional conduction)	$\frac{2\pi}{\cosh^{-1}(h/R)}$	none	[5.16]
7. An isothermal sphere of radius R transfers heat to an isothermal plane; $R/h < 0.8$ (see item 4) 	$\frac{4\pi R}{1 - R/2h}$	meter	[5.16, 5.17]

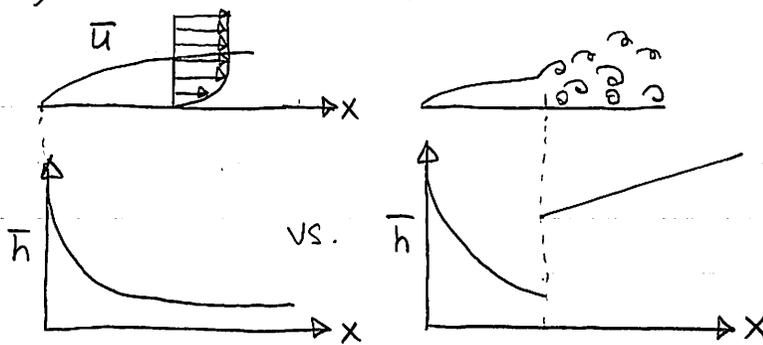
Convection Heat Transfer

Convection is the transfer of thermal energy due to both conduction and by bulk "carrying" of the energy through the velocity of the fluid.

Four Categories:

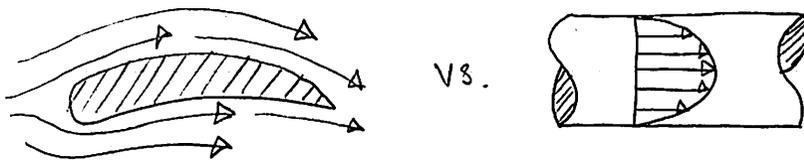
1) Forced vs. Free (Natural)

2) Laminar vs. Turbulent

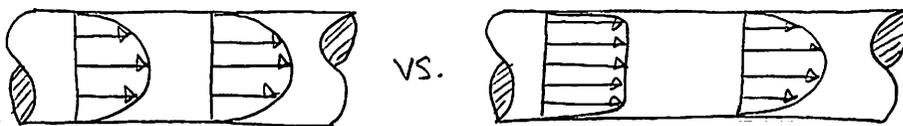


⇒ We will find out why.

3) External vs. Internal

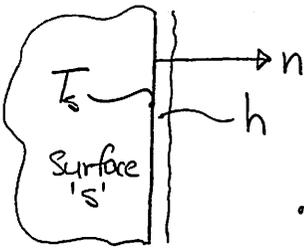


4) Fully Developed vs. Developing



We assume:

- 1) Steady State
- 2) Constant properties
- 3) Incompressible



$$q'' = h(T_s - T_f) \quad (1)$$

or

$$q'' = -k_f \left. \frac{\partial T}{\partial n} \right|_s \quad (2)$$

where:

h = heat trans. coeff.

T_s = surface temp.

T_f = fluid temp faraway

k_f = fluid thermal cond.

$T =$

Non-dimensionalizing:

Let:

$$\theta = \frac{T - T_f}{T_s - T_f}, \quad n^* = \frac{n}{L}$$

Back substitute and equate (1) & (2)

$$-k_f \frac{(T_s - T_f)}{L} \left. \frac{\partial \theta}{\partial n^*} \right|_s = h (T_s - T_f)$$

$$\boxed{-\frac{\partial \theta}{\partial n^*} = \frac{hL}{k_f} = Nu \Rightarrow \text{Nusselt number}}$$

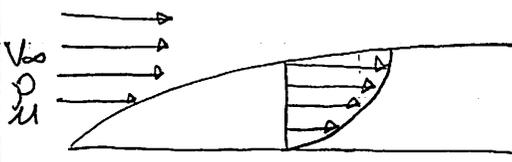
$$\boxed{Nu = \frac{hL}{k_f} = \frac{\text{convective heat transfer rate}}{\text{conductive heat transfer rate}}}$$

Since $Nu \propto \frac{\partial \theta}{\partial n^*} \propto \theta$, it typically depends on:

- 1) Flow cond.
- 2) Fluid properties
- 3) Geometry
- 4) Boundary conditions

External Flow

Assuming $\rho, \mu, c = \text{constant}$ (density, viscosity, heat capacity)
 Linked to heat transfer because of the boundary layer



\Rightarrow Where all of the gradients happen in velocity

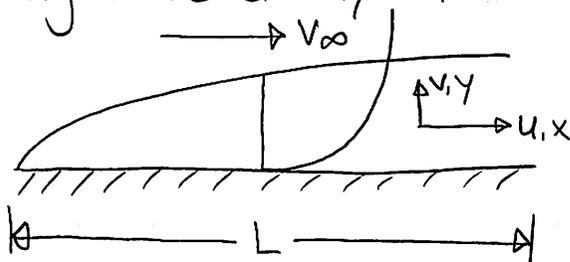
2-D Laminar Flow: (Flat Plate, steady)

x-momentum: $u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$

y-momentum: $u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$

continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Looking more closely at our boundary layer:



Let: $\bar{u} = \frac{u}{V_\infty}$, $\bar{v} = \frac{v}{V_\infty}$, $\bar{p} = \frac{p}{\rho V_\infty^2}$

$\bar{x} = \frac{x}{L}$, $\bar{y} = \frac{y}{L}$

Now our equations become:

$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re_L} \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]$, $Re_L = \frac{\rho V_\infty L}{\mu}$

$\bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{1}{Re_L} \left[\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right]$

$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$

So what can we say about these equations and their relative terms? $u \sim V_\infty$, $x \sim L$, $y \sim \delta$ (boundary layer thick.)

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{V_\infty}{L} + \frac{v}{\delta} = 0 \Rightarrow v \sim \left(\frac{\delta}{L} \right) V_\infty$

Looking back at our momentum equations

↳ Note dropped the negative sign since scaling analysis.

Going to x-momentum

$$V_\infty \frac{V_\infty}{L} + V_\infty \left(\frac{\delta}{L}\right) \cdot \frac{V_\infty}{\delta} = -\frac{1}{\rho} \frac{\cancel{\rho} V_\infty^2}{L} + \nu \left[\frac{V_\infty}{L^2} + \frac{V_\infty}{\delta^2} \right]$$

≈ 0 since $\frac{\delta}{L} \ll 1$

$$\underbrace{\sim \frac{V_\infty^2}{L}} \quad \underbrace{\sim \frac{V_\infty^2}{L}} \quad \underbrace{\sim \frac{\nu V_\infty}{\delta^2}}$$

Can rewrite as:

$$\frac{V_\infty}{L^2} = \frac{\delta^2}{L^2} \cdot \frac{V_\infty}{\delta^2} \approx 0 \text{ since } \frac{\delta}{L} \ll 1$$

y-momentum:

$$V_\infty \frac{\delta}{L} V_\infty \cdot \frac{1}{L} + V_\infty^2 \left(\frac{\delta}{L}\right)^2 \frac{1}{\delta} = -\frac{1}{\rho} \cdot \frac{\cancel{\rho} V_\infty^2}{\delta} + \nu \frac{\delta}{L} V_\infty \left[\frac{1}{L^2} + \frac{1}{\delta^2} \right]$$

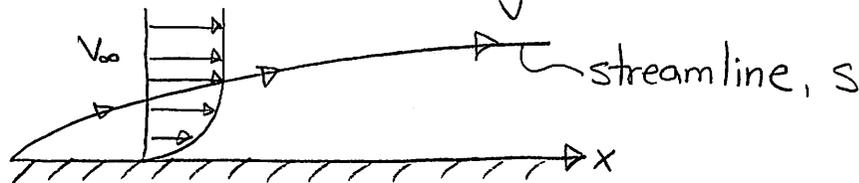
$$\underbrace{\sim \frac{V_\infty^2}{L} \cdot \left(\frac{\delta}{L}\right)} \quad \underbrace{\sim \frac{V_\infty^2}{L} \cdot \left(\frac{\delta}{L}\right)} = \frac{V_\infty^2}{\delta} + \nu V_\infty \left(\frac{\delta}{L}\right) \left[\frac{1}{L^2} + \frac{1}{\delta^2} \right]$$

Since $\frac{\delta}{L} \ll 1$, we can drop the inertia terms, & viscous terms

Our y-momentum equation becomes:

$$\frac{\partial p}{\partial y} = 0 \Rightarrow \boxed{p(y) = \text{Constant}} \Rightarrow \text{Pressure is not a function of } y$$

If we look at the edge of our boundary layer



Our x-momentum equation along the streamline s becomes

$$u \frac{\partial u}{\partial x} + \underbrace{v \frac{\partial u}{\partial y}}_{0 \text{ at } s} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \underbrace{\nu \left(\frac{\partial^2 u}{\partial y^2} \right)}_{0 \text{ at } s}$$

$$V_\infty \frac{\partial V_\infty}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \Rightarrow \boxed{\rho + \frac{1}{2} \rho V_\infty^2 = \text{constant}}$$

↳ Bernoulli equation

Note from this, we can see that the pressure in the x-direction is constant ($P \neq f(x)$) for steady flow past a flat plate, since $V_\infty = \text{constant}$ at the b.l. edge. Note, not true for cylinder, sphere, etc., where V_∞ changes.

$$\frac{\partial v_0}{\partial x} = 0 \quad \text{or} \quad \frac{1}{\rho} \frac{\partial p}{\partial x} = -V_\infty \frac{\partial V_\infty}{\partial x} = 0$$

So our boundary layer equations become:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (2)$$

$$P(y) = \text{constant} \quad (3)$$

Boundary conditions:

$$u = v = 0 \quad @ \quad y = 0$$

$$u = U_\infty \quad @ \quad y \rightarrow \infty$$

So now we must convert our PDE into a solvable ODE. Cannot use separation of variables (infinite medium). Let's try a similarity solution.

Assume: $\frac{u}{V_\infty} = f(\eta)$ where $\eta = \frac{y}{\delta}$

So how do we define δ ?

Looking back at our non-dimensional x-momentum:

$$\underbrace{\bar{u}}_{\sim O(1)} \frac{\partial \bar{u}}{\partial \bar{x}} + \underbrace{\bar{v}}_{\sim O(1)} \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{Re_L} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}, \quad \bar{u} = \frac{u}{V_\infty} \sim \frac{V_\infty}{V_\infty} \sim O(1)$$

Has to be on order of 1.

$$\bar{x} = \frac{x}{L} \sim \frac{L}{L} \sim O(1)$$

$$\frac{1}{Re_L} \frac{V_\infty}{V_\infty} \cdot \left(\frac{\delta}{L}\right)^2 \sim 1$$

$$\bar{y} = \frac{y}{L} \sim \frac{\delta}{L}$$

$$\frac{1}{Re_L} \frac{\delta^2}{L^2} \sim 1 \Rightarrow \frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}} \quad \text{or} \quad \frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}}$$

$$\boxed{\delta \sim \sqrt{\frac{\nu x}{V_\infty}}}$$

so now we get:

$$\boxed{\eta = y \sqrt{\frac{V_\infty}{\nu x}}}$$

↳ Will see an easier way to get this later.

So what about v ? Looking at continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \Rightarrow \bar{u} = \phi(\eta) = \frac{u}{V_\infty}$$

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial \bar{u}}{\partial x} = -\frac{1}{2x} \eta \phi'$$

Back substituting, we obtain:

$$\frac{\partial \bar{v}}{\partial y} = \frac{1}{2x} \phi' \eta$$

$$\bar{v} = \frac{1}{2x} \int_0^\eta \phi' \eta \, dy = \frac{1}{2x} \sqrt{\frac{xV}{V_\infty}} \int_0^\eta \phi' \eta \, d\eta$$

$$\frac{1}{2} \frac{1}{\sqrt{Re_x}}$$

Aside:

We know $\eta = y \sqrt{\frac{V_\infty}{\nu x}}$
 $d\eta = dy \sqrt{\frac{V_\infty}{\nu x}}$

Now we need to solve our integral:

$$\int_0^\eta \eta \phi' \, d\eta = u v - \int v \, du \quad (\text{Integration by parts: } \int u \, dv = uv - \int v \, du)$$

$$= \eta \phi - \int_0^\eta \phi \, d\eta, \quad u = \eta, \quad v = \phi$$

$$\text{let } \int_0^\eta \phi \, d\eta = F(\eta)$$

$$\text{So now we have: } \bar{v} = \frac{1}{2} \frac{1}{\sqrt{Re_x}} (\eta F' - F)$$

Putting everything together we get:

$$F''' + \frac{1}{2} F F'' = 0; \quad \eta = y \sqrt{\frac{V_\infty}{\nu x}}$$

$$\text{At the wall: } \eta = 0, \quad F' = F = 0$$

$$\text{Outside the b.l: } \eta \rightarrow \infty, \quad F' = 1$$

Rewriting our equation:

$$\frac{\partial^2 \bar{u}}{\partial \eta^2} + \frac{1}{2} F \frac{\partial \bar{u}}{\partial \eta} = 0$$

To solve, we need to assume an infinite series solution

$$F = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + \dots +$$

$$F' = a_1 + 2a_2 \eta + 3a_3 \eta^2 + \dots +$$

$$F'' = 2a_2 + 6a_3 \eta + \dots +$$

$$F''' = 6a_3 + \dots +$$

Back substitute and solve for your coefficients

$$F''' + \frac{1}{2} F F'' = 0$$

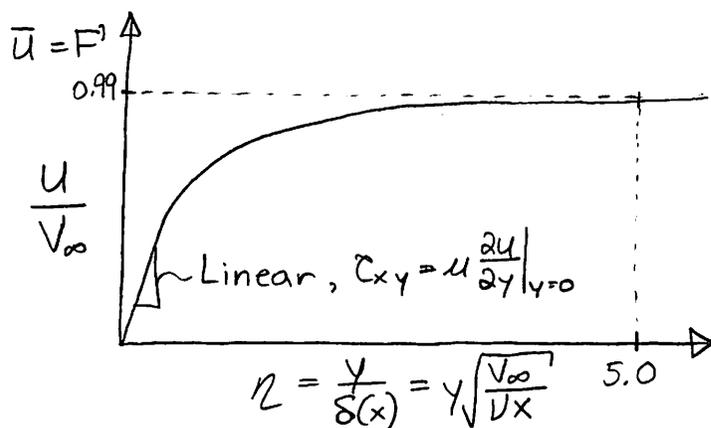
$$\underbrace{(\dots)}_0 \eta^0 + \underbrace{(\dots)}_0 \eta^1 + \underbrace{(\dots)}_0 \eta^2 + \underbrace{(\dots)}_0 \eta^3 + \dots + \underbrace{(\dots)}_0 \eta^n = 0$$

You obtain a recursion formula relating your constants

$$F = \frac{a_2 \eta^2}{2!} - \frac{a_2 \eta^5}{2 \cdot 5!} + \frac{11}{4} \frac{a_3^3 \eta^8}{8!} + \dots$$

$a_2 = 0.332$ \Rightarrow Heinrich Blasius solved this in 1911 for his Ph.D. work with Ludwig Prandtl.

$$\text{So } F = \int_0^\eta \phi d\eta, \quad F' = \phi = \bar{u} = \frac{u}{V_\infty}$$



$$5.0 = \delta \sqrt{\frac{V_\infty}{\nu x}} = \frac{\delta}{x} \sqrt{Re_x}$$

$$\delta = \frac{5x}{\sqrt{Re_x}}$$

\hookrightarrow Hydrodynamic boundary layer thickness.

This is very usefull because now we can calculate things like shear stress

$$\begin{aligned} \tau(x) &= \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \rho \nu V_\infty \left. \frac{\partial \bar{u}}{\partial y} \right|_{y=0} \\ &= \rho \nu V_\infty \left. \frac{\partial \bar{u}}{\partial \eta} \right|_{\eta=0} \cdot \left(\frac{\partial \eta}{\partial y} \right) \\ &= \rho \nu V_\infty a_2 \sqrt{\frac{V_\infty}{x \nu}} \end{aligned}$$

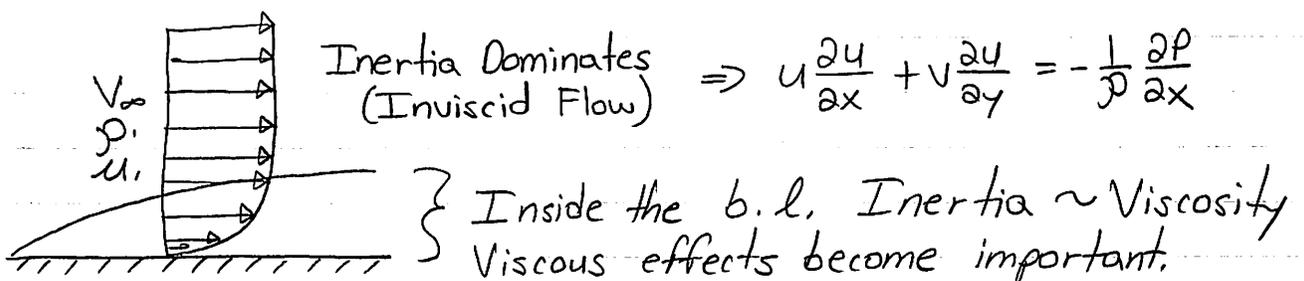
$$\boxed{\frac{\tau(x)}{\frac{1}{2} \rho V_\infty^2} = \frac{2a_2}{\sqrt{Re_x}} = \frac{0.664}{Re_x^{1/2}}} \Rightarrow \text{Skin friction coefficient for a flat plate in laminar flow conditions.}$$

If we want the average:

$$\begin{aligned} \bar{\tau} &= \frac{1}{L} \int_0^L \tau(x) dx \Rightarrow \tau(x) = C \frac{1}{\sqrt{x}} \\ &= C \cdot \frac{1}{L} \int_0^L \frac{dx}{\sqrt{x}} = \frac{2C}{\sqrt{L}} \end{aligned}$$

$$\boxed{C_c = \frac{\bar{\tau}}{\frac{1}{2} \rho V_\infty^2} = \frac{1.328}{Re_L^{1/2}}} \Rightarrow \text{Average skin friction coeff.}$$

So is there an easier way to see some simple things about boundary layers?



$$\rho \frac{V_\infty^2}{L} \sim \mu \frac{V_\infty}{\delta^2} \Rightarrow \delta^2 = \frac{\mu L}{\rho V_\infty} \Rightarrow \delta \sim \sqrt{\frac{\nu L}{V_\infty}} \left(\frac{L}{L} \right)$$

$$\boxed{\delta \sim \frac{L}{\sqrt{Re_L}}}$$

Example Calculate the boundary layer thickness for a jetliner.

$$V_{\infty} = 400 \text{ miles/hour (177 m/s) (Plane speed)}$$

$$\nu_{\text{air}} = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$$

$$L = 5 \text{ m (Length of wing in the fuselage direction)}$$

$$Re_L \approx \frac{V_{\infty} L}{\nu} = \frac{(177 \text{ m/s})(5 \text{ m})}{1.5 \times 10^{-5} \text{ m}^2/\text{s}} = 5.9 \times 10^7$$

$$\delta \approx \frac{L}{\sqrt{Re_L}} = \frac{5 \text{ m}}{\sqrt{5.9 \times 10^7}}$$

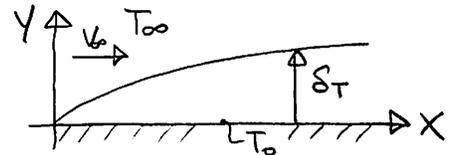
$$\boxed{\delta = 0.65 \text{ mm}} \Rightarrow \text{Less than 1 mm thick! Very hard to see.}$$

END OF LECTURE 9

Heat Transfer

Note we are looking for heat transfer on the external surface

$$h = \frac{q''_{y=0}}{\Delta T} = \frac{q''_{y=0}}{T_0 - T_{\infty}} = ?$$



For a flat plate, laminar flow: $k, \rho, \mu, C_p = \text{constant}$
 Writing out our energy equation for a fluid element, which is the heat equation but extended to allow for fluid motion

$$\underbrace{u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}}_{\text{Convection of heat}} = \underbrace{\alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{Conduction of heat}} \Rightarrow \text{Usually we have } \nabla^2 T = 0$$

$$\text{or } \nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

0 since steady.

For a boundary layer at $T_0 = T_{\text{wall}} = \text{constant}$

$$\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$$

Boundary conditions: $T(y=0) = T_0$
 $T(y \rightarrow \infty) = T_{\infty}$

Non-dimensionalizing: Let $\Theta = \frac{T - T_0}{T_{\infty} - T_0}$

$$u \frac{\partial \Theta}{\partial x} + v \frac{\partial \Theta}{\partial y} = \alpha \frac{\partial^2 \Theta}{\partial y^2} \Rightarrow \text{Note the similarity to the boundary layer eqns.}$$

Previously we had:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow \text{Hydrodynamic boundary layer}$$

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \Rightarrow \text{Thermal boundary layer}$$

$$\begin{array}{l} \text{B.C.'s : } \theta(y=0) = 0 \\ \theta(y \rightarrow \infty) = 1 \\ \left. \frac{\partial \theta}{\partial y} \right|_{y \rightarrow \infty} = 0 \end{array} \quad \text{vs.} \quad \begin{array}{l} \bar{u}(y=0) = 0 \\ \bar{u}(y \rightarrow \infty) = 1 \\ \left. \frac{\partial \bar{u}}{\partial y} \right|_{y \rightarrow \infty} = 0 \end{array}$$

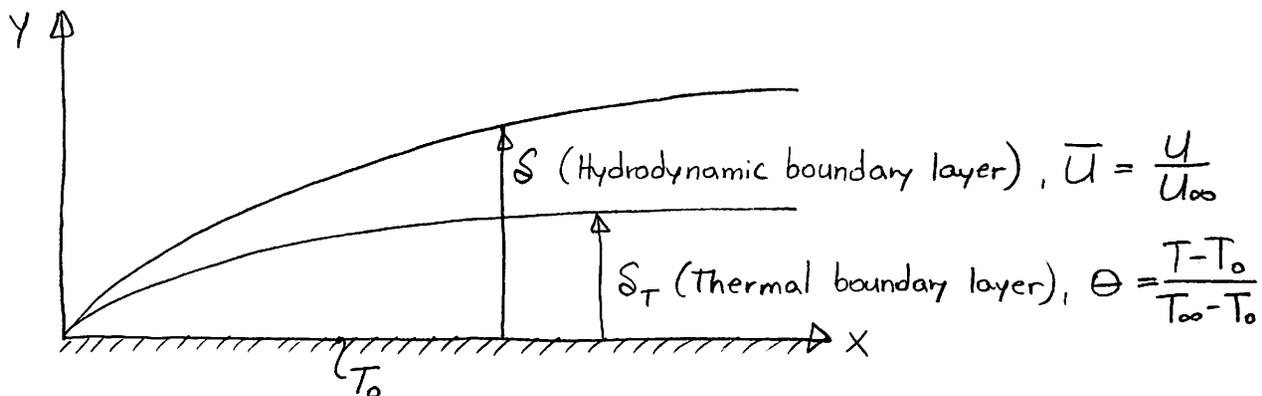
So our solution is already solved. Just use the b.l. soln's.

Here, we can define a usefull quantity called the Prandtl number

$$\text{Pr} = \frac{\nu}{\alpha} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}}$$

If $\text{Pr} = 1$, $\theta = \bar{u}$, $\delta = \delta_T \Rightarrow$ Hydrodynamic & thermal boundary layers are same.

But usually $\nu \neq \alpha$. If $\nu > \alpha$, the hydrodynamic boundary layer is thicker since you transfer momentum more efficiently than thermal energy.



Now lets solve the thermal boundary layer equation.

If $Pr \neq 1$; $\theta(\eta)$, $\eta = y \sqrt{\frac{V_\infty}{xU}}$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \quad \frac{\partial \eta}{\partial x} = \frac{\partial y}{\partial x} \sqrt{\frac{V_\infty}{xU}} = \frac{\partial y}{\partial x} \left(\frac{\partial \eta}{\partial y} \right)$$

$$\frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial^2 \theta}{\partial \eta^2} \cdot \left(\frac{\partial \eta}{\partial y} \right)^2$$

Now substituting back into our energy equation PDE,

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2} F Pr \frac{\partial \theta}{\partial \eta} = 0, \quad \text{note } F = \int_0^\eta \phi d\eta = F(\eta)$$

$$\bar{u} = \phi(\eta) = \frac{y}{V_\infty} = F'$$

To solve we usually integrate but here we can use a trick.

Let:

$$Pr^{2/3}, F(\eta) = F(\eta^*); \quad \eta^* = \eta Pr^{1/3}$$

$$\frac{\partial \eta^*}{\partial \eta} = Pr^{1/3}$$

Now our PDE becomes:

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) Pr^{1/3} \frac{\partial \theta}{\partial \eta} = 0$$

$$\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) Pr^{2/3} \underbrace{\frac{\partial \theta}{\partial \eta Pr^{1/3}}}_{\partial \eta^*} = 0$$

Multiplying through by $Pr^{-2/3}$

$$\frac{1}{Pr^{2/3}} \left(\frac{\partial^2 \theta}{\partial \eta^2} + \frac{1}{2} F(\eta^*) Pr^{2/3} \frac{\partial \theta}{\partial \eta^*} \right) = 0$$

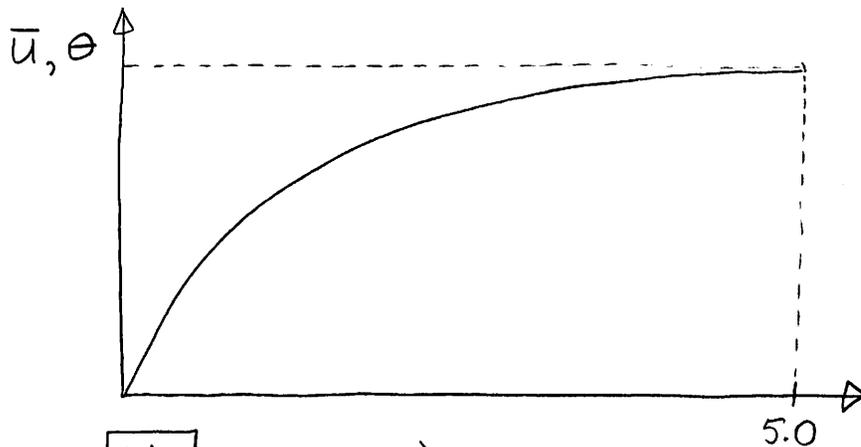
$$\frac{\partial^2 \theta}{\partial \eta^{*2}} + \frac{1}{2} F(\eta^*) \frac{\partial \theta}{\partial \eta^*} = 0$$

Now our PDE becomes identical to before

$$\frac{\partial^2 \theta}{\partial \eta^{*2}} + \frac{1}{2} F(\eta^*) \frac{\partial \theta}{\partial \eta^*} = 0 \Rightarrow \theta(\eta^*) = \bar{u}(\eta)$$

$$\text{B.C.'s: } \theta(\eta^* = 0) = 0, \quad \theta = \frac{T - T_0}{T_\infty - T_0}$$

$$\theta(\eta^* \rightarrow \infty) = 1$$



$$\delta \sqrt{\frac{V_\infty}{xU}} = \eta(y = \delta) = 5.0$$

$$Pr^{1/3} \delta_T \sqrt{\frac{V_\infty}{xU}} = \eta^*(y = \delta_T) = 5.0$$

$\eta, \eta^* = \eta Pr^{1/3}$
 \hookrightarrow Thermal boundary layer
 \hookrightarrow Hydrodynamic boundary lay.

Taking the ratio of our two boundary layer thicknesses:

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}} \Rightarrow \text{Makes sense, the only difference is } U \text{ \& } \alpha$$

$$Pr = \frac{\nu}{\alpha}$$

Now we can solve for heat transfer

$$q''_{x=0} = -k \left. \frac{\partial T}{\partial y} \right|_{y=0} \Rightarrow \text{we know } \Theta = \frac{T - T_0}{T_\infty - T_0}, \quad \partial \Theta = \frac{\partial T}{T_\infty - T_0}$$

$$q''_{x=0} = -k (T_\infty - T_0) \left(\frac{V_\infty}{xU} \right)^{1/2} Pr^{1/3} \left. \frac{\partial \Theta}{\partial \eta^*} \right|_{\eta^*=0}$$

$$\eta^* = \eta Pr^{1/3} = y \left(\frac{V_\infty}{xU} \right)^{1/2} Pr^{1/3}$$

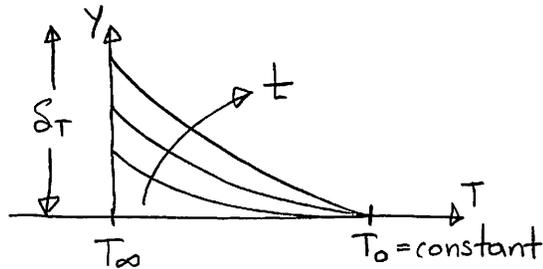
$$\partial \eta^* = \partial y \left(\frac{V_\infty}{xU} \right)^{1/2} Pr^{1/3}$$

$$= \frac{k (T_0 - T_\infty)}{x} \underbrace{\left(\frac{V_\infty x}{U} \right)^{1/2}}_{Re_x^{1/2} \Rightarrow \text{Reynolds number}} Pr^{1/3} \underbrace{F''(0)}_{a_2 = 0.332}$$

$$q''_{x=0} = \frac{k \Delta T}{x} Re_x^{1/2} Pr^{1/3} a_2$$

$$Nu_x = \frac{hx}{k} = \underbrace{\frac{q''}{\Delta T}}_h \cdot \frac{x}{k} = a_2 Re^{1/2} Pr^{1/3} \Rightarrow \boxed{Nu_x = 0.332 Re_x^{1/2} Pr^{1/3}}$$

We can rethink this as:



$$q''_{x=0} = \frac{k \Delta T}{\sqrt{\pi \alpha t}} \Rightarrow \text{Solved before for semi-infinite conduction. Check Lecture 8, Page 66 of my notes.}$$

$$Nu_x = \left(\frac{q''}{\Delta T} \cdot \frac{x}{k} \right) = \frac{x}{\sqrt{\pi \alpha t}}$$

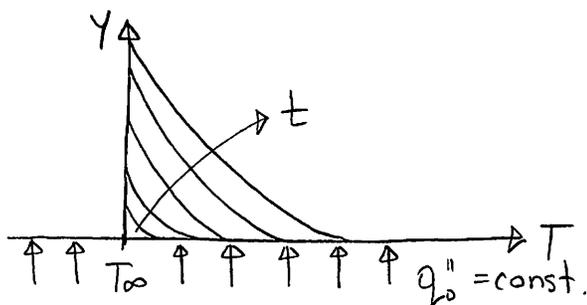
we know $t = \frac{x}{V_\infty}$

$$Nu_x = \frac{1}{\sqrt{\pi}} \cdot \left(\frac{V_\infty x}{\alpha} \right)^{1/2} \Rightarrow \boxed{Nu_x = \frac{1}{\sqrt{\pi}} \cdot Re_x^{1/2} Pr^{1/2}}$$

↳ For constant wall temperature, T_0 , $Pr \ll 1$

Now if we have a constant heat flux: ($Pr \ll 1$)

$$q''_0 = \text{constant}$$



We know from our previous solution that: (Lecture 8, Page 67)

$$T - T_\infty = \frac{q''_0}{k} \left(\frac{4 \alpha t}{\pi} \right)^{1/2} e^{-\frac{x^2}{4 \alpha t}} - x \operatorname{erfc} \left(\frac{x}{2 \sqrt{\alpha t}} \right) \Rightarrow \text{Evaluate at } x=0$$

$$\underbrace{T_0 - T_\infty}_{\Delta T} = \frac{q''_0}{k} \left(\frac{4 \alpha t}{\pi} \right)^{1/2}$$

$$Nu_x = \left(\frac{q_0''}{\Delta T} \cdot \frac{x}{k} \right) = \frac{4}{\sqrt{\pi}} \cdot \frac{x}{\sqrt{\alpha t}} = \frac{4}{\sqrt{\pi}} \cdot \left(\frac{V_\infty x}{\alpha} \right)^{1/2}, \quad t = \frac{x}{V_\infty}$$

$$\boxed{Nu_x = \frac{4}{\sqrt{\pi}} \cdot Re^{1/2} Pr^{1/2}} \Rightarrow \text{Constant heat flux, } q_0'', \quad Pr \ll 1$$

Note that for these solutions, we've already defined $Pe = Re Pr$

$$Pe_x = \frac{V_\infty x}{\alpha} = \frac{\rho C_p V_\infty \Delta T}{k \Delta T} = \frac{\text{heat storage rate in the b.l.}}{\text{heat conductance through the b.l.}}$$

$$\boxed{Nu_x \propto Pe^{1/2}}$$

Average Heat Transfer Coefficient (\bar{h})

$$\bar{h} = \frac{\bar{q}''}{\Delta T} \quad \text{or} \quad \bar{h} = \frac{q''}{\Delta T}$$

Const. wall T.

Const. heat flux problems

Uniform Wall Temperature:

$$\bar{h} = \frac{\bar{q}''}{\Delta T} = \frac{1}{\Delta T} \left[\frac{1}{L} \cdot \int_0^L q'' dx \right] = \frac{1}{L} \int_0^L h(x) dx$$

Uniform Heat Flux

$$\bar{h} = \frac{q''}{\Delta T} = \frac{q''}{\frac{1}{L} \int_0^L \Delta T(x) dx}$$

The Nusselt number based on \bar{h} and L is \overline{Nu}_L

This is not the average of Nu_x .

For a flat plate case: ($x_0 = x(0) = 0$)

$$\bar{h} = \frac{1}{L} \int_0^L \underbrace{h(x)}_{\frac{k}{x} Nu_x} dx = \frac{0.332 k Pr^{1/3} \sqrt{V_\infty}}{L \sqrt{\nu}} \cdot \int_0^L \frac{\sqrt{x}}{x} dx = 0.664 Re_L^{1/2} Pr^{1/3} \left(\frac{k}{L} \right)$$

So now we see: $\bar{h} = 2h(x=L)$ in laminar flow

$$\boxed{\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.664 Re_L^{1/2} Pr^{1/3}}$$

For $Pr \ll 1$:

$$\boxed{\overline{Nu}_L = 1.13 Pe_L^{1/2}}, \quad \boxed{Pe_L = Re_L Pr}$$

Some Observations and Notes

Previous results are valid under the following conditions:

- 1) Re_x or $Re_L < 5.0 \times 10^5$ (Laminar flow)
- 2) $Ma = \frac{V_\infty}{\text{sound speed}} < 0.3$ (Incompressible flow)
- 3) $Ec = \text{Eckert number} = \frac{V_\infty^2}{c_p(T_0 - T_\infty)} \ll 1$ (Viscous dissipation heating is negligible)
 $= \frac{\text{Kinetic Energy}}{\text{Enthalpy}}$

The higher the kinetic energy, the larger the effect of viscous dissipation.

- 4) We have always assumed that properties are constant. Need to evaluate properties at the average temperature of the boundary layer, or the film temperature:

$$T_f = \frac{T_0 + T_\infty}{2}$$

- 5) h or $\bar{h} \propto \frac{1}{\sqrt{x}}$ or $\frac{1}{\sqrt{L}}$, $Nu_x \propto \sqrt{x}$

Thus $h \rightarrow \infty$ and $Nu_x \rightarrow 0$ at $x \rightarrow 0$. Of course, $h \rightarrow \infty$ will not occur at $x \rightarrow 0$ since the b.l. model breaks down at $x=0$

D'Alembert's Paradox (Back to it)

Note in our derivation, we mentioned D'Alembert's paradox. The drag force goes to zero on a body moving through a fluid at high Reynolds number.

What is the source of the problem & how was it solved? Well, let's take a look at what we did.

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{u} = \frac{u}{V_\infty}, \quad \bar{v} = \frac{v}{V_\infty}, \quad \bar{p} = \frac{p}{\rho V_\infty^2}$$

X-momentum:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} \cdot \frac{1}{\rho} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

If we non-dimensionalize like before

$$\bar{u} V_\infty \frac{\partial \bar{u} \cdot V_\infty}{\partial \bar{x} \cdot L} + \bar{v} V_\infty \frac{\partial \bar{u} \cdot V_\infty}{\partial \bar{y} \cdot L} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \nu \frac{V_\infty^2}{L} \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]$$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} \cdot \frac{V_\infty^2}{L} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \frac{V_\infty^2}{L} = -\frac{V_\infty^2}{L} \frac{\partial \bar{p}}{\partial \bar{x}} + \nu \frac{V_\infty^2}{L^2} \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]$$

Divide through by $\frac{V_\infty^2}{L}$ on both sides

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \underbrace{\frac{\nu V_\infty^2}{V_\infty^2 L^2}}_{\frac{\nu}{V_\infty L}} \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]$$

$$\frac{\nu}{V_\infty L} = \frac{\mu}{\rho V_\infty L} = \frac{1}{Re_L}$$

$$\boxed{\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re_L} \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]}$$

Herein lies the problem. As $Re_L \rightarrow \infty$, our viscous terms completely drop out, so viscosity is negligible.

If the viscous terms are negligible, the potential flow solution is valid and appropriate, which says there is no drag on the body. We all know this is wrong since drag increases with Re_L .

So where did we go wrong?

Turns out hydrodynamicists back then were not aware of the boundary layer, and scaled the N-S equation incorrectly just like we did.

What we should have used is: $V_s = \frac{\delta}{L} V_\infty$, δ is critical y -dim.

$$\bar{u} = \frac{u}{V_\infty}, \quad \bar{v} = \frac{v}{V_s} = \left(\frac{L}{\delta}\right) \frac{v}{V_\infty}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{\delta}, \quad \bar{p} = \frac{p}{\rho V_\infty^2}$$

This is the correct scaling
This is the correct scaling.

So let's try this again: $\partial u = V_\infty \partial \bar{u}$; $\partial v = \frac{\delta}{L} V_\infty \partial \bar{v}$, $\partial x = L \partial \bar{x}$, $\partial y = \delta \partial \bar{y}$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} \left(V_\infty^2 \frac{1}{L} \right) + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} \left(V_\infty \left(\frac{\delta}{L} \right) V_\infty \cdot \frac{1}{\delta} \right) = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \bar{x}} \cdot \frac{\rho V_\infty^2}{L}$$

$$+ \nu \left[\frac{V_\infty}{L^2} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{V_\infty}{\delta^2} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]$$

Dividing through by $\left(\frac{V_\infty^2}{L}\right)$ on both sides

$$\boxed{\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{Re_L} \left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \left(\frac{L^2}{\delta^2}\right) \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right]}$$

↳ This is the correct scaled equation.

We can see that if we assume $\frac{\delta}{L} \ll 1$, $\Rightarrow \frac{L}{\delta} \gg 1$

As $Re_L \rightarrow \infty$, we can't say much about the last viscous term.

Now if we wanted an order of magnitude, we can use our correct scaled equation to say some things:

$$\bar{u} = \frac{u}{V_\infty} \sim \frac{V_\infty}{V_\infty} \sim O(1), \quad \bar{v} = \left(\frac{L}{\delta}\right) \frac{v}{V_\infty} \sim \left(\frac{L}{\delta}\right) \left(\frac{\delta}{L}\right) \frac{V_\infty}{V_\infty} \sim O(1)$$

$$\bar{x} = \frac{x}{L} \sim \frac{L}{L} \sim O(1), \quad \bar{y} = \frac{y}{\delta} \sim \frac{\delta}{\delta} \sim O(1)$$

$$\bar{p} = \frac{p}{\rho V_\infty^2} \sim \frac{\rho V_\infty^2}{\rho V_\infty^2} \sim O(1)$$

$$\underbrace{\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}}}_{\sim O(1)} + \underbrace{\bar{v} \frac{\partial \bar{v}}{\partial \bar{y}}}_{\sim O(1)} = - \underbrace{\frac{\partial \bar{p}}{\partial \bar{x}}}_{\sim O(1)} + \frac{1}{Re_L} \left[\underbrace{\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}}_{\sim O(1)} + \left(\frac{L}{\delta}\right)^2 \underbrace{\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}_{\sim O(1)} \right]$$

For this equation to balance, which it needs to for the boundary layer approximation (where viscosity \sim inertia)

$$\text{Then: } \frac{1}{Re_L} \left(\frac{L}{\delta}\right)^2 \sim O(1)$$

$$\frac{L^2}{\delta^2} = Re_L$$

$$\frac{\delta}{L} = \frac{1}{Re_L} \Rightarrow \boxed{\delta \sim \frac{L}{\sqrt{Re_L}}} \Rightarrow \text{Boundary layer thickness}$$

So we see how Prandtl resolved D'Alembert's paradox by elucidating the boundary layer concept.

So now we can write down the more generalized boundary layer equations (without the flat plate assumptions)
 We know $\frac{\delta}{L} \ll 1$, this helps us simplify our N-S eqn.

$$\underbrace{\bar{u} \frac{\partial \bar{u}}{\partial x}}_{\sim O(1)} + \underbrace{\bar{v} \frac{\partial \bar{v}}{\partial y}}_{\sim O(1)} = - \underbrace{\frac{\partial \bar{p}}{\partial x}}_{\sim O(1)} + \underbrace{\frac{1}{Re_L} \frac{\partial^2 \bar{u}}{\partial x^2}}_{\sim O\left(\frac{1}{Re_L}\right)} + \underbrace{\frac{L^2}{\delta^2} \frac{1}{Re_L} \frac{\partial^2 \bar{u}}{\partial y^2}}_{\sim O(1)}$$

$$Re_L \sim \frac{L^2}{\delta^2}$$

So our x-viscosity term becomes: $\frac{\delta^2}{L^2} \frac{\partial^2 \bar{u}}{\partial x^2} \ll 1$, since $\frac{\delta}{L} \ll 1$

Therefore, our equation simplifies to:

$$\boxed{\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{v}}{\partial y} = - \frac{\partial \bar{p}}{\partial x} + \frac{L^2}{\delta^2} \frac{1}{Re_L} \frac{\partial^2 \bar{u}}{\partial y^2}} \Rightarrow \text{x-momentum}$$

Now we can look at y-momentum:

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} \cdot \frac{1}{\rho} + \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$$

Non-dimensionalizing:

$$V_\infty^2 \bar{u} \left(\frac{\delta}{L} \right) \left(\frac{1}{L} \right) \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \left(\frac{\delta}{L} \right) V_\infty^2 \left(\frac{1}{\delta} \right) \frac{\partial \bar{v}}{\partial \bar{y}} = - \frac{\rho V_\infty^2}{\rho \delta} \frac{\partial \bar{p}}{\partial \bar{y}} + \nu_\infty \frac{\delta}{L} \left[\frac{1}{L^2} \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{1}{\delta^2} \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right]$$

$$\left(\frac{\delta}{L} \right) \frac{V_\infty^2}{L} \left[\bar{u} \frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{v}}{\partial \bar{y}} \right] = - \frac{V_\infty^2}{\delta} \frac{\partial \bar{p}}{\partial \bar{y}} + \nu_\infty \frac{\delta}{L} \left[\frac{1}{L^2} \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{1}{\delta^2} \frac{\partial^2 \bar{v}}{\partial \bar{y}^2} \right]$$

We know that $\frac{\delta}{L} \ll 1$, so both inertia & viscosity ~ 0

$$\cancel{\frac{V_\infty^2}{\delta} \frac{\partial \bar{p}}{\partial \bar{y}}} = 0 \Rightarrow \frac{\partial \bar{p}}{\partial \bar{y}} = 0, \quad \boxed{\bar{p} = f(\bar{y}) = \text{constant}}$$

↳ y-momentum

And now for continuity: $\bar{u} = \frac{u}{V_\infty}$, $\bar{v} = \frac{v}{V_s} = \frac{L}{\delta} \frac{v}{V_\infty}$, $\bar{x} = \frac{x}{L}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

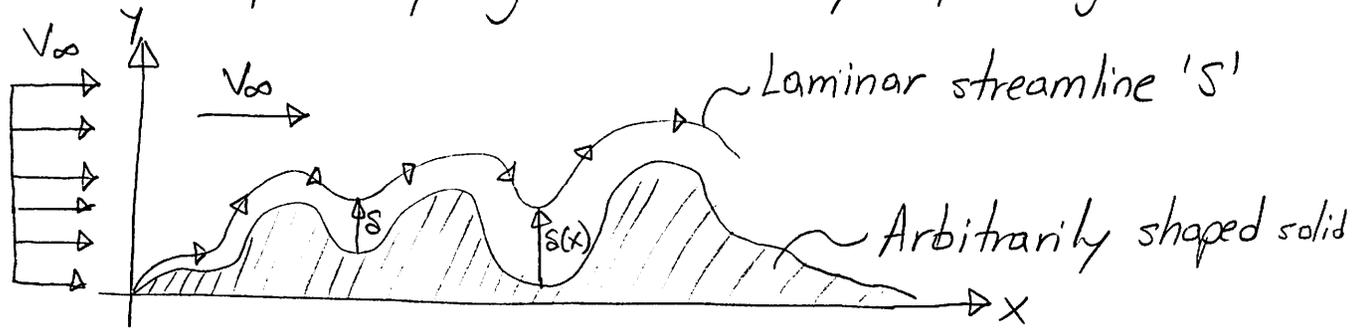
$$\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial u}{V_\infty} = \frac{\partial u}{\partial x} \frac{1}{L}, \quad \frac{\partial \bar{v}}{\partial \bar{y}} = \frac{\partial v}{V_s} = \frac{\partial v}{\partial y} \frac{L}{\delta V_\infty}, \quad \frac{\partial \bar{x}}{\partial x} = \frac{1}{L}$$

Non-dimensionalizing

$$\frac{V_\infty \partial \bar{u}}{L \partial \bar{x}} + \left(\frac{\delta}{L}\right) \frac{V_\infty \partial \bar{v}}{\delta \partial \bar{y}} = 0$$

$$\frac{V_\infty}{L} \left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} \right) = 0 \Rightarrow \boxed{\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0} \Rightarrow \text{Continuity}$$

Now finally analyzing the boundary layer edge streamline



At 's' $\Rightarrow \frac{\partial u}{\partial y} = 0$, $u = V_\infty$, viscosity ~ 0 (potential flow)

So our N-S equation on the streamline is:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

$$u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \Rightarrow \boxed{\frac{V_\infty}{V_\infty} \frac{\partial V_\infty}{\partial x} = -\frac{\partial p}{\partial x}} \Rightarrow \text{Non-dimensional form. } V_\infty = \frac{u}{V_\infty} \Big|_{y=s}$$

$$\boxed{V_\infty \frac{\partial V_\infty}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}} \Rightarrow \text{Solves for } \frac{\partial p}{\partial x} \text{ in x-momentum.}$$

So we have 4 generalized boundary layer equations for laminar flow over any solid body: (in dimensional form)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$p(y) = \text{constant} \quad (2)$$

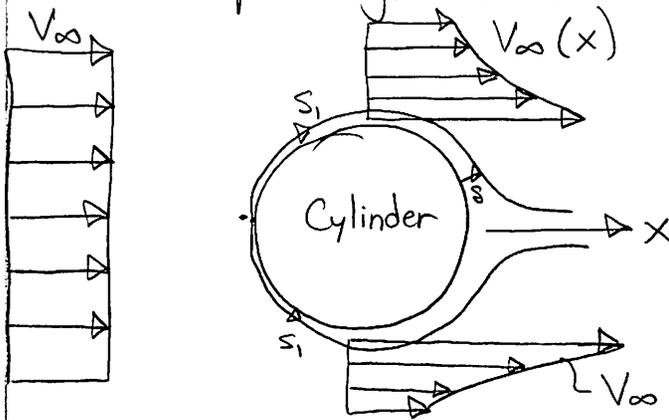
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad (3)$$

$$V_\infty \frac{\partial V_\infty}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 \quad (4)$$

Sometimes (1) is written with (4) as:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = V_\infty \frac{\partial V_\infty}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (5)$$

You can see that the above equations are very cumbersome and difficult to solve. No analytical solutions exist for more complex geometries other than a flat plate. For ex:



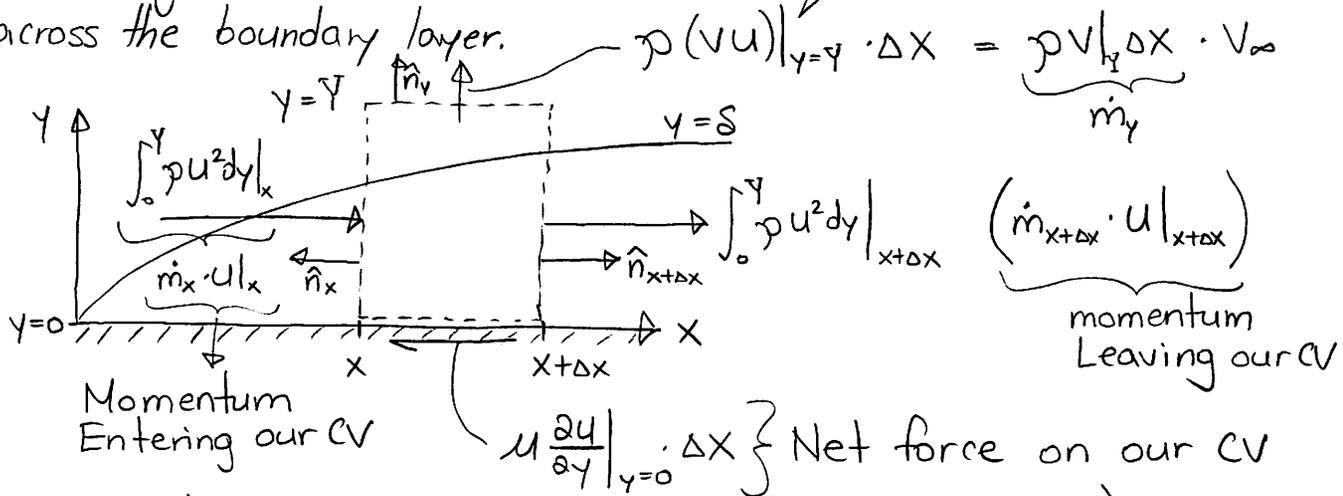
This is a case where equation (4) has to be solved and we cannot assume $\frac{\partial V_\infty}{\partial x} = 0$.

V_∞ changes with x-coordinate.

Thankfully, a dude by the name of Theodore von Kármán found a much simpler way to solve these problems. He just integrated across the boundary layer to solve the conservation of momentum and energy. His technique is called the momentum integral technique.

Integral Technique (Very usefull for arbitrary shapes where N-S not)

The integral form of the momentum equation can be derived across the boundary layer.



Our momentum eqn. becomes: (From Control Volume analysis)

$$\underbrace{\int_0^Y \rho u^2 dy|_{x+\Delta x} + \rho v u|_Y \Delta x - \int_0^Y \rho u^2 dy|_x}_{\text{net momentum outflow (m}\ddot{a}\text{)}} = \underbrace{-\mu \frac{\partial u}{\partial y}|_0}_{\Sigma F} \Delta x$$

Divide through by $\rho \cdot \Delta x$ and let $\Delta x \rightarrow 0$ (Do a Taylor series expansion as well)

$$\frac{\partial}{\partial x} \int_0^Y u^2 dy + v u|_Y = -\nu \frac{\partial u}{\partial y}|_0$$

We know $u|_Y = V_\infty$, and from continuity: $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$

$$v(Y) = v|_0 - \int_0^Y \frac{\partial u}{\partial x} dy = - \int_0^Y \frac{\partial u}{\partial x} dy \Rightarrow \text{Substitute back}$$

$$\frac{d}{dx} \int_0^Y u^2 dy - \int_0^Y V_\infty \frac{\partial u}{\partial x} dy = -\nu \frac{\partial u}{\partial y}|_0 \Rightarrow \text{We can re-write this as: let } \delta = Y \text{ since as } y > \delta, \text{ our integral} = 0 \text{ so not impor.}$$

$$\frac{d}{dx} \int_0^\delta u(V_\infty - u) dy = \nu \left(\frac{\partial u}{\partial y} \right)_0 \Rightarrow \text{Momentum integral}$$

If we non-dimensionalize our momentum integral

$$\bar{u} = \frac{u}{V_\infty} = \phi(\eta), \quad \eta = \frac{y}{\delta}$$

$$\frac{d}{dx} \int_0^\delta \bar{u}(1-\bar{u}) dy = \frac{u}{V_\infty} \cdot \frac{\partial \bar{u}}{\partial y} \Rightarrow d\eta = \frac{dy}{\delta}, \quad \bar{u} = \phi \text{ (back substitute)}$$

$$\frac{\partial \delta}{\partial x} \int_0^1 \phi(1-\phi) d\eta = \frac{u}{V_\infty \delta} \left(\frac{\partial \phi}{\partial \eta} \right) \Big|_0$$

$$\delta \frac{\partial \delta}{\partial x} = \frac{u}{V_\infty} \beta, \quad \beta = \frac{\phi'(0)}{\int_0^1 \phi(1-\phi) dy}$$

Integrating, we obtain:

$$\frac{\delta^2}{2} = \frac{u x}{V_\infty} \cdot \beta$$

$$\delta^2 = \frac{u x}{V_\infty} 2\beta$$

$$\left(\frac{\delta}{x} \right)^2 = \frac{2\beta}{Re_x} \Rightarrow \frac{\delta}{x} = \frac{\sqrt{2\beta'}}{\sqrt{Re_x}} \Rightarrow \text{Note same form as before but } 5.0 = \sqrt{2\beta'}$$

The way to solve the momentum integral equation is to assume the shape of a velocity profile. Developed by von Karman (another of Prandtl's students). We don't care what happens in the b.l.
Looking at our B.C.'s

$$\left. \begin{aligned} \phi(0) = \bar{u}(0) = \frac{u}{V_\infty} \Big|_0 = 0 \\ \phi(1) = 1 \\ \phi'(1) = 0 \Rightarrow \frac{\partial \phi}{\partial \eta} = 1 \neq 0 \times \\ \phi''(0) = 0 \Rightarrow \frac{\partial^2 \phi}{\partial \eta^2} = -2 \times \end{aligned} \right\} \phi = \eta \quad \left. \begin{aligned} \phi = 2\eta - \eta^2 \\ \phi = \sin\left(\frac{\pi}{2}\eta\right) \\ \text{or} \\ \phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \end{aligned} \right\}$$

Since $\phi'(0) = \text{constant}$ (constant wall shear stress) Satisfies all B.C.'s

Now we can solve for $\sqrt{2\beta'}$, we obtain

ϕ	$\sqrt{2\beta'}$
η	3.464
$2\eta - \eta^2$	5.477
$\sin\left(\frac{\pi}{2}\eta\right)$	4.795
$\frac{3}{2}\eta - \frac{1}{2}\eta^3$	4.641

Note all of our solutions for $\sqrt{2\beta'}$ are close to 5.0 that we got before. So we chose our ϕ based on our B.C.'s. Let's say we wanted to solve for shear:

$$\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3$$

$$\tau(x) = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{\rho \nu V_\infty}{\delta} \cdot \frac{\partial \bar{u}}{\partial \eta} = \frac{3}{2} \cdot \frac{\nu V_\infty}{\left(\frac{\delta}{x}\right)x}$$

$$\bar{u} = \frac{u}{V_\infty}, \quad \eta = \frac{y}{\delta}$$

$$\partial u = V_\infty \partial \bar{u}, \quad \partial y = \delta \partial \eta$$

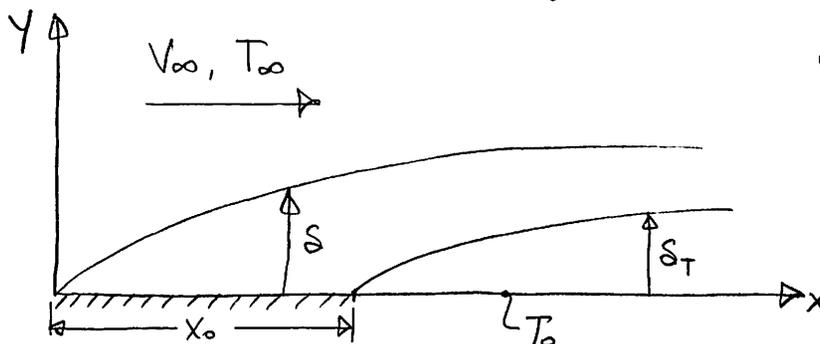
$$\frac{\tau(x)}{\frac{1}{2}\rho V_\infty^2} = \frac{3\nu}{V_\infty \left(\frac{\delta}{x}\right)x} = \frac{3}{Re_x \left(\frac{\delta}{x}\right)} = \frac{3}{\sqrt{2\beta'}} \cdot \frac{1}{\sqrt{Re_x}}$$

$$C_{f,x} = \frac{\tau(x)}{\frac{1}{2}\rho V_\infty^2} = \frac{0.646}{Re_x^{1/2}}$$

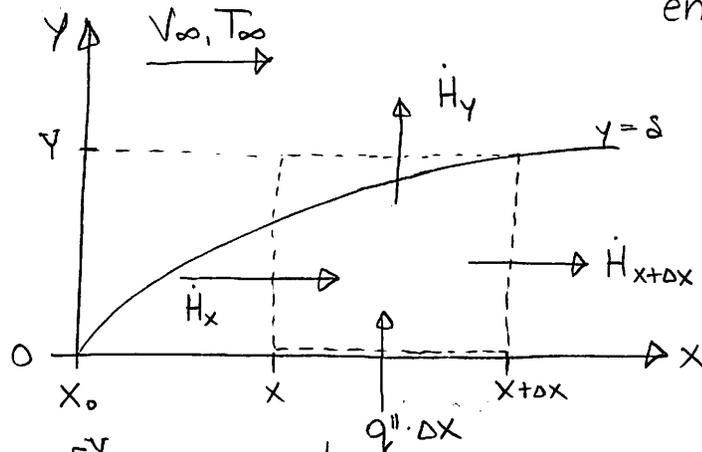
\Rightarrow Before we had $\frac{0.664}{Re_x^{1/2}}$
Very close to exact sol'n!

Energy Integral:

What if we had the following problem: (Difficult to solve using analytical technique developed before)



Looking at our control volume in the thermal b.l., and considering energy flow in and out:



$$\dot{H}_x = \int_0^y \rho c_p u T dy \Big|_{x=x_0} ; \quad \dot{H}_y = \rho v c_p T \Big|_y \cdot \Delta x$$

$$\dot{H}_{x+\Delta x} + \dot{H}_y - \dot{H}_x = q'' \Delta x$$

$$\int_0^y \rho c_p u T dy \Big|_{x+\Delta x} + \rho v c_p T \Big|_y \Delta x - \int_0^y \rho c_p u T dy \Big|_x = -k \frac{\partial T}{\partial y} \Big|_0 \Delta x$$

Divide through by $\rho c_p \Delta x$ and let $\Delta x \rightarrow 0$ (Do Taylor series exp)

$$\frac{d}{dx} \int_0^y u T dy + v T \Big|_y = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

We know $T \Big|_y = T_\infty$, and from continuity: $v(y) = -\int_0^y \left(\frac{\partial u}{\partial x} \right) dy$
So we have:

$$\frac{d}{dx} \int_0^y u T dy - \int_0^y T_\infty \frac{\partial u}{\partial x} dy = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

Rearranging, we obtain:

$$\boxed{\frac{d}{dx} \int_0^y u (T - T_\infty) dy = -\alpha \frac{\partial T}{\partial y} \Big|_0} \Rightarrow \text{Energy integral}$$

Integrating from 0 to δ_T (the integral above δ_T is zero, so below δ_T is the only useful solution)

$$\frac{d}{dx} \int_0^{\delta_T} \bar{u} \left(\frac{T - T_\infty}{T_0 - T_\infty} \right) dy = - \frac{\alpha}{V_\infty} \left. \frac{\partial \left(\frac{T - T_\infty}{T_0 - T_\infty} \right)}{\partial y} \right|_0 ;$$

Let:

$$\theta = \frac{T - T_0}{T_\infty - T_0} = \theta(\eta_T)$$

$$\eta_T = \eta Pr^{1/3} = \frac{y}{\delta_T}$$

we see that:

$$\frac{T - T_\infty}{T_0 - T_\infty} = 1 - \frac{T - T_0}{T_\infty - T_0} = 1 - \theta$$

$$\frac{d}{dx} \int_0^{\delta_T} \bar{u} (1 - \theta) dy = \frac{\alpha}{V_\infty} \left. \frac{\partial \theta}{\partial y} \right|_0 \quad (1)$$

So our boundary conditions are:

$$\theta(0) = \frac{T_0 - T_0}{T_\infty - T_0} = 0$$

$$\theta(1) = \theta(\eta_T = 1) = 1 = \frac{T_\infty - T_0}{T_\infty - T_0}$$

$$\theta'(1) = 0 \Rightarrow \text{No heat transfer at or above } \delta_T$$

$$\theta''(0) = 0 \Rightarrow \text{Linear temperature profile at } y=0 \Rightarrow \left. \frac{\partial \theta}{\partial y} \right|_{y=0} = \text{const.}$$

We know $\eta_T = \frac{y}{\delta_T} \Rightarrow d\eta = \frac{1}{\delta} dy \Rightarrow$ Back substituting into (1)

$$\frac{d}{dx} (\delta_T) \int_0^1 \bar{u} (1 - \theta) d\eta_T = \frac{\alpha}{V_\infty} \theta'(0)$$

Using a clever mathematical trick: (Assume the following solution)

$$\bar{u} \frac{\delta}{\delta_T} = \theta(\eta^*), \quad \eta^* = \eta Pr^{1/3} \text{ (From before)}$$

This assumption is OK since: (check)

$$\bar{u} = \phi = \frac{3}{2} \eta - \frac{1}{2} \eta^3 = \frac{3}{2} \frac{y}{\delta} - \frac{1}{2} \left(\frac{y}{\delta} \right)^3$$

Multiply through by $\frac{\delta}{\delta_T}$ on both sides:

$$\theta = \frac{3}{2} \left(\frac{y}{\delta_T} \right) - \frac{1}{2} \left(\frac{y^3}{\delta^2 \delta_T} \right)$$

We can neglect this term since

at $y \Rightarrow$ small, $y \ll \delta_T$
 $y^3 \rightarrow 0$, integral goes to 0.
 at $y \Rightarrow$ large, $y \rightarrow \delta_T$
 $(1 - \theta) \rightarrow 0$, so our integral goes to 0.

Now our solution becomes: $\theta(\eta^*) = \frac{3}{2} \left(\frac{\eta}{\delta_T} \right)$

$$\frac{d}{dx} \left(\frac{\delta_T^2}{\delta} \right) \int_0^1 \theta(1-\theta) d\eta_T = \frac{\alpha}{U \delta_T} \left(\frac{U}{V_\infty} \theta'(0) \right) \Rightarrow \theta(\eta^*) = \bar{u} \frac{\delta}{\delta_T}$$

$$\bar{u} = \frac{\delta_T}{\delta} \theta$$

$$\boxed{\frac{d}{dx} \left(\frac{\delta_T^2}{\delta} \right) = \frac{1}{Pr \cdot \delta_T} \left[\frac{U}{V_\infty} \cdot \frac{\theta'(0)}{\int_0^1 \theta(1-\theta) d\eta} \right]}$$

But note that this is nothing but:

$$\delta_T \frac{d}{dx} \left(\frac{\delta_T^2}{\delta} \right) = \frac{1}{Pr} \left(\frac{U}{V_\infty} \cdot \beta \right) \Rightarrow \text{We defined } \beta \text{ before (pg. 101)}$$

$$\delta_T \frac{d}{dx} \left(\frac{\delta_T^2}{\delta} \right) = \frac{1}{Pr} \cdot \delta \frac{\partial \delta}{\partial x} \quad (1)$$

$= \beta \left(\frac{U}{V_\infty} \right) \Rightarrow$ Derived this previously

Doing another math trick: (Divide both sides of (1) by $\delta^{1/2}$)

$$\frac{\delta_T}{\delta^{1/2}} \frac{d}{dx} \left(\frac{\delta_T^2}{\delta} \right) = \frac{1}{Pr} \delta^{1/2} \frac{\partial \delta}{\partial x}$$

$$\int_{x_0}^x \frac{\delta_T}{\delta^{1/2}} \frac{d}{dx} \left(\frac{\delta_T^2}{\delta} \right) = \int_{x_0}^x \frac{1}{Pr} \delta^{1/2} \frac{\partial \delta}{\partial x} \Rightarrow \text{Integrate both sides:}$$

$$\frac{2}{3} \left(\frac{\delta_T^2}{\delta} \right)^{3/2} \Big|_{x_0}^x = \frac{1}{Pr} \cdot \frac{2}{3} \delta^{3/2} \Big|_{x_0}^x$$

$$\left(\frac{\delta_T^2}{\delta} \right)^{3/2} = \frac{1}{Pr} \cdot \delta^{3/2} \left[1 - \left(\frac{\delta_0}{\delta} \right)^{3/2} \right]$$

But we know in our problem that at $x = x_0$, $\delta_T = 0$, and $\delta_0 = \underbrace{C \sqrt{x}}$

$$\text{since } \frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \Rightarrow \delta \sim C \sqrt{x}$$

Aside:

$$\int_{x_0}^x \frac{\delta_T}{\delta^{3/2}} d(\delta_T^2)$$

$$= \int_{x_0}^x \frac{\delta_T}{\delta^{3/2}} 2\delta_T d\delta_T$$

$$= \int_{x_0}^x \frac{2\delta_T^2}{\delta^{3/2}} d\delta_T = \frac{2}{3} \frac{\delta_T^3}{\delta^{3/2}} \Big|_{x_0}^x$$

$$\left(\frac{\delta_T}{\delta}\right)^3 = \frac{1}{Pr} \left(1 - \left(\frac{x_0}{x}\right)^{3/4}\right)$$

$$\frac{\delta}{\delta_T} = \frac{Pr^{1/3}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}$$

⇒ Note, we obtain this analytical profile without assuming anything about the δ_T profile.

So now we can calculate the heat transfer:

$$q''_{y=0} = -k \left. \frac{\partial T}{\partial y} \right|_0 = -\frac{k}{\delta_T} \cdot \frac{\partial \left(\frac{T-T_0}{T_\infty-T_0}\right)}{\partial \eta_T} \cdot \underbrace{(T_\infty-T_0)}_{\Delta T}$$

$$\theta = \theta(\eta_T)$$

$$\eta_T = \frac{y}{\delta_T}$$

$$d\eta_T = \frac{dy}{\delta_T}$$

For $x > x_0$

$$q''_{y=0} = \frac{k\Delta T}{\delta_T} \theta'(0) = \frac{k\Delta T}{x} \cdot \frac{\theta'(0)}{\left(\frac{\delta_T}{\delta}\right)\left(\frac{\delta}{x}\right)} \Rightarrow \text{We know } \frac{\delta_T}{\delta}$$

$$q''_{y=0} = \left(\frac{k\Delta T}{x}\right) \frac{Pr^{1/3} Re_x^{1/2} \theta'(0)}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3} \sqrt{2\beta}}$$

$$\frac{\delta}{x} = \frac{\sqrt{2\beta'}}{Re_x^{1/2}}$$

So our complete solution becomes:

$$q''_{y=0} = \left(\frac{k\Delta T}{x}\right) \left[\frac{\theta'(0)}{2} \int_0^1 \theta(1-\theta) d\eta_T \right]^{1/2} \cdot \frac{Re_x^{1/2} \cdot Pr^{1/3}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}$$

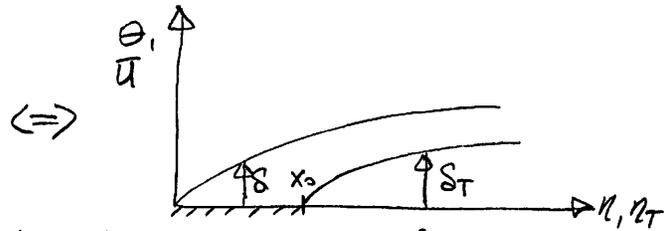
$$q''_{y=0} = h\Delta T = k_f \frac{\Delta T}{x}$$

$$\left(\frac{x q''_{y=0}}{k\Delta T}\right) = \frac{hx}{k_f} = Nux$$

To solve this integral, we need to assume a velocity profile and solve.

However, there is a trick, since the integral is not a function of x_0 at all, it has to equal to 0.332. Let's see what I mean.

$$Nu_x = 0.332 \frac{Re_x^{1/2} Pr^{1/3}}{\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}$$



To see why $= 0.332$, let's look at the limit of $x_0 = 0$

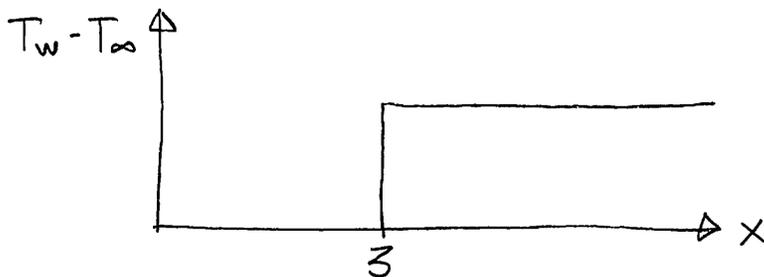
$$Nu_x = 0.332 \frac{Re_x^{1/2} Pr^{1/3}}{\left[1 - \left(\frac{0}{x}\right)^{3/4}\right]^{1/3}} = 0.332 Re_x^{1/2} Pr^{1/3}$$

The solution collapses to our initial solution solved on page 89 of notes (go check). The only way it does this is if the integral on the previous page $= 0.332$.

END OF LECTURE II

Show b.l. NSF video

So what if we had steps of heating instead of only 1 step?



$$\theta(x, y, z) = \frac{T - T_w}{T_\infty - T_w}, \quad z < x$$

↳ This will satisfy the differential equation

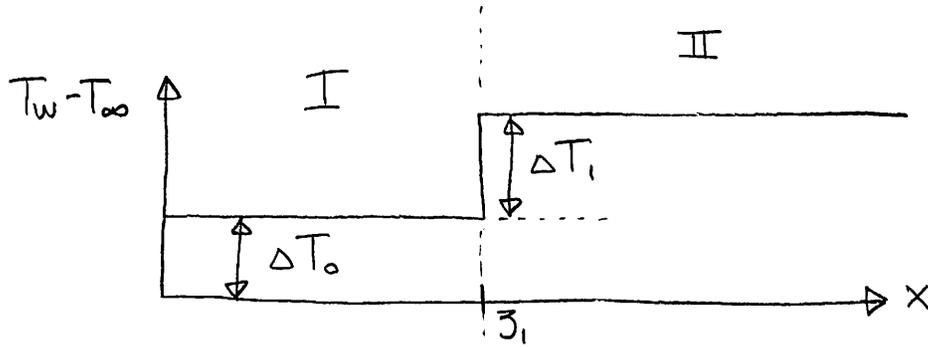
$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \theta = f(x, y, z) \Rightarrow \theta = \theta(\eta_T) = f\left(\frac{y}{\delta_T}\right) = \frac{3}{2} \frac{y}{\delta_T}$$

$$T - T_\infty = (T_w - T_\infty) \cdot f(x, y, z) \quad \frac{\delta}{\delta_T} = f(x_0, x) \Rightarrow \text{page 100 of notes}$$

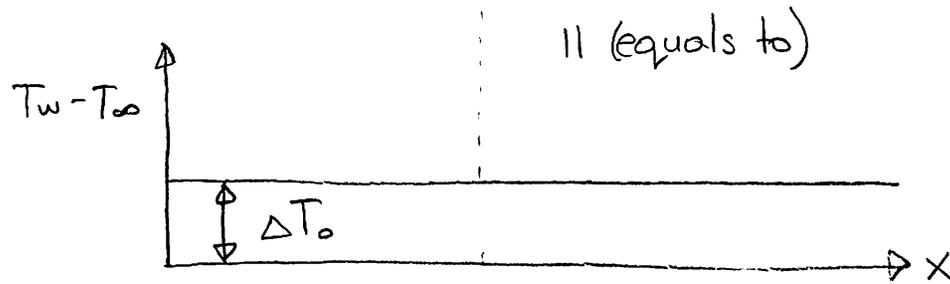
$$f(x, 0, z) = 1 \Rightarrow \text{Wall temperature at } y=0 \Rightarrow T_w$$

$$f(x, \infty, z) = 0 \Rightarrow \text{at } y \rightarrow \infty, T = T_\infty$$

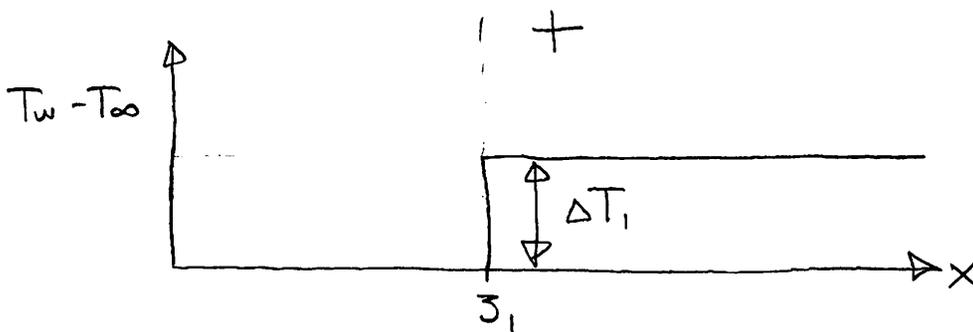
So now if we have temperature jumps, we can use superposition to solve.



$$(T - T_{\infty}) = \Delta T_0 f(x, y, 0) + \Delta T_1 f(x, y, z_1)$$

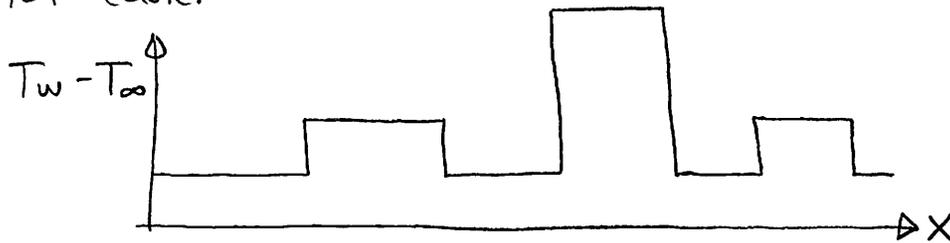


$$(T - T_{\infty})_I = \Delta T_0 f(x, y, 0)$$



$$(T - T_{\infty})_{II} = \Delta T_1 f(x, y, z_1)$$

So we can use superposition to solve, makes our lives a lot easier:

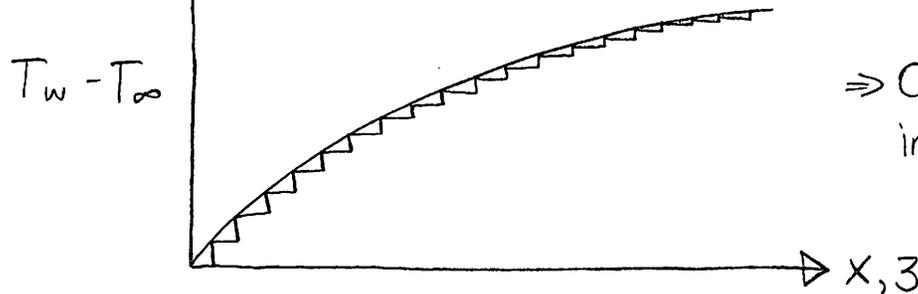


$$(T - T_{\infty}) = \sum_{j=0}^n \Delta T_j \cdot f(x, y, z_j), \quad z_n < x < z_{n+1}$$

Similarly we can show:

$$q''_{r, y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \sum_{j=0}^n \frac{\Delta T_j}{\left[1 - \left(\frac{z_j}{x}\right)^{3/4}\right]^{1/3}}, \quad z_n < x < z_{n+1}$$

What if our wall temperature difference was continuously changing:



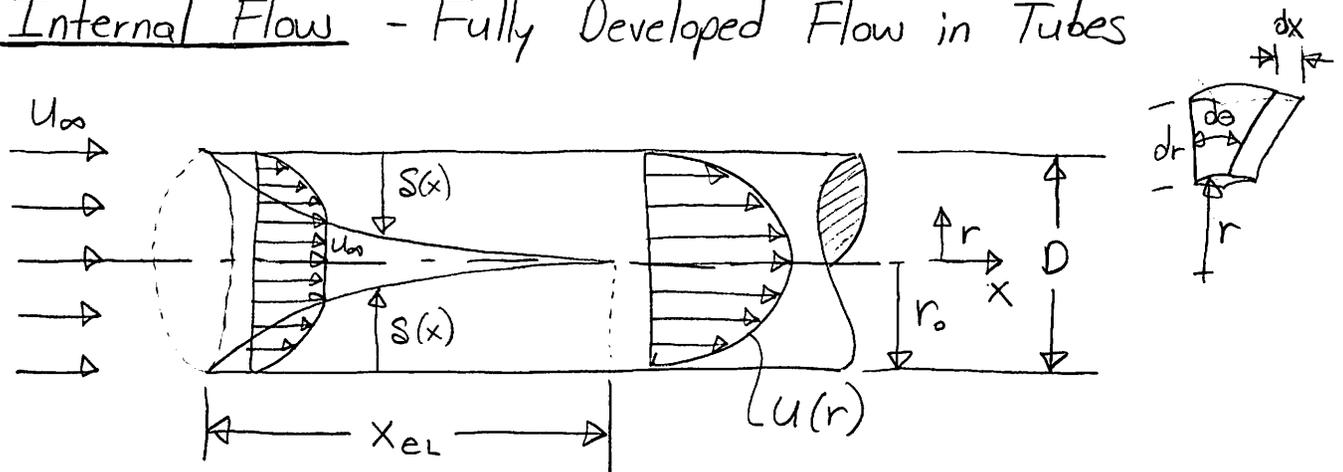
⇒ Our summation becomes integration.

$$q''_{r=y=0} = C \int_0^x \frac{dT}{\left[1 - \left(\frac{z}{x}\right)^{3/4}\right]^{1/3}} dz$$

$\frac{dz}{dz}$
 $\frac{.23}{.23}$

$$q''_{r=y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \int_0^x \frac{d(T_w - T_\infty)}{\left[1 - \left(\frac{z}{x}\right)^{3/4}\right]^{1/3}} dz$$

Internal Flow - Fully Developed Flow in Tubes



x_{eL} = entrance length or developing length. Velocity profile varies with radial position, r , and axial location, x .

We can estimate the magnitude of the entrance length, x_{eL} . We know from previous solution that the b.l. thickness in a laminar flow on a flat plate is:

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \Rightarrow \text{Blasius solution}$$

Extra Derivation

Solve for the wall heat flux ($q''|_{y=0}$) if $(T_w - T_\infty) = \beta\sqrt{x}$.

We just figured out how to deal with this problem. Since the temperature change is continuous, let:

$$z = x$$

$$(T_w - T_\infty) = \beta\sqrt{z}$$

$$\frac{d(T_w - T_\infty)}{dz} = \frac{1}{2}\beta \frac{1}{\sqrt{z}}$$

Let $s = \frac{z}{x} \Rightarrow q''|_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \beta \frac{1}{2} \int_0^x \frac{dz}{z^{1/2} \left[1 - \left(\frac{z}{x}\right)^{3/4}\right]^{1/3}}$

Multiply by $\left(\frac{x}{x}\right)$, we will obtain

$$q''|_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} x^{1/2} \beta \frac{1}{2} \int_0^1 \frac{ds}{s^{1/2} \left[1 - s^{3/4}\right]^{1/3}}$$

$\int_0^1 \frac{ds}{s^{1/2} \left[1 - s^{3/4}\right]^{1/3}}$, let $\lambda = (1 - s^{3/4})$ We need to solve this
 $s = (1 - \lambda)^{4/3}$, $s^{1/2} = (1 - \lambda)^{2/3}$, $\left[1 - s^{3/4}\right]^{1/3} = \lambda^{1/3}$
 $= \frac{4}{3} \int_0^1 \frac{(1 - \lambda)^{1/3} d\lambda}{(1 - \lambda)^{2/3} \lambda^{1/3}} = \frac{4}{3} \int_0^1 \frac{d\lambda}{\lambda^{1/3} (1 - \lambda)^{1/3}} = \frac{4}{3} \int_0^1 \lambda^{2/3 - 1} (1 - \lambda)^{2/3 - 1} d\lambda$

From integral tables: $\int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{1.355^2}{0.893} = 2.06$

$$\approx q''|_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \underbrace{x^{1/2} \beta}_{\Delta T(x)} \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot (2.06)$$

$$Nu_x = \left(\frac{q''|_{y=0} \cdot x}{\Delta T(x) \cdot k_f}\right) = 0.455 Re_x^{1/2} Pr^{1/3}$$

↳ Note we only get constant heat flux if $T(x) = \beta(x)$

We can estimate that $\delta \sim \frac{D}{2}$ when the two b.l.'s merge
 $\frac{D}{2x_{eL}} \sim \frac{5.0}{\sqrt{Re_{x_{eL}}}} \Rightarrow$ Note, I don't use equals (=) since not a flat plate

$$\frac{D}{x_{eL}} \sim \frac{10}{\sqrt{Re_{x_{eL}}}} = \frac{10}{\sqrt{\frac{\rho U_{\infty} x_{eL}}{\mu}}} = \frac{10}{\sqrt{\frac{\rho U_{\infty} x_{eL} \cdot D}{\mu D}}} = \frac{10}{\sqrt{\frac{\rho U_{\infty} D}{\mu}} \cdot \sqrt{\frac{x_{eL}}{D}}}$$

$$\sqrt{\frac{D}{x_{eL}}} \sim \frac{10}{\sqrt{Re_D}}$$

$$\boxed{\frac{x_{eL}}{D} \sim \frac{Re_D}{100} \sim 0.01 Re_D}, \quad \boxed{Re_D = \frac{\rho U_{\infty} D}{\mu}}$$

Note, the actual solution, experimentally verified is

$$\boxed{\frac{x_{eL}}{D} = 0.05 Re_D} \Rightarrow \text{We were fairly close given our assumptions.}$$

If $x > x_{eL}$, the flow is fully developed.

Now looking at the fully developed region, with Navier-Stokes: (x-momentum)

$$\rho \left(\frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + \frac{u_{\phi}}{r} \frac{\partial u_x}{\partial \phi} + u_x \frac{\partial u_x}{\partial x} \right) = -\frac{\partial p}{\partial x} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_x}{\partial \phi^2} + \frac{\partial^2 u_x}{\partial x^2} \right] + \rho g_x = 0$$

We know for fully developed flow that $u_{\phi} = u_r = 0$
 $\frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial \phi} = 0$

So most of our terms drop out and we are left with
 $-\frac{\partial p}{\partial x} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_x}{\partial r} \right) = 0$

B.C.'s: $\left. \frac{\partial u_x}{\partial r} \right|_{r=0} = 0 \quad (1)$

$$u_x(r=r_0) = 0 \quad (2)$$

From here, I will use $u_x(r) = u(r)$. Dropping the x subscript for simplicity. (110)

Integrating twice and applying our boundary conditions:

$$u(r) = \frac{r_0^2}{4\mu} \left(-\frac{\partial P}{\partial x}\right) \left(1 - \frac{r^2}{r_0^2}\right) \Rightarrow \text{Velocity profile in a pipe}$$

Now if we solve for our average velocity:

$$\bar{u} = \frac{1}{\pi r_0^2} \int_0^{r_0} u \cdot 2\pi r dr = \frac{r_0^2}{4\mu} \left(-\frac{\partial P}{\partial x}\right) \underbrace{\int_0^1 (1-\lambda) 2\lambda}_{\frac{1}{2}}, \quad \lambda = \frac{r^2}{r_0^2}$$

$$\bar{u} = \frac{r_0^2}{8\mu} \left(-\frac{\partial P}{\partial x}\right)$$

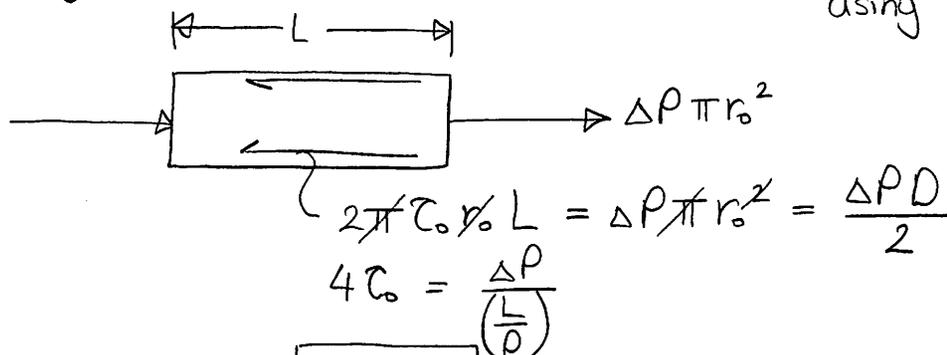
$$u(r) = 2\bar{u} \left(1 - \frac{r^2}{r_0^2}\right)$$

Typically, we want to solve for our friction coefficient and pressure loss

$$f = \frac{\Delta P}{\left(\frac{L}{D}\right) \frac{1}{2} \rho V^2} \Rightarrow \text{Tube friction factor. Easy way to calculate pressure loss in tubes.}$$

$$C_f = \frac{\tau}{\frac{1}{2} \rho V^2} \Rightarrow \text{Tube friction coefficient. We've determined before. It's related to the friction factor.}$$

Looking at a finite differential element in our flow and using a force balance.



We can say: $4C_f = f$

So what is our friction factor in a pipe

$$f = \frac{\Delta P}{\left(\frac{L}{D}\right) \frac{1}{2} \rho V^2} \quad (1)$$

We've solved before that $\bar{u} = V = \frac{r_0^2}{8\mu} \left(-\frac{\partial P}{\partial x}\right)$

$$\frac{8\mu V}{r_0^2} = -\frac{\partial P}{\partial x} = \frac{\Delta P}{L} \quad (2)$$

From (1): $\frac{\Delta P}{L} \cdot \frac{2D}{\rho V^2} = f = \frac{8\mu V}{r_0^2} \cdot \frac{2D}{\rho V^2}$

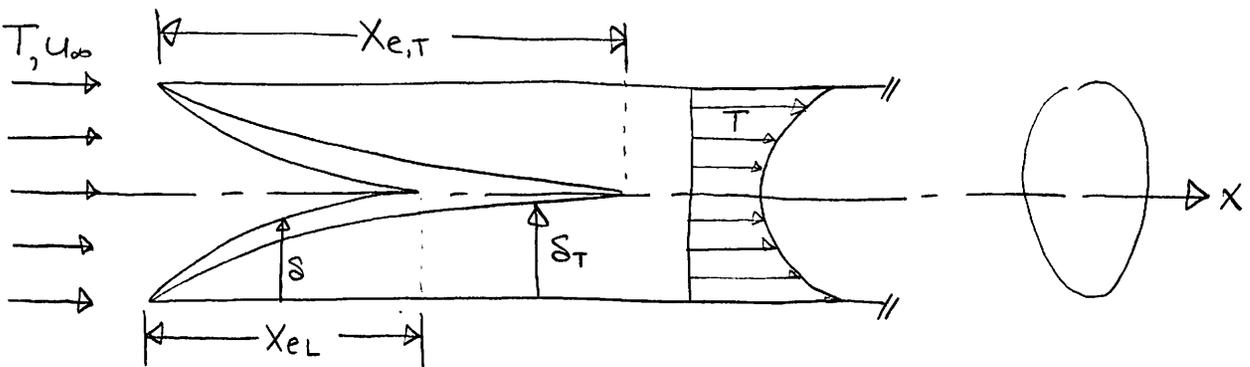
$$\frac{16\mu D}{r_0^2 \rho V} = f = \frac{16\mu D}{\left(\frac{D}{2}\right)\left(\frac{D}{2}\right)\rho V} = \frac{64\mu}{\rho V D} = \frac{64}{Re_D}$$

$$\boxed{f = \frac{64}{Re_D}} \Rightarrow \text{Pipe friction factor for laminar flow. Also known as the Darcy-Weisbach eqn.}$$

Note, $\boxed{f \cdot Re = \text{constant}} \Rightarrow$ For any cross section pipe.

See Table 4.5, page 307 of Mills. $\boxed{D_H = \frac{4A}{P}}$; $A = \text{area}$, $P = \text{perimeter}$

Heat Transfer in the Pipe



Here we have a similar situation as the hydrodynamic developing (or entrance) length, but with temperature.

$$\boxed{\frac{X_{e,T}}{D} = 0.017 Re_D Pr} \Rightarrow \text{Thermal developing length.}$$

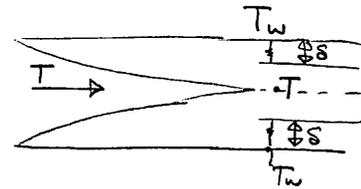
To estimate the heat transfer, let's try a simple analysis

$$h = \frac{q''_{wall}}{\Delta T}$$

$$h \propto \frac{k_f}{\delta_T} \quad \text{since} \quad h \Delta T = k_f \frac{\Delta T}{\delta_T} \Rightarrow h \sim \frac{k_f}{\delta_T}$$

Assuming $\delta_T \approx \frac{r_0}{2}$ (since pipe flow) \Rightarrow

$$\bar{h} \sim \frac{2k}{r_0} = 4 \frac{k}{D}$$



We know that $\overline{Nu}_0 = \frac{\bar{h}D}{k_f} \Rightarrow \boxed{Nu_0 \approx 4}$ \Rightarrow Just from a very simple analysis

We'll see how accurate we are in a little bit.

It's important to note here that heat transfer for internal flow problems is calculated using the bulk fluid temperature.

$$\bar{h} = \frac{q''_{wall}}{T_w - T_b}, \quad T_b = \text{bulk fluid temperature}$$

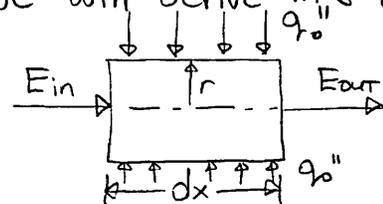
Think of T_b as the uniform temperature of the pipe fluid if it was allowed to mix and come to an equilibrium temp. in an adiabatic way.

$$\boxed{T_b = \frac{1}{A\bar{V}} \int_A u(r)T dA}, \quad \text{where } A = \text{cross sectional area}$$

$$\bar{V} = \text{average velocity}$$

Constant Wall Heat Flux ($q''_0 = \text{constant}$, Fully developed flow)
 Writing out our energy equation: (we will derive this a little later)

$$\underbrace{\rho C_p u \frac{\partial T}{\partial x}}_{\text{convection}} = \underbrace{k \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)}_{\text{conduction}}$$



We know already that: $u = 2\bar{u} \left(1 - \frac{r^2}{r_0^2} \right)$

Our B.C.'s are: $\left. \frac{\partial T}{\partial r} \right|_{r=0} = 0$ (no heat flux across centerline due to symmetry)
 $T(r=r_0) = T_w(x)$

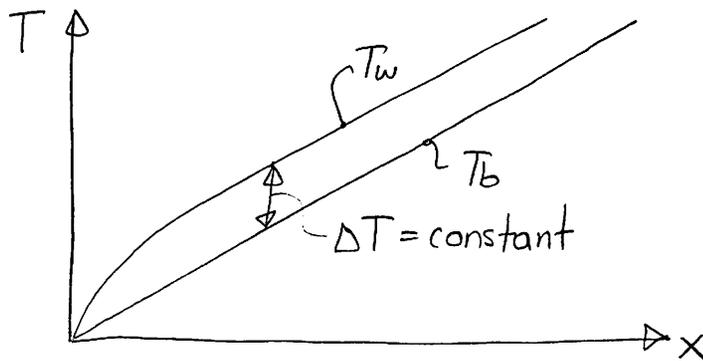
Let: $\frac{T - T_b}{T_w - T_b} = f(r)$, $\partial f = \partial T \left(\frac{1}{T_w - T_b} \right)$

$$q''|_{r=r_0} = -k \left. \frac{\partial T}{\partial r} \right|_{r_0} = -k \left. \frac{\partial f}{\partial r} \right|_{r_0} \cdot (T_w - T_b) = \text{constant}$$

For fully developed flow, the temperature profile shape does not change: $\left. \frac{\partial f}{\partial r} \right|_{r_0} = \text{constant}$. So we have:

$$-k \left. \frac{\partial f}{\partial r} \right|_{r_0} (T_w - T_b) = q''_0 = \text{constant}$$

constant \Rightarrow This means $\frac{\partial T_b}{\partial x} = \frac{\partial T_w}{\partial x}$

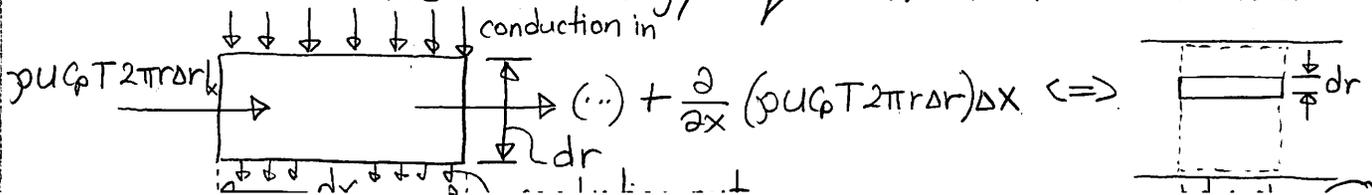


Note, if $\frac{\partial T_b}{\partial x} \neq \frac{\partial T_w}{\partial x}$, then the slopes would cross one another and this is a clear violation of conservation of energy.

Note:
 Did informal
 feed back, &
 discussed quiz.

END OF LECTURE 12

Now we can solve our energy equation on a fluid element:



Now we can write our conduction terms:

$$-k \frac{\partial T}{\partial r} 2\pi r \Delta x \Big|_{r+\Delta r} \Rightarrow \text{Heat conduction at } r+\Delta r$$

$$-k \frac{\partial T}{\partial r} 2\pi r \Delta x \Big|_r \Rightarrow \text{Heat conduction at } r$$

Putting everything together (energy balance on our control volume)

$$\rho u c_p T 2\pi r \Delta r \Big|_{x+\Delta x} - \rho u c_p T 2\pi r \Delta r \Big|_x = -k \frac{\partial T}{\partial r} 2\pi r \Delta x \Big|_r + k \frac{\partial T}{\partial r} 2\pi r \Delta x \Big|_{r+\Delta r}$$

Rearranging and dividing by $2\pi \Delta x \Delta r$

$$\rho u c_p r \frac{\partial T}{\partial x} = k \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$

or

$$\boxed{u \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)} \Rightarrow \text{Governing energy equation we must solve.}$$

We know that $u = 2\bar{u} \left[1 - \frac{r^2}{r_0^2} \right]$

$$2\bar{u} \left[1 - \frac{r^2}{r_0^2} \right] \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \Rightarrow \text{PDE since } T = T(x, r)$$

But note that we've already proven that $\frac{\partial T}{\partial x} = \frac{\partial T_b}{\partial x} = \text{constant}$ by simple energy balance:

$$q_0'' \pi D \Delta x = \rho \bar{u} c_p \frac{\pi D^2}{4} \frac{\partial T}{\partial x} \cdot \Delta x, \text{ note, } \bar{u} \text{ here, not } u(r), \text{ since we are doing the whole cross section of flow.}$$

$$\boxed{\frac{\partial T}{\partial x} = \frac{2q_0''}{\rho \bar{u} c_p r_0}} = \text{constant}$$

So now we have to solve for the radial temperature profile:

$$2\bar{u} \left(1 - \frac{r^2}{r_0^2}\right) \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right)$$

Rearranging and subbing in α :

$$\left(1 - \frac{r^2}{r_0^2}\right) \frac{4q_0''}{kr_0} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \Rightarrow \text{ODE, we can solve now}$$

B.C.'s: $\frac{\partial T}{\partial r} \Big|_{r=0} = 0$ (Symmetry)

$$T(r=r_0) = T_w,$$

Integrating once:

$$r \frac{\partial T}{\partial r} = \frac{4q_0''}{kr_0} \left[\frac{r^2}{2} - \frac{r^4}{4r_0^2} \right] + C_1$$

$$\frac{dT}{dr} = \frac{4q_0''}{kr_0} \left[\frac{r}{2} - \frac{r^3}{4r_0^2} \right] + \frac{C_1}{r}$$

$C_1 = 0$ since at $r \rightarrow 0$, this term explodes

$$T(r) = \frac{4q_0''}{kr_0} \left[\frac{r^2}{4} - \frac{r^4}{16r_0^2} \right] + C_2$$

Using our second boundary condition ($T(r=r_0) = T_w$)

$$T(r_0) = \frac{4q_0''}{kr_0} \left[\frac{r_0^2}{4} - \frac{r_0^4}{16r_0^2} \right] + C_2 = T_w$$

$$C_2 = T_w - \frac{4q_0''}{kr_0} \left[\frac{3}{16} r_0^2 \right]$$

So our solution becomes:

$$T(r) = T_w - \frac{4q_0''}{kr_0} \left(\frac{3r_0^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_0^2} \right) \Rightarrow \text{Quartic temperature profile}$$

Now we can solve for $T_b = \frac{\int_0^{r_0} uT2\pi r dr}{\pi r_0^2 \bar{u}}$

Solving (I'll leave the steps for you), we obtain:

$$T_b = T_w - \frac{11}{24} \frac{q_s'' r_0}{k}$$

or

$$\boxed{T_w - T_b = \frac{11}{24} \frac{q_s'' r_0}{k}} \Rightarrow \text{Not a function of } x, \text{ makes sense, since } \frac{\partial T_b}{\partial x} = \frac{\partial T_w}{\partial x} = \text{constant.}$$

Now we can solve for our Nusselt number

$$\bar{h} = \frac{q_s''}{T_w - T_b} = \frac{24k}{11r_0} = \frac{48k}{11D}$$

$$\overline{Nu}_0 = \frac{\bar{h}D}{k} = \frac{48}{11} = 4.364$$

$$\boxed{\overline{Nu}_0 = 4.364} \Rightarrow \text{Laminar flow in a tube with constant heat flux conditions.}$$

We can do a similar analysis to show that for a constant wall temperature boundary condition:

$$\boxed{\overline{Nu}_0 = \frac{hD}{k} = 3.66} \Rightarrow T_w = \text{constant}$$

We see that our previous solution of $Nu_0 \approx 4$ was pretty close to the exact analytical solutions.

Note, both of these solutions are valid only if:

$$\boxed{Re_0 = \frac{\rho \bar{u} D}{\mu} \lesssim 2300} \Rightarrow \text{Laminar flow in a smooth pipe, fully developed.}$$

We will discuss turbulence a little later.

Some physical insights:

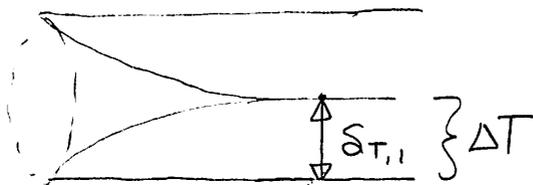
On page 79 of the notes, we defined the Nusselt number as the non-dimensional temperature gradient

$$\underbrace{-\frac{\partial \Theta}{\partial n^*}}_{\text{Measure of the gradient}} = \underbrace{\frac{hL}{k_f}}_{\text{Nusselt \#}} = Nu, \quad n^* = \frac{n}{L} \text{ or } n^* = \frac{n}{D} \text{ for a pipe}$$

$$\Theta = \frac{T - T_f}{T_s - T_f}$$

This tells us why it's so advantageous to go to mini or micro channel flows for cooling. By reducing the channel dimension (diameter), we reduce the effective boundary layer thickness (thermal, δ_T) and pump up our heat transfer.

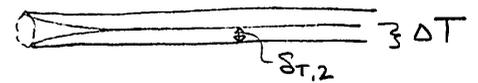
Large Channel:



$$-\frac{\partial \Theta}{\partial n^*} = \frac{h_1 D_1}{k_f} = Nu \approx 4$$

$$h_1 = \frac{4k_f}{D_1}$$

Micro Channel:



$$-\frac{\partial \Theta}{\partial n^*} = \frac{h_2 D_2}{k_f} = Nu \approx 4$$

$$h_2 = \frac{4k_f}{D_2}$$

Since $D_2 \ll D_1$, $h_2 \gg h_1$

This is the direction of research these days, pushing dimensions to the micron length scale to maximize h . One drawback to this is the excess pumping power required to drive the fluid.

$$\Delta P = f \cdot \left(\frac{L}{D}\right) \frac{1}{2} \rho \bar{V}^2, \quad \text{as } D \downarrow, \Delta P \uparrow \text{ non-linearly}$$

since $f = \frac{64}{Re_0} = \frac{64\mu}{\rho \bar{V} D}$ and $\bar{V} \propto D^2$ for a constant m .

To get an order of magnitude feel for the enhancement

$$k_f = 0.6 \text{ W/m}\cdot\text{K}$$

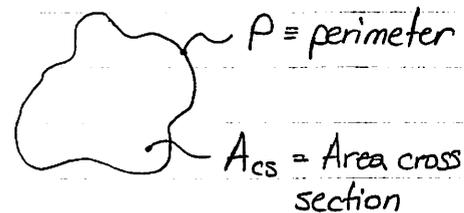
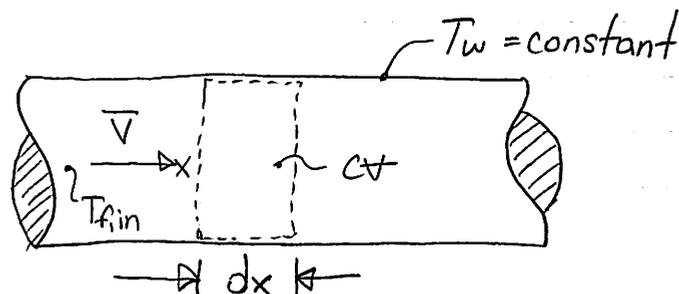
$$D_1 = 1 \text{ cm}, \quad D_2 = 100 \mu\text{m}$$

$$Nu = 4$$

$$h_1 = \frac{Nu \cdot k_f}{D_1} = \frac{(4)(0.6 \text{ W/m}\cdot\text{K})}{0.01 \text{ m}} = 240 \text{ W/m}^2\cdot\text{K}$$

$$h_2 = \frac{Nu \cdot k_f}{D_2} = \frac{(4)(0.6 \text{ W/m}\cdot\text{K})}{0.0001} = 24\,000 \text{ W/m}^2\cdot\text{K} = 24 \text{ kW/m}^2\cdot\text{K}$$

Example Heat Exchange to a liquid flowing in a tube, with heat transfer coefficient h . Find T_{out} given that $T_w = \text{constant}$, and fully developed flow.



Apply an energy balance on our control volume

$$\underbrace{\rho A_{cs} \bar{V} c_p}_{\dot{m}} \cdot dT_f = \underbrace{h P dx}_{dA_{\text{surface}}} (T_w - T_f)$$

$$\text{Let } T^* = \frac{T_f - T_w}{T_{f,in} - T_w}, \quad x^* = \frac{x}{L}, \quad L = \text{length of pipe}$$

$$dT^* = \frac{dT_f}{T_{f,in} - T_w}, \quad dx^* = \frac{dx}{L}$$

$$\rho A_{cs} \bar{V} c_p (T_{f,in} - T_w) dT^* = h P L dx^* (-T^*) (T_{f,in} - T_w)$$

$$\rho A_{cs} \bar{V} c_p dT^* + h P L T^* dx^* = 0$$

$$\int_{T_{f,in}}^{T_f} \frac{dT^*}{T^*} + \int_{x=0}^x \frac{hPL}{\rho A_{cs} \bar{V} c_p} dx^* = 0$$

$$\ln \left(\frac{T^*}{T_{f,in}^*} \right) + \frac{hPLx^*}{\rho A_{cs} \bar{V} c_p} = 0$$

$$T^* = T_{f,in}^* e^{-\frac{hPL}{\rho A_{cs} \bar{V} c_p} x^*} \Rightarrow T_{f,in}^* = \frac{T_{f,in} - T_w}{T_{f,in} - T_w} = 1$$

$$T_f = T_w + (T_{f,in} - T_w) e^{-\frac{hPL}{\rho A_{cs} \bar{V} c_p} x}$$

So remember here, $T_f = T_b$ (bulk fluid temperature)

$$T_b = \frac{1}{\rho A_{cs} \bar{V} c_p} \int_{A_{cs}} \rho c_p V(x,y) \cdot T(x,y) dA$$

\Rightarrow Note, already defined on page 113 of notes

Note also $q = hP dx (T_w - T_b)$

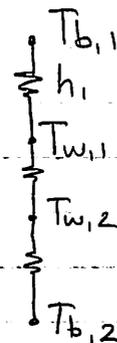
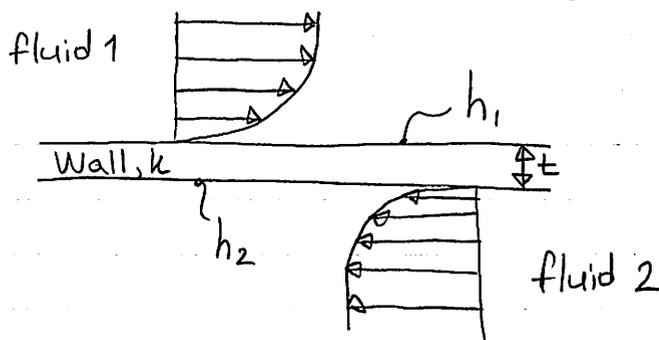
\hookrightarrow But by definition, h (or Nu) = $f(T_b)$

For problems like this, always base property values on:

$$\bar{T} = \frac{T_{b,in} + T_{b,out}}{2}, \text{ where } T_{b,in} = T_{f,in}$$

$$T_{b,out} = T_{f,out}$$

Example #2 | Heat exchanger.



Using our thermal resistance concepts:

From our resistance network, we can write

$$U = \frac{1}{\left\{ \frac{1}{h_1} + \frac{t}{k} + \frac{1}{h_2} \right\}} \Rightarrow \frac{1}{UA} = \frac{t}{kA} + \frac{1}{h_1 A} + \frac{1}{h_2 A}$$

So now we can analyze our heat exchanger: (Perimeter, P)

$$\textcircled{1} \longrightarrow \dot{m}_1, T_{b1}, c_{p,1} \quad (\text{or: } \dot{M}, T, C)$$

$$\textcircled{2} \longrightarrow \dot{m}_2, T_{b2}, c_{p,2} \quad (\text{or: } \dot{m}, t, c)$$

Across our element dx : $dq = -\dot{M}C dT$

$$dq = +\dot{m}c dt$$

$$dT = -\frac{dq}{\dot{M}C} \quad \text{and} \quad dt = +\frac{dq}{\dot{m}c}$$

$$d(T-t) = dq \left\{ -\frac{1}{\dot{M}C} - \frac{1}{\dot{m}c} \right\}$$

$$d(T-t) = \frac{dq}{\dot{M}C} \left\{ -1 - \frac{\dot{M}C}{\dot{m}c} \right\}$$

Aside:
 $d(T-t) = dT - dt$

But we know that $dq = UA dx (T-t)$, where $P dx = dA$

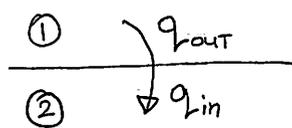
$$\frac{d(T-t)}{(T-t)} = \frac{U}{\dot{M}C} \left\{ -1 - \frac{\dot{M}C}{\dot{m}c} \right\} dA$$

Integrating both sides from inlet (a) to outlet (b), we obtain

$$\ln \left\{ \frac{T_b - t_b}{T_a - t_a} \right\} = \frac{UA}{\dot{M}C} \left\{ -1 - \frac{\dot{M}C}{\dot{m}c} \right\} \quad \textcircled{1}$$

For the entire heat exchanger, we know:

Energy out of stream ①, $q_{out} = -q$ Energy in to stream ②, $q_{in} = q$

$$-\dot{M}C(T_b - T_a) = \dot{m}c(t_b - t_a)$$


rearranging:

$$-\frac{\dot{M}C}{\dot{m}c} = \frac{t_b - t_a}{T_b - T_a} \quad (2)$$

Back substituting (2) into (1)

$$\ln \left\{ \frac{T_b - t_b}{T_a - t_a} \right\} = \frac{UA}{\dot{M}C} \left\{ -1 + \frac{t_b - t_a}{T_b - T_a} \right\}$$

Rewriting this, we obtain:

$$\ln \left\{ \frac{T_b - t_b}{T_a - t_a} \right\} = \frac{UA}{\dot{M}C(T_b - T_a)} \left\{ (T_a - T_b) + t_b - t_a \right\}$$

$-q$ (net heat transfer)

Since $q = UA \Delta T$, now we can write

$$\Delta T = \text{LMTD}$$

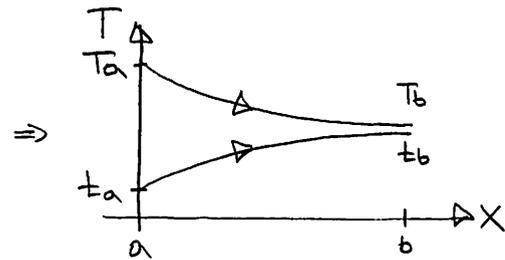
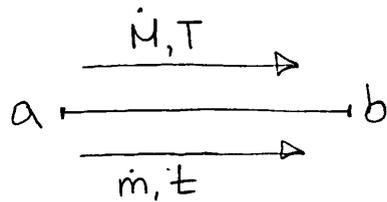
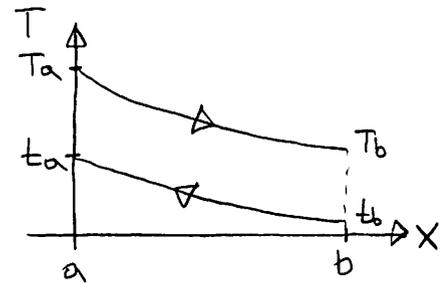
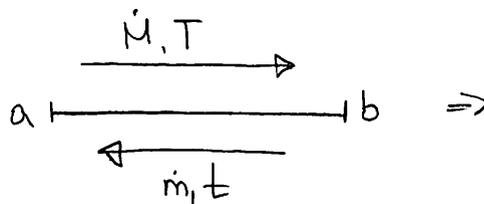
$$\Delta T = \frac{(T_b - t_b) - (T_a - t_a)}{\ln \left\{ \frac{T_b - t_b}{T_a - t_a} \right\}}$$

$$\frac{\dot{M}C}{\dot{m}c} \frac{T_a - T_b}{t_a - t_b}$$

$$q = UA (\text{LMTD}) \quad \Downarrow \quad \text{Log Mean Temperature Difference}$$

Note, it doesn't matter which side is a and b. You can reverse it and still get the same answer. (i.e. switch a & b)

It also doesn't matter which direction the flows are traveling in, as long as "a" and "b" refer to some physical end of the heat exchanger.

Parallel Flow:Counter Flow:

Think of the LMTD as a convenient way to define a ΔT between two streams whose temperature is continuously varying.

END OF LECTURE 13

Special Case \Rightarrow Balanced counter flow heat exchanger
Assuming $\dot{M} = \dot{m}$, and $C = c$, then the LMTD becomes undefined:

$$\text{LMTD} = \frac{\Delta T_b - \Delta T_a}{\ln\left(\frac{\Delta T_b}{\Delta T_a}\right)} \quad ; \quad \Delta T = T - t$$

In this case, $\Delta T_b = \Delta T_a$, so $\text{LMTD} = \frac{0}{0} \Rightarrow$ undefined

The way to resolve this is the following:
Suppose $\Delta T_b = \Delta T_a + \epsilon$, where $\epsilon \ll 1$

$$\text{LMTD} = \frac{\Delta T_a + \epsilon - \Delta T_a}{\ln\left\{\frac{\Delta T_a + \epsilon}{\Delta T_a}\right\}} = \frac{\epsilon}{\ln\left(1 + \frac{\epsilon}{\Delta T_a}\right)}$$

But we know that $\ln(1+x) \approx x$ for $x \ll 1$

$$\ln\left(1 + \frac{\epsilon}{\Delta T_a}\right) \approx \frac{\epsilon}{\Delta T_a} \Rightarrow \text{LMTD} = \frac{\epsilon}{\epsilon/\Delta T_a} = \Delta T_a = \Delta T_b = \Delta T \quad \text{for } \epsilon \ll 1$$

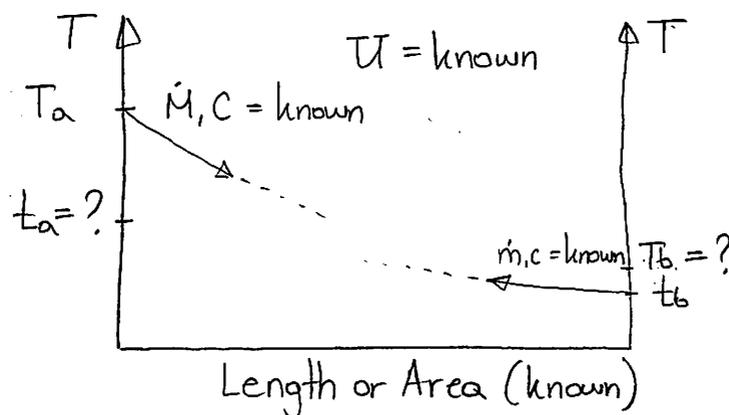
So our undefined problem is resolved.

In general, we can say the following:

Parallel Flow	Counter Flow
<p><u>Disadvantage #1</u>: Large temp. difference at one end of the heat exchanger causes added thermal stresses & early failure</p> <p><u>Disadvantage #2</u>: The outlet temp. of the cold fluid never exceeds the outlet temp. of the hot fluid. Less efficient.</p>	<p><u>Advantage #1</u>: More uniform ΔT minimizes the thermal stresses throughout the exchanger.</p> <p><u>Advantage #2</u>: The outlet temp. of the cold fluid can approach the highest temperature of the hot fluid. More efficient.</p> <p><u>Advantage #3</u>: More uniform ΔT produces a more uniform q.</p>

ϵ -NTU Method (Effectiveness - NTU method)

Note, in most heat exchanger design problems, we don't know the fluid outlet temperatures, i.e. $T_{1,out}$ or $T_{2,out}$



Now we can define a quantity called the capacity

$$C_h = (\dot{m}c)_{hot} [W/K]; \quad C_c = (\dot{m}c)_{cold} [W/K]$$

To solve, we could guess an exit temp., solve for $Q_h = Q_c = C_h \Delta T_h = C_c \Delta T_c$ (124)

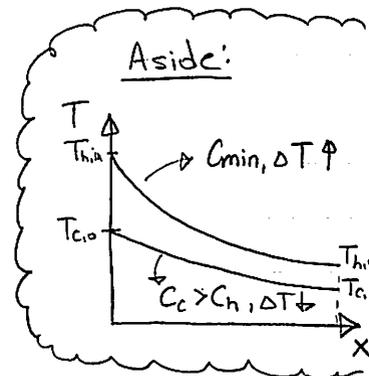
Then we would have to calculate Q from $UA LMTD$ and check against our previous answer. If differing, we would guess another exit temp. and try again.

We can make our lives much easier with ϵ -NTU.

$$\epsilon = \frac{\text{actual heat transferred}}{\left(\begin{array}{l} \text{maximum heat that could possibly} \\ \text{be transferred from one stream to the other} \end{array} \right)}$$

Mathematically, this is equal to:

$$\epsilon = \frac{C_h (T_{h,in} - T_{h,out})}{C_{min} (T_{h,in} - T_{c,in})} = \frac{C_c (T_{c,out} - T_{c,in})}{C_{min} (T_{h,in} - T_{c,in})}$$



where C_{min} is the smaller of C_h & C_c .

The reason we use C_{min} is because the fluid with the lower capacity will be the one which undergoes the maximal temperature change, i.e. $q = C_{min} \Delta T_{max}$, $C_{min} \uparrow, \Delta T \downarrow$

So we can write: $Q = \epsilon C_{min} (T_{h,in} - T_{c,in})$

We can also define: $NTU = \frac{UA}{C_{min}} = \frac{\text{heat rate capacity of h.x.}}{\text{heat capacity rate of flow}}$
(dimensionless)

Using energy balances & simplifying, we can solve for our two cases:

Parallel Flow:

$$-\left(\frac{C_{min}}{C_c} + \frac{C_{min}}{C_h} \right) NTU = \ln \left[-\left(1 + \frac{C_c}{C_h} \right) \epsilon \frac{C_{min}}{C_c} + 1 \right]$$

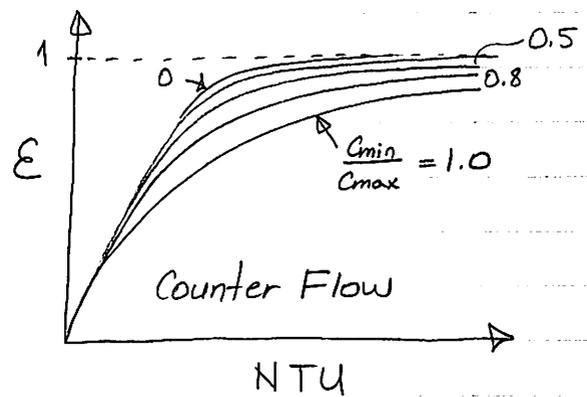
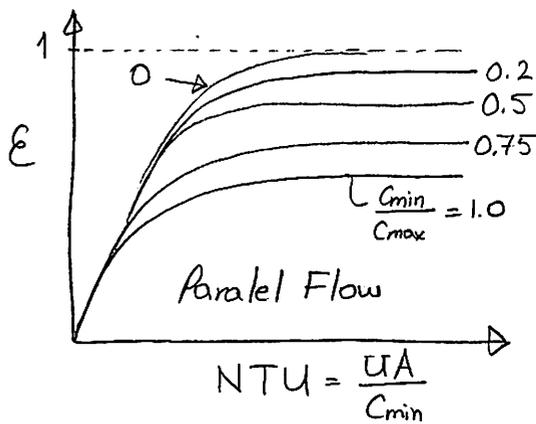
Solving for ϵ , we obtain:

$$\epsilon = \frac{1 - \exp \left[-\left(1 + C_{min}/C_{max} \right) NTU \right]}{1 + C_{min}/C_{max}} = f \left(\frac{C_{min}}{C_{max}}, NTU \text{ only} \right)$$

Counter Flow:

$$\epsilon = \frac{1 - \exp[-(1 + C_{min}/C_{max})NTU]}{1 - (C_{min}/C_{max}) \exp[-(1 - C_{min}/C_{max})NTU]} = f\left(\frac{C_{min}}{C_{max}}, NTU, \text{only}\right)$$

We can plot our results in a more useful form



Note, the ϵ -NTU method can be applied to any arbitrarily shaped heat exchanger with more difficult flow patterns (i.e. cross flow, mixed flow, etc...)

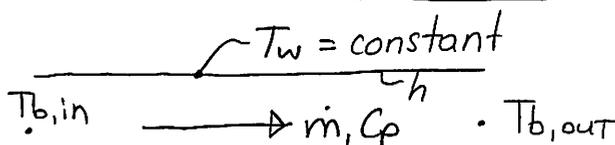
For a great book, see "Compact Heat Exchangers", Kays & London, (1955.)

Note that our solution for the pipe flow with constant wall temperature will also simplify to this:

$$\underbrace{\frac{T_{b,out} - T_{b,in}}{T_w - T_{b,in}}}_{\epsilon} = 1 - e^{\underbrace{\left(-\frac{hPL}{mC_p}\right)}_{\substack{hPL = UA \\ mC_p = C_{min}}}} \Rightarrow \text{See page 120 of notes.}$$

$P = \text{perimeter}$

$$\boxed{\epsilon = 1 - e^{-NTU}} \Rightarrow f(NTU \text{ only})$$



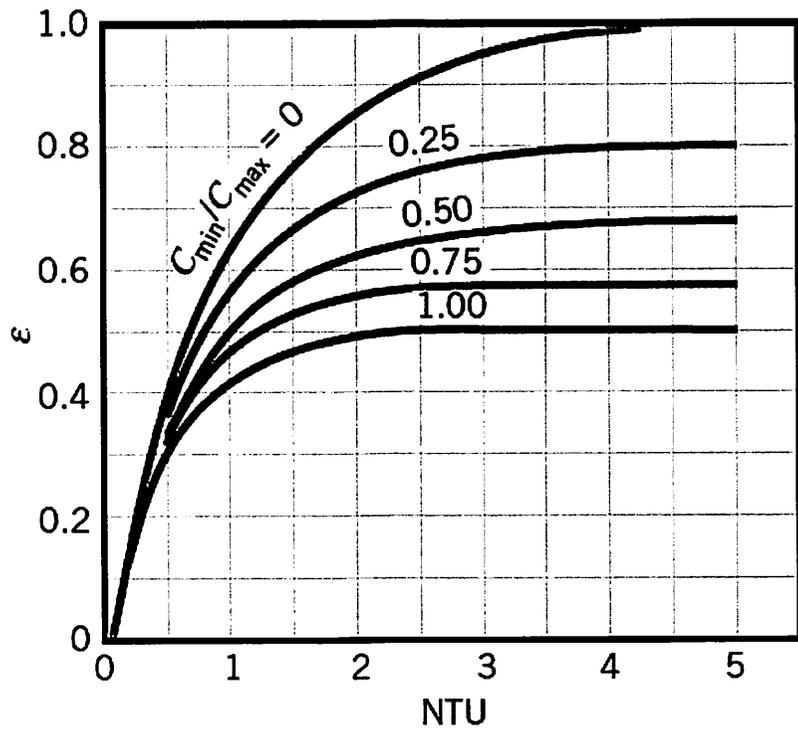


FIGURE 11.10 Effectiveness of a parallel-flow heat exchanger (Equation 11.28).

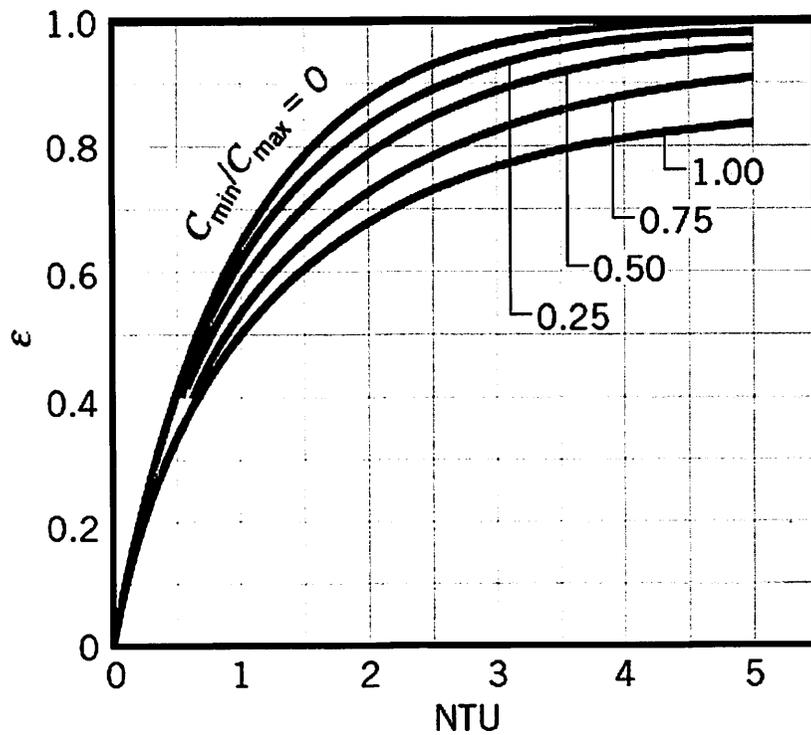


FIGURE 11.11 Effectiveness of a counterflow heat exchanger (Equation 11.29).

Example | Consider the following parallel-flow h.x. specification

Cold flow enters at 40°C , $C_c = 20,000 \text{ W/K}$

Hot flow enters at 150°C , $C_h = 10,000 \text{ W/K}$

$$A = 30 \text{ m}^2, \quad U = 500 \text{ W/m}^2 \cdot \text{K}$$

Determine the heat transfer and the exit temperatures.

Here, we can't use LMTD since we don't have exit conditions so ϵ -NTU must be used.

$$\text{NTU} = \frac{UA}{C_{\min}} = \frac{500(30)}{10,000} = 1.5$$

$$\frac{C_{\min}}{C_{\max}} = 0.5$$

From our tables, we obtain $\epsilon = 0.596$

$$\begin{aligned} Q &= \epsilon C_{\min} (T_{h,\text{in}} - T_{c,\text{in}}) = 0.596(10,000)(110) \\ &= 655,600 \text{ W} = 655.6 \text{ kW} \end{aligned}$$

From energy balances:

$$T_{h,\text{out}} = T_{h,\text{in}} - \frac{Q}{C_h} = 150 - \frac{655,600}{10,000} = 84.44^\circ\text{C}$$

$$T_{c,\text{out}} = T_{c,\text{in}} + \frac{Q}{C_c} = 40 + \frac{655,600}{20,000} = 72.78^\circ\text{C}$$

Example | Suppose we had the same problem, but $A = \text{unknown}$ and we want $T_{h,\text{out}} = 90^\circ\text{C}$. Calculate A .

Here we can use the ϵ -NTU method or LMTD since we have an exit temperature. Applying an energy balance:

$$T_{c,\text{out}} = T_{c,\text{in}} + \frac{C_h}{C_c} (T_{h,\text{in}} - T_{h,\text{out}}) = 40 + \frac{1}{2}(150 - 90) = 70^\circ\text{C}$$

Using the ϵ -NTU method:

$$\epsilon = \frac{C_h (T_{h,in} - T_{h,out})}{C_{min} (T_{h,in} - T_{c,in})} = \frac{10,000 (150 - 90)}{10,000 (150 - 40)} = 0.5455$$

From our charts, $NTU \cong 1.15 = UA/C_{min}$

$$A = \frac{10,000 (1.15)}{500} = 23.00 \text{ m}^2$$

We could have also used LMTD (note doesn't matter which side is a and b)

$$\begin{aligned} LMTD &= \frac{\Delta T_a - \Delta T_b}{\ln \left(\frac{\Delta T_a}{\Delta T_b} \right)} = \frac{(T_{h,in} - T_{c,in}) - (T_{h,out} - T_{c,out})}{\ln \left(\frac{T_{h,in} - T_{c,in}}{T_{h,out} - T_{c,out}} \right)} \\ &= \frac{(150 - 40) - (90 - 70)}{\ln \left(\frac{150 - 40}{90 - 70} \right)} = 52.79 \text{ K} \end{aligned}$$

So from $Q = UA(LMTD)$

$$A = \frac{Q}{U(LMTD)} = \frac{10,000 (150 - 90)}{(500)(52.79)} = 22.73 \text{ m}^2$$

The answers differ by 1%, which reflects graph reading innacuracy.

Example / On page 109 of the notes, we did external flow with variable wall temperature. We solved for:

$$q''|_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \int_0^x \frac{d(T_w - T_\infty)}{d\zeta} \frac{d\zeta}{\left[1 - \left(\frac{\zeta}{x}\right)^{3/4}\right]^{1/3}}$$

Solve for $q''|_{y=0}$ if $(T_w - T_\infty) = \beta\sqrt{x}$

Since it's a continuous temperature change, $\zeta \approx x$

$$(T_w - T_\infty) = \beta\sqrt{\zeta}$$

$$\frac{d(T_w - T_\infty)}{d\zeta} = \frac{1}{2}\beta\frac{1}{\sqrt{\zeta}}$$

Let $s = \frac{\zeta}{x} \Rightarrow q''|_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \beta \cdot \frac{1}{2} \int_0^x \frac{d\zeta}{\zeta^{1/2} \left[1 - \left(\frac{\zeta}{x}\right)^{3/4}\right]^{1/3}}$

Multiply by $\left(\frac{x}{x}\right)$, we will obtain

$$q''|_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} x^{1/2} \beta \cdot \frac{1}{2} \int_0^1 \frac{ds}{s^{1/2} \left[1 - s^{3/4}\right]^{1/3}}$$

$$\int_0^1 \frac{ds}{s^{1/2} \left[1 - s^{3/4}\right]^{1/3}}, \quad \lambda = (1 - s^{3/4})$$

$$s = (1 - \lambda)^{4/3}$$

$$s^{1/2} = (1 - \lambda)^{2/3}$$

$$\left[1 - s^{3/4}\right]^{1/3} = \lambda^{1/3}$$

We need to solve this

$$= \frac{4}{3} \int_0^1 \frac{(1-\lambda)^{1/3} d\lambda}{(1-\lambda)^{2/3} \lambda^{1/3}} = \frac{4}{3} \int_0^1 \frac{d\lambda}{\lambda^{1/3} (1-\lambda)^{1/3}} = \frac{4}{3} \int_0^1 x^{\frac{2}{3}-1} (1-x)^{\frac{2}{3}-1} dx$$

So we know from integral tables: $\int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$$q''|_{y=0} = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} x^{1/2} \beta \underbrace{\frac{1}{2} \cdot \frac{4}{3}}_{\Delta T(x)} (2.06) = \frac{(1.355)^2}{0.893} = 2.06$$

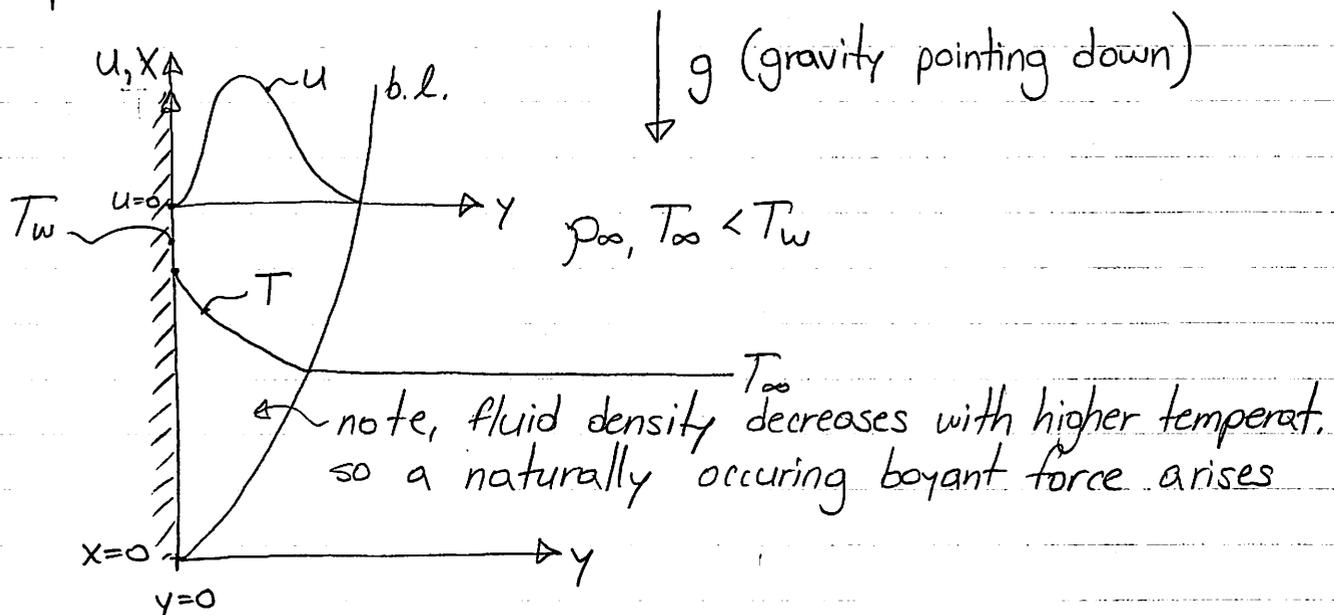
$$Nu_x = \left(\frac{q''|_{y=0} \cdot x}{\Delta T(x) \cdot k_f}\right) = 0.455 Re_x^{1/2} Pr^{1/3}$$

\Rightarrow Note, we get constant heat flux only if $T(x) = \beta\sqrt{x}$

Natural Convection

Unlike forced convection, in which the fluid motion driving force is external to the fluid, natural convection processes are driven by body forces exerted directly within the fluid as a result of heating or cooling.

What is natural convection?
Simplest case:

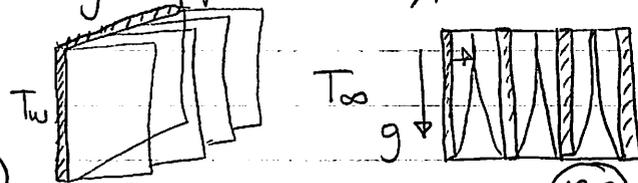


This flow is similar to a flow past a flat plate with a boundary layer developing, however here $u=0$ at the wall and at $y \rightarrow \infty$.

We see natural convection all around us. When we are stationary, we are undergoing natural convection heat loss to the environment.

Many passive heat sinks (fins) are natural convection. I.e. transformer cooling (high voltage applications), home appliances, electronics cooling.

&
Cooling towers in nuclear power plants (Stack Effect, Natural Convection.)



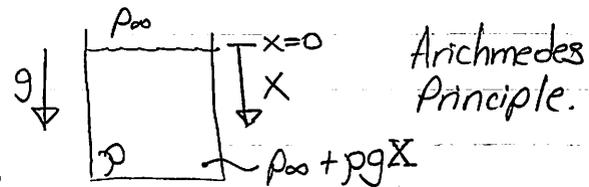
We can now solve for our governing equations.
x-momentum: (steady state)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - g \quad (1)$$

We know in the x-direction
 for a fluid that:

$$p(x) = p_{\infty} + \rho g x$$

Body force (gravity in -x)



Developed by Archimedes of Syracuse.

$$\frac{\partial p}{\partial x} = -\rho_{\infty} g \quad (\text{negative since } x \text{ pointing against gravity})$$

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{\rho_{\infty} g}{\rho} \quad (2)$$

Back substitute (2) into (1)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \left(\frac{\rho_{\infty} - \rho}{\rho} \right) + \nu \frac{\partial^2 u}{\partial y^2} \quad (3)$$

However, we need to know the relation between density, ρ
 and temperature, T . In general for a fluid we can define
 a quantity called the isothermal compressibility, β .

$$\beta = \frac{1}{v} \frac{\partial v}{\partial T} \Big|_p \quad (\text{or coefficient of thermal expansion})$$

specific volume. \Rightarrow

$$= -\frac{1}{\rho} \frac{\partial \rho}{\partial T} \Big|_p \quad \Rightarrow \text{proof}$$

We can rewrite as:

$$\beta = -\frac{1}{\rho} \frac{(\rho_{\infty} - \rho)}{T_{\infty} - T}$$

$$\frac{\rho_{\infty} - \rho}{\rho} = \beta (T - T_{\infty}) \quad (4)$$

Aside: $\rho = \frac{\rho}{R_{\text{specific}} T}, \quad \frac{\partial \rho}{\partial T} = \frac{\rho - 1}{R_s T^2}$

$$\frac{1}{\rho} = v, \quad v = \frac{R_s T}{\rho}, \quad \frac{\partial v}{\partial T} = \frac{R_s}{\rho}$$

$$\frac{\rho}{R_s T} \cdot \frac{R_s}{\rho} = \frac{1}{T} = \frac{1}{v} \frac{\partial v}{\partial T} \Big|_p$$

$$-\frac{R_s T}{\rho} \cdot \frac{\rho}{R_s} \left(-\frac{1}{T^2} \right) = \frac{1}{T} = \frac{1}{\rho} \frac{\partial \rho}{\partial T} \Big|_p$$

Back substitute ④ into ③, we obtain

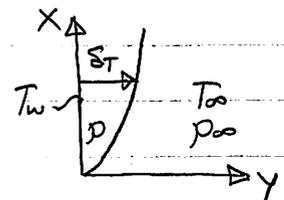
$$\underbrace{u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}}_{\text{Inertia Forces, I}} = \underbrace{g\beta(T-T_\infty)}_{\text{Boyancy Forces, } \beta} + \underbrace{\nu \frac{\partial^2 u}{\partial y^2}}_{\text{Viscous Forces, } \nu}$$

Now if we do our simple and tested scaling analysis

$$\delta_T \sim \sqrt{\alpha L} \sim \sqrt{\frac{\alpha x}{u}} \quad (\text{Speed of heat propagation via conduction})$$

$$u \sim \frac{\alpha x}{\delta_T^2}, \quad \partial u \sim \frac{\alpha \partial x}{\delta_T^2}$$

$$y \sim \delta_T, \quad dy \sim \delta_T$$



Inertia, I	Viscosity, \nu	Boyancy, \beta
$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \sim \frac{u^2}{x}$	$\nu \frac{\partial^2 u}{\partial y^2} \sim \nu \frac{u}{y^2}$	$g\beta(T-T_\infty)$
$\sim \frac{\alpha^2 x}{\delta_T^4}$	$\sim \frac{\nu \alpha x}{\delta_T^4}$	$= g\beta \Delta T$

Now if we take a ratio of our forces with respect to our boyant force term:

I/\beta	\nu/\beta	\beta/\beta
$\frac{\alpha^2 x}{\delta_T^4 g\beta \Delta T}$	$\frac{\nu \alpha x}{\delta_T^4 g\beta \Delta T}$	1

Assuming that $\nu/\beta \sim O(1)$ (Inside the natural convection b.l.)

$$\frac{\nu \alpha x}{\delta_T^4 g\beta \Delta T} \sim 1$$

Multiplying both sides by $\frac{x^3}{x^3}$

$$\frac{U \propto}{g \beta \Delta T x^3} \left(\frac{x}{\delta_T} \right)^4 \sim 1$$

$$\boxed{\frac{x}{\delta_T} \sim \left(\frac{g \beta \Delta T x^3}{\alpha U} \right)^{1/4}} = (\text{Rayleigh Number})^{1/4}$$

$$\boxed{Ra = \frac{g \beta \Delta T x^3}{\alpha U}}, \quad \boxed{\frac{x}{\delta_T} \sim Ra^{1/4}}$$

$$Ra = \underbrace{\frac{\text{Buoyancy Force}}{\text{Viscous Force}}}_{Gr} \cdot \underbrace{\frac{\text{Momentum Diffusivity}}{\text{Thermal Diffusivity}}}_{Pr} = Gr \cdot Pr$$

↳ Grashof number

If: $Ra < Ra_{crit}$, conduction dominates ($Gr \ll 1, Pr \ll 1$)
 $Ra > Ra_{crit}$, convection dominates ($Gr > 1, Pr > 1$)

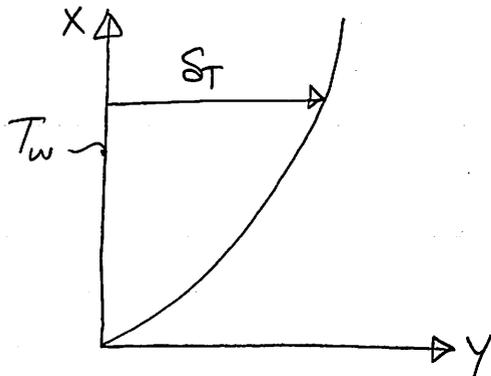
So what about heat transfer?

$$q''|_{y=0} = -k \frac{\partial T}{\partial y} \Big|_{y=0} \sim k \frac{\Delta T}{\delta_T}$$

$$\left(\frac{q''|_{y=0}}{\Delta T} \right) \cdot \frac{x}{k_f} = Nu_x \sim \frac{x}{\delta_T} \sim Ra_x^{1/4}, \quad \boxed{Nu_x \sim Ra_x^{1/4}}$$

So we can tell that our Nusselt number will be proportional to the Rayleigh number to the $1/4$ power just by looking at our Fourier conduction equation.

Looking back at our natural convection b.l.



$$\delta_T \sim x^{1/4} \quad \left(\text{since } \frac{U \alpha x}{\delta_T^4 \beta \Delta T g} \sim 1 \right)$$

Our heat transfer will slowly decrease with increasing x .

Constant Heat Flux Case

Instead of $T_w = \text{constant}$, what if $q''|_{y=0} = \text{constant}$

$$q''|_{y=0} \sim \frac{k \Delta T}{\delta_T}$$

$$\Delta T \sim \frac{q'' \delta_T}{k}$$

We can re-do our scaling analysis with our new ΔT definition

Inertia, I	Viscosity, ν	Boyancy, β
$\frac{\alpha^2 x}{\delta_T^4}$	$\frac{\nu \alpha x}{\delta_T^4}$	$g \beta \delta_T (q''/k)$

Again, assuming (Viscous Force)/(Boyancy Force) ~ 1 (inside b.l.)

$$\frac{\nu \alpha x}{\delta_T^5 g \beta (q''/k)} \sim 1 \cdot \left(\frac{x^4}{x^4} \right)$$

$$\frac{x}{\delta_T} = \left[\frac{g \beta q'' x^4}{\nu \alpha k} \right]^{1/5}$$

$$Ra^* = \frac{g \beta q'' x^4}{\alpha \nu k}$$

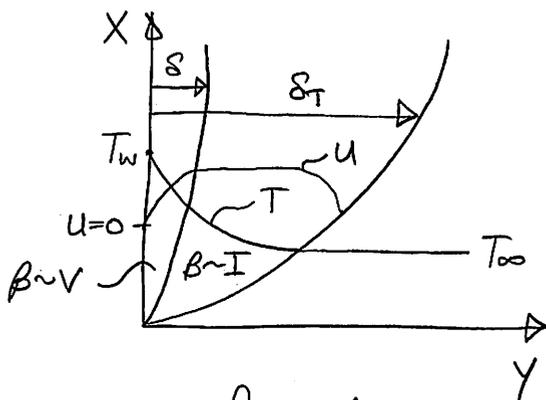
\Rightarrow Modified Rayleigh Number.

\hookrightarrow For constant heat flux

$$Nu_x \sim Ra_x^{*1/5}$$

Note our two previous solutions are valid if $\frac{\text{Viscosity}}{\text{Boyaney}} \sim 1$

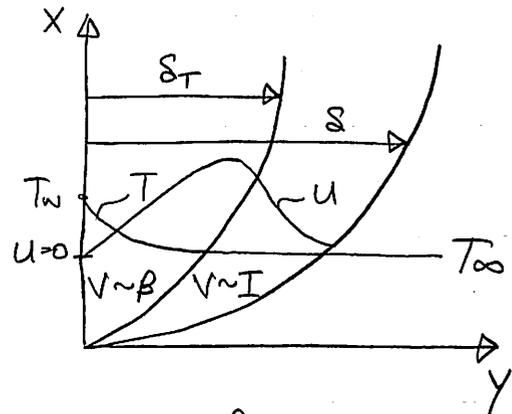
If we have a very viscous fluid, we must solve our boundary layer equations fully.



$Pr \ll 1$
(Liquid Metal)



Temperature diffuses far and boyancy moves fluid up as wall drags a thin b.l. (hydrodynamic) where $\text{Boyaney} \sim \text{Viscosity}$. More like flow over a flat plate with U_∞ where in U_∞ region $\text{Boyaney} \sim \text{Inertia}$



$Pr \gg 1$
(Synthetic Oil)



Temperature has small diffusion but momentum diffusion dominates and creates a thick hydrodynamic b.l. where $\text{Viscosity} \sim \text{Inertia}$, $\beta = 0$ since no temperature change.

Now we can go back to our analysis and solve for our x-momentum equation. Using the integral technique we solved for earlier, we can solve.

Energy Integral Technique (for natural convection)

Assume $\delta \sim \delta_T$ ($\text{Viscosity} \sim \text{Boyaney}$), $T_w = \text{constant}$

$$\delta \sim \delta_T, \quad \theta = \frac{T - T_w}{T_\infty - T_w}, \quad \eta = \frac{y}{\delta_T}, \quad \theta = f(\eta)$$

$$u = \Gamma(x) \phi(\eta), \quad \Gamma(x) = \text{function of } x \text{ only} \quad (135)$$

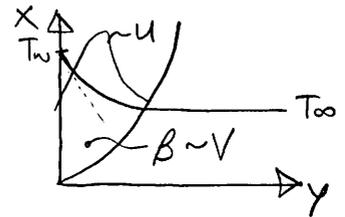
So our boundary conditions are:

$$\theta(0) = 0 \quad (T = T_w \text{ at } y=0)$$

$$\theta(1) = 1 \quad (T = T_\infty \text{ at } y = \delta_T)$$

$$\theta'(1) = 0 \quad (\partial T / \partial y|_{y=\delta_T} = 0)$$

$$\theta''(0) = 0 \quad (\partial^2 T / \partial y^2|_{y=0} = \text{constant at the wall})$$



$$\phi(0) = 0 \quad (u = 0 \text{ at } y=0)$$

$$\phi(1) = 0 \quad (u = 0 \text{ at } y = \delta_T)$$

$$\phi'(1) = 0 \quad (\partial u / \partial y|_{y=\delta_T} = 0)$$

Our energy integral equation for this b.l. is:

$$\frac{d}{dx} \int_0^{\delta_T} u(T - T_\infty) dy = -\alpha \left. \frac{\partial T}{\partial y} \right|_{y=0} \Rightarrow \text{Check page 103 of notes for derivation.}$$

Non-dimensionalizing

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \theta, \quad u = \Gamma(x) \phi(\eta), \quad \eta = \frac{y}{\delta_T} \Rightarrow dy = \delta_T d\eta$$

$$\theta = \frac{T - T_w}{T_\infty - T_w} \Rightarrow d\theta = \frac{dT}{T_\infty - T_w}$$

Back substituting into our energy integral equation:

$$\frac{d}{dx} (\delta_T \Gamma) \int_0^1 \phi(\eta) (1 - \theta(\eta)) d\eta = \frac{\alpha}{\delta_T} \left. \frac{\partial \theta}{\partial \eta} \right|_{\eta=0} \quad (1)$$

Note, we can use a trick to solve for $\Gamma(x)$ since we know at the wall, Buoyancy \sim Viscosity (one of our assumptions)

$$g\beta\Delta T + \nu \frac{\partial^2 u}{\partial y^2} = 0 \quad (\text{Inertia is negligible since } u|_{y=0} = 0)$$

$$dy = \delta_T d\eta; \quad \partial^2 u \Rightarrow u = \Gamma(x) \phi(\eta) \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\Gamma(x)}{\delta_T^2} \underbrace{\frac{\partial^2 \phi}{\partial \eta^2}}_{\phi''(0)}$$

$$\Gamma(x) = \frac{g\beta\Delta T \delta_T^2}{\nu [-\phi''(0)]} \quad (2)$$

Back substituting ② into ①

$$\frac{g\beta\Delta T}{\nu[-\phi''(0)]} \cdot \frac{\partial \delta_T^3}{\partial x} = \frac{\alpha}{\delta_T} \frac{\theta'(0)}{\int_0^{\delta_T} \phi(1-\theta) d\eta}$$

Rewriting:

$$\delta_T \frac{\partial \delta_T^3}{\partial x} = \frac{\alpha \nu}{g\beta\Delta T} [\gamma]$$

$$[\gamma] = \frac{\theta'(0) [-\phi''(0)]}{\int_0^{\delta_T} \phi(1-\theta) d\eta}$$

Note, γ only depends on the temperature and velocity profiles that we assume, so it's useful to take it out and call it γ separately by itself.

$$\delta_T \frac{\partial \delta_T^3}{\partial x} = \delta_T 3\delta_T^2 \frac{\partial \delta_T}{\partial x} = 3\delta_T^3 \frac{\partial \delta_T}{\partial x}$$

Aside:

$$\left(\text{Think of } \delta_T = f(x), \frac{d}{dx} [f(x)]^3 = 3f(x)^2 \frac{df}{dx}\right)$$

$$3\delta_T^3 \frac{\partial \delta_T}{\partial x} = \frac{\alpha \nu}{g\beta\Delta T} \gamma$$

$$\int_0^{\delta_T} 3\delta_T^3 \partial \delta_T = \frac{\alpha \nu}{g\beta\Delta T} \gamma \int_0^x dx$$

$$\frac{3}{4} \delta_T^4 = \frac{\alpha \nu \gamma}{g\beta\Delta T} \cdot \left(\frac{x^4}{x^4}\right)$$

$$\left(\frac{\delta_T}{x}\right)^4 = \left[\frac{\nu \alpha}{g\beta\Delta T x^3}\right] \cdot \left(\frac{4}{3}\gamma\right) \Rightarrow Ra = \frac{g\beta\Delta T x^3}{\nu \alpha}$$

$$\frac{\delta_T}{x} = \frac{1}{Ra_x^{1/4}} \left(\frac{4}{3}\gamma\right)^{1/4}$$

\Rightarrow Note, before we had $\frac{\delta_T}{x} \sim \frac{1}{Ra^{1/4}}$
So we are very close!
See page 133 of notes.

We're almost there, we need to assume velocity and temperature profiles ($\phi(\eta)$, $\theta(\eta)$, $\eta = \frac{y}{\delta_T}$) and solve for γ .

Our assumed profiles must satisfy our boundary conditions
Let:

$$\left. \begin{aligned} \theta(\eta) &= \frac{3}{2}\eta - \frac{1}{2}\eta^3 \\ \phi(\eta) &= \eta(1-\eta^2) \end{aligned} \right\} \frac{\delta_T}{x} = \left(\frac{210}{Ra_x}\right)^{1/4} \Rightarrow \text{After solving for } \eta \text{ and back substituting.}$$

Now we can solve for heat transfer

$$\begin{aligned} q''|_{y=0} &= -k_f \left. \frac{\partial T}{\partial y} \right|_{y=0} = -\frac{k_f}{\delta_T} \cdot \frac{d}{d\eta} (1-\theta) \Delta T \\ &= \Delta T \frac{k_f}{\delta_T} \theta'(0) \\ &= \frac{3}{2} \frac{k_f}{\delta_T} \Delta T \end{aligned} \quad \left. \begin{aligned} \theta &= \frac{T-T_w}{T_\infty-T_w} \\ 1-\theta &= \frac{T-T_\infty}{T_w-T_\infty} \\ dT &= d(1-\theta) \underbrace{(T_w-T_\infty)}_{\Delta T} \end{aligned} \right\}$$

Now we can formulate our Nusselt number for natural convection on a heated vertical wall with $T_w = \text{constant}$.

$$\frac{\left(\frac{q''|_{y=0}}{\Delta T}\right) \cdot x}{k_f} = Nu_x = \frac{3}{2} \left(\frac{x}{\delta_T}\right) = \frac{1.5}{(210)^{1/4}} \cdot Ra_x^{1/4}$$

$$\boxed{Nu_x = 0.394 Ra_x^{1/4}} \quad , \quad \boxed{Ra_x = \frac{g\beta\Delta T x^3}{\alpha\nu}}$$

We know that $h_x = C \cdot \frac{1}{x^{1/4}}$ since $Nu_x = \frac{h_x \cdot x}{k}$

$$\bar{h} = \frac{1}{L} \int_0^L h_x dx = \frac{C}{L} \int_0^L \frac{dx}{x^{1/4}} = \frac{4}{3} C \frac{L^{3/4}}{L} = \frac{4}{3} \frac{C}{L^{1/4}}$$

$$\bar{h} = \frac{4}{3} h_L$$

$$\boxed{Nu_L = \frac{\bar{h}L}{k_f} = 0.525 Ra_L^{1/4}} \Rightarrow \text{Note experiments show a very accurate result of}$$

$$\boxed{Nu_{L,exp} = 0.52 Ra_L^{1/4}} \quad (138)$$

Now if we wanted to solve for the constant heat flux case ($q''|_{y=0} = \text{constant}$). Our energy integral would change a bit.

$$\frac{d}{dx} \int_0^{\delta} u(T-T_{\infty}) dy = \frac{1}{\rho c_p} q''_0 \quad \left. \vphantom{\frac{d}{dx}} \right\} \text{before was } -\frac{k}{\rho c_p} \frac{\partial T}{\partial y} \Big|_{y=0}$$

Non-dimensionalizing just like before, we obtain:

$$\frac{d}{dx} (\delta_T \Delta T \Gamma(x)) \int_0^1 \phi(1-\theta) d\eta = \alpha \left(\frac{q''_0}{k} \right)$$

Substituting in our $\Gamma(x)$

$$\frac{g\beta}{\nu [-\phi''(0)]} \cdot \frac{d}{dx} (\delta_T^3 \Delta T^2) \int_0^1 \phi(1-\theta) d\eta = \alpha \left(\frac{q''_0}{k} \right) \quad (1)$$

$$q''_{y=0} = -k \frac{\partial T}{\partial y} \Big|_{y=0} = -\frac{k}{\delta_T} \left[\frac{2 \left(\frac{T-T_{\infty}}{\Delta T} \right)}{2(y/\delta)} \right] \Big|_{y=0} \cdot \Delta T$$

$$= \frac{k}{\delta_T} \theta'(0) \cdot \Delta T$$

$$\boxed{\Delta T = \left(\frac{q''_0}{k} \right) \cdot \frac{\delta_T}{\theta'(0)}} \Rightarrow \text{Back substituting into (1)}$$

$$\frac{g\beta \left(\frac{q''_0}{k} \right)^2}{\nu [-\phi''(0)] [\theta'(0)]^2} \cdot \frac{d\delta_T^5}{dx} = \alpha \left(\frac{q''_0}{k} \right) \cdot \frac{1}{\int_0^1 \phi(1-\theta) d\eta}$$

$$\frac{d\delta_T^5}{dx} = \frac{\nu \alpha}{g\beta (q''_0/k)} \cdot \left[\frac{[-\phi''(0) \cdot [\theta'(0)]^2]}{\int_0^1 \phi(1-\theta) d\eta} \right]$$

Depends only on our assumed profiles for ϕ and θ . Call this $[\gamma']$

$$\left(\frac{\delta_T}{x} \right)^5 = \left[\frac{\nu \alpha}{g\beta (q''_0/k) x^4} \right] \cdot [\gamma']$$

$$\frac{x}{\delta_T} = \frac{(Ra_x^*)^{1/5}}{[\gamma']},$$

$$Ra_x^* = \frac{g\beta(q''_0/k)x^4}{\nu\alpha}$$

$$\left(\frac{q''|_{y=0}}{\Delta T}\right) \cdot \frac{x}{k_f} = Nu_x = \frac{q''|_{y=0}}{\frac{q''|_{y=0}}{k_f} \cdot \frac{\delta_T}{\theta'(0)}} \cdot \frac{x}{k_f}$$

$$Nu_x = \frac{3}{2} \frac{x}{\delta_T}$$

Assuming the same temperature and velocity profiles as before (ϕ, θ) and solving for $[\gamma']$, we get:

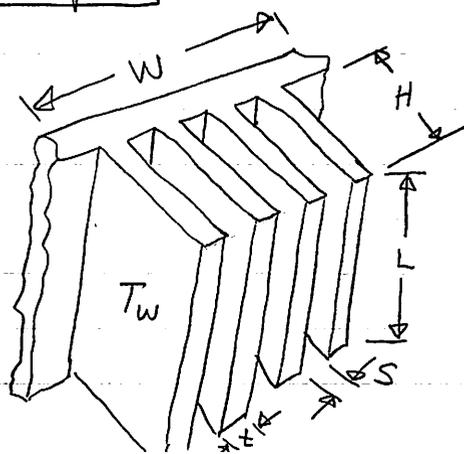
$$Nu_x = 0.503 Ra_x^{*1/5}$$

In general:

- $Ra_L > 10^9 \Rightarrow$ Turbulent free convection
- $Ra_L \leq 10^9 \Rightarrow$ Eq. 4.85 on pg. 326 in Mills
- $10^9 \leq Ra_L \leq 10^{12} \Rightarrow$ Eq. 4.86
- $10^6 \leq Ra_L \leq 10^9 \Rightarrow$ Eq. 4.87

Note: 1) All properties for natural convection problems are evaluated at $T = (T_\infty + T_w)/2$
 2) β is evaluated at T_∞ . $\beta = \frac{1}{T}$ for ideal gases where T is in Kelvin.

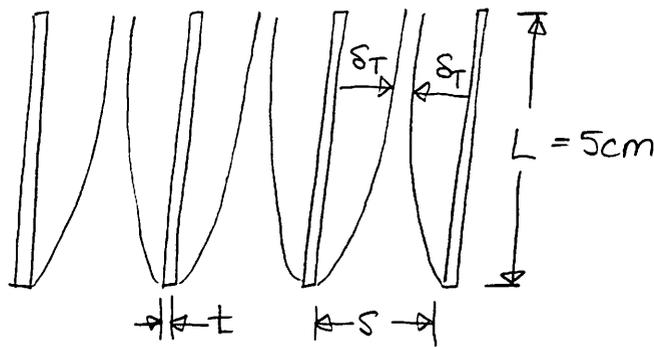
Example | Electronics cooling heat sink



~ Quiescent air ($U_\infty = 0$)
 at T_∞
 $\downarrow g$

How much heat can we dissipate if we assume $T_w = 80^\circ\text{C}$, $t \ll S$, $S = 3\text{mm}$, $L = 5\text{cm}$, $n = 20$ (number of fins), and $H = 3\text{cm}$, $T_\infty = 25^\circ\text{C}$.

Here, since $t \ll S$, we can assume isothermal vertical flat plates. We also assume that $\delta_T < \frac{S}{2}$ (non interacting b.l.'s). This makes the isolated vertical flat plate approx. correct. Also, $S < \frac{S}{2}$



$$n = 20 \text{ fins}$$

$$H = 3 \text{ cm}$$

$$\sim T_\infty = 25^\circ\text{C}$$

Using our developed correlation (constant wall temperature)

$$T_\infty = 25^\circ\text{C} = 298.15 \text{ K}, \quad \beta = \frac{1}{T_\infty} = \frac{1}{298.15 \text{ K}} = 0.00335 \text{ K}^{-1}$$

$$\bar{T} = \frac{T_\infty + T_w}{2} = 325.65 \text{ K}, \Rightarrow \text{Air Properties: } \begin{cases} k_f = 0.0277 \text{ W/m}\cdot\text{K} \\ \nu = 17.9 \times 10^{-6} \text{ m}^2/\text{s} \\ Pr = 0.71 = \frac{\nu}{\alpha} \end{cases}$$

(at \bar{T})

$$\Delta T = 80^\circ\text{C} - 25^\circ\text{C} = 55^\circ\text{C}$$

$$\text{We first calculate } Ra_L = \frac{g\beta\Delta TL^3}{\alpha\nu} = \frac{(9.81)(0.00335)(55)(0.05)^3(0.71)}{(17.9 \times 10^{-6})^2}$$

$$Ra_L = 5.0 \times 10^5 \text{ (Laminar Flow } < 10^9)$$

$$\text{Now we can calculate } \bar{h}_L \text{ from } Nu_L = \frac{\bar{h}_L L}{k_f} = 0.525 Ra_L^{1/4}$$

$$\bar{h}_L = \frac{Nu_L k_f}{L} = \frac{0.525 Ra_L^{1/4} k_f}{L} = \frac{0.525 (5.0 \times 10^5)^{1/4} (0.0277)}{0.05}$$

$$\bar{h}_L = 7.737 \text{ W/m}^2 \cdot \text{K}$$

$$A_T = n \cdot 2 \cdot LH = (20 \text{ fins})(2 \text{ sides/fin})(0.03 \text{ m})(0.05 \text{ m}) = 0.06 \text{ m}^2$$

$$Q_{TOT} = \bar{h}_L \cdot A_T \cdot \Delta T = (7.737 \text{ W/m}^2 \cdot \text{K})(0.06 \text{ m}^2)(55^\circ \text{C})$$

$$\boxed{Q_{TOT} = 25.53 \text{ W}}$$

Note, we can check our $\delta_T < \frac{S}{2}$ and $\delta < \frac{S}{2}$ approximation
We calculated previously that $\frac{\delta_T}{L} = \frac{(210)^{1/4}}{(Ra_L)^{1/4}}$

$$\begin{aligned} \delta_T &= L \cdot \left(\frac{210}{Ra_L} \right)^{1/4} \quad (\text{see page 138 of notes}) \\ &= (0.05 \text{ m}) \left(\frac{210}{5.0 \times 10^5} \right)^{1/4} = 0.0072 \text{ m} = 7.2 \text{ mm} > \frac{S}{2} \end{aligned}$$

So our previous approximation was incorrect since the thermal boundary layers will overlap. However, the calculation is a good approximation or starting point. Luckily, many researchers have looked at this problem and have developed more accurate correlations \Rightarrow Let's re-calculate and see how off we were:

$$Nu_s = \frac{\bar{h}_L S}{k_f} = \left[\frac{576}{(Ra_s \cdot \frac{S}{L})^2} + \frac{2.873}{(Ra_s \cdot \frac{S}{L})^{0.5}} \right]^{-0.5}$$

$$\begin{aligned} Ra_s &= \frac{g\beta\Delta T S^3 Pr}{\nu^2} = Gr_s \cdot Pr \quad ; \quad \frac{S}{L} = \frac{0.003}{0.05} = 0.06 \\ &= \frac{(9.81)(0.00335)(55)(0.003)^3(0.71)}{(17.9 \times 10^{-6})^2} = 108.14 \end{aligned}$$

$$Nu_s = \frac{\bar{h}_L S}{k_f} = \left[\frac{576}{(108.14 \cdot 0.06)^2} + \frac{2.873}{(108.14 \cdot 0.06)^{0.5}} \right]^{-0.5} = 0.259$$

$$\bar{h}_L = \frac{(0.259)(0.0277)}{0.003} = 2.4 \text{ W/m}^2 \cdot \text{K}$$

$$Q_{TOT} = \bar{h}_L \cdot A_T \cdot \Delta T = (2.4 \text{ W/m}^2 \cdot \text{K})(0.06 \text{ m}^2)(55^\circ \text{C})$$

$$\boxed{Q_{TOT} = 7.92 \text{ W}} < Q_{TOT} \text{ from our first try.}$$

Makes sense since overlapping b.l.'s act to diminish heat transfer. (142)

In general:

Closely Packed Fins - Greater surface area - Smaller heat transfer coefficient (h_L)	Widely Spaced Fins - Higher heat transfer coefficient (h_L) - Smaller surface area
--	--

For those of you who are interested, for $T_w = \text{constant}$

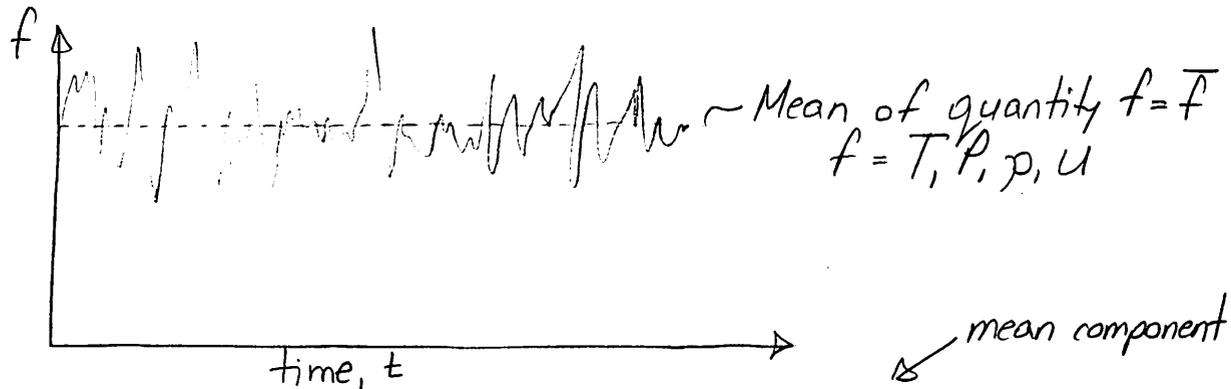
$$S_{opt} = 2.714 \cdot \frac{L}{Ra_L^{1/4}}$$

⇒ See paper on Dropbox by Bar-Cohen and Rohsenow (1984)

Turbulent Boundary Layers

Turbulence in a fluid can be seen as a spectrum of coexisting vortices (eddies) in which kinetic energy from larger ones is dissipated to successively smaller ones until the very smallest of these vortices are damped out by viscous shear stress.

Turbulent flow causes fluctuations of the velocity components, pressure, temperature, and in compressible flows, density.



We can define our quantity f as: $f = \bar{f} + f'$

We can prove that $\overline{f'} = 0$ ↳ fluctuating component

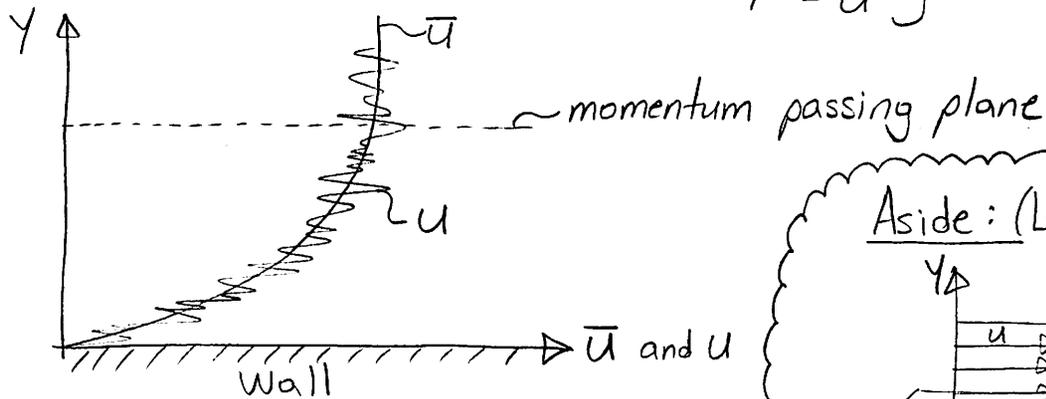
Additive law of expectation

$$\bar{f} = \overline{\bar{f} + f'} = \bar{\bar{f}} + \overline{f'} = \bar{f} + \overline{f'} \Rightarrow \overline{f'} = 0$$

$$\bar{f}(t_0) = \frac{1}{\tau} \int_{t_0 - \tau/2}^{t_0 + \tau/2} f dt \quad (\text{Definition of } \bar{f})$$

Note also that: $\overline{f f'} = \overline{f} \overline{f'} = 0 \Rightarrow \overline{f f'} = 0$
 \downarrow
 $\tau \rightarrow \text{Constant}$

Now if we focus on velocity, i.e. $\left. \begin{matrix} f = u \\ \bar{f} = \bar{u} \\ f' = u' \end{matrix} \right\} u = \bar{u} + u'$



The average shear stress is written as:

$$\bar{\tau} = \underbrace{\mu \frac{\partial u}{\partial y}}_{\text{Laminar Component}} - \underbrace{\overline{\rho u v}}_{\text{Turbulent Component (momentum leaving cv)}} \Rightarrow \text{Control volume on a fluid element inside the boundary layer, momentum balance. See page 100 of notes.}$$

Note we typically write $\tau = \mu \frac{\partial u}{\partial y}$ for laminar fully developed flow over a flat plate since we evaluate $\partial u / \partial y$ at the wall, where $v = 0$. Also, in the fluid, $v' = 0$, and $\bar{v} \ll \bar{u}, \bar{v} \approx 0$.

$$\bar{\tau} = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u v}$$

$$u = \bar{u} + u'$$

$$v = \bar{v} + v'$$

For fully developed flow, $\bar{v} = 0, \bar{v}' = 0$

$$\overline{u v} = \overline{(\bar{u} + u')(\bar{v} + v')} = \overline{\bar{u} \bar{v}} + \overline{\bar{u} v'} + \overline{u' \bar{v}} + \overline{u' v'} = \overline{\bar{u} \bar{v}} + \overline{\bar{u} v'} + \overline{u' \bar{v}} + \overline{u' v'} = \overline{\bar{u} \bar{v}} + \overline{u' v'}$$

$\overline{u v} = \overline{\bar{u} \bar{v}} + \overline{u' v'} \Rightarrow$ Not trivial to calculate, but we can model

Note, $\overline{u'} = 0$ and $\overline{v'} = 0$, however $\overline{u' v'} \neq 0$.

Note also, $\bar{v} \ll \bar{u}$, and $\bar{v} < v'$ (for boundary layer flow)
 $\frac{\overline{u'v'}}{\bar{u}\bar{v}} \gg 1$, So we can assume $\bar{u}\bar{v} \approx 0$ since $\bar{v} \approx 0$
 We see from before that: (Using some intuition)

$$\overline{u'v'} \rightarrow 0 \text{ as } \frac{\partial \bar{u}}{\partial y} \rightarrow 0 \text{ (Outside the b.l.)}$$

$$\overline{u'v'} \uparrow \text{ as } \frac{\partial \bar{u}}{\partial y} \uparrow \text{ (Shear at the wall is highest)}$$

To build a useful model by assuming $\overline{u'v'}$ is proportional to $\frac{\partial \bar{u}}{\partial y}$

Aside: $\overline{u'v'}$ = negative hence the turbulent stress increases overall shear $\bar{\tau}$.

$$\begin{aligned} \bar{\tau} &= \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{v'u'} \\ &= \mu \frac{\partial \bar{u}}{\partial y} - \underbrace{\left(\text{factor which reflects turbulent mixing} \right)}_{\equiv -\rho \epsilon} \cdot \frac{\partial \bar{u}}{\partial y} \end{aligned}$$

or

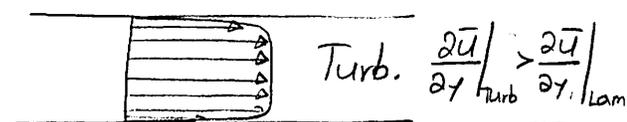
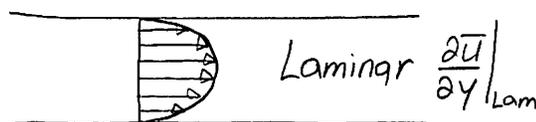
$$\boxed{\bar{\tau} = \rho(\nu + \epsilon) \frac{\partial \bar{u}}{\partial y}} \Rightarrow \epsilon = \text{eddy diffusivity} \left[\frac{m^2}{s} \right]$$

This model is often used in computational fluid dynamics (CFD) software packages to deal with turbulent flows.

We can also re-write this as:

$$\bar{\tau} = \underbrace{\rho \nu}_{\mu} + \underbrace{\rho \epsilon}_{\mu_t} \Rightarrow \text{Turbulent viscosity}$$

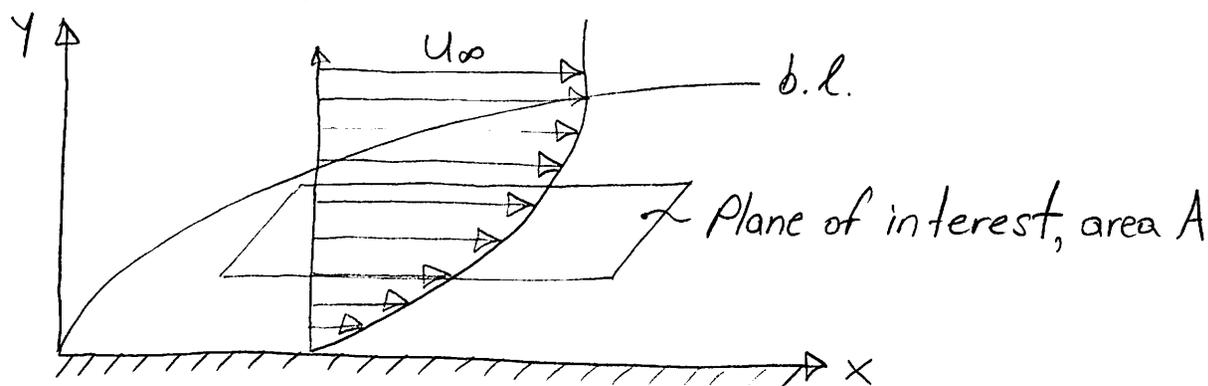
Looking at our experimental velocity profiles:



Since $\tau_0 = \rho \overline{u'v'}$

$$u', v' \sim \sqrt{\frac{\tau_0}{\rho}} \Rightarrow \tau_0 \cong \text{Turbulent shear stress or Reynolds stress}$$

We can now relate our viscosity to mixing length, l , by creating an analogy with molecular viscosity.



We know that shear stress is the time rate of change of momentum. Looking at our plane of interest, and considering the rate of molecular motion across the plane:

- Assuming:
- 1) n molecules per unit volume
 - 2) $\frac{1}{3}$ of molecules have velocities along the y -dir.
 - 3) Half of these, or $\frac{1}{6}n$ molecules per unit volume have mean velocity \bar{v} in the $+y$ -direction; the other half have a mean velocity \bar{v} in the $-y$ direction.

From this, we can say at any given time there are $\frac{1}{6}n\bar{v}$ molecules that cross our plane from below, and $\frac{1}{6}n\bar{v}$ from above.

Also, molecules that cross have, on average, experienced their last collisions at a distance λ (λ = mean free path) from the plane.

Since the mean velocity $u_x = u_x(y)$ is a function of y , we can write a balance equation for momentum.

The mean x-component of momentum transported per unit time per unit area across the plane upwards = $\frac{1}{6} n \bar{v} [m u_x (y - \lambda)]$ ①

The mean x-component of momentum transported per unit time per unit area across the plane downwards = $\frac{1}{6} n \bar{v} [m u_x (y + \lambda)]$ ②

$$\tau = \frac{1}{6} n \bar{v} m [u_x (y - \lambda) - u_x (y + \lambda)] \Rightarrow \text{Taylor series exp. H.O.T. terms drop}$$

$$\tau = \frac{1}{6} n \bar{v} m \left(-2 \frac{\partial u_x}{\partial y} \lambda \right) = -\mu \frac{\partial u_x}{\partial y} \Rightarrow n m = \rho \text{ (density)}$$

$$\boxed{\mu = \frac{1}{3} \rho \bar{v} \lambda} \Rightarrow \text{Note if you substitute for } \bar{v}, \lambda, \text{ you will see that } \mu \neq f(\rho, \text{ or } \rho)!$$

Now back to our turbulence derivation:

$$\mu_t \sim \rho v_t l, \text{ where } v_t = \text{turbulent or friction velocity}$$

$$\frac{\mu_t}{\rho} = \epsilon = v_t \cdot l \sim \sqrt{\frac{\tau_0}{\rho}} \cdot l, \quad v_t \sim v' \text{ or } u'$$

$$\frac{\tau_0}{\rho} = \epsilon \frac{\partial \bar{u}}{\partial y}$$

$$\frac{\tau_0}{\rho} = \sqrt{\frac{\tau_0}{\rho}} \cdot l \frac{\partial \bar{u}}{\partial y} = \sqrt{\frac{\tau_0}{\rho}} \cdot K y \frac{\partial \bar{u}}{\partial y} \quad \text{①}$$

$$\boxed{l = K y}, \quad K = 0.41 \Rightarrow \text{Von Karman constant (from } l = \text{mixing length or eddy size experiment)}$$

This says that turbulent eddies at a location y must be no bigger than the distance to the wall. Makes sense since eddies cannot cross into the wall

If we further non-dimensionalize:

$$u^+ = \frac{u}{\sqrt{\tau_0/\rho}} \quad , \quad y^+ = \frac{y\sqrt{\tau_0/\rho}}{\nu}$$

$$\partial u = \sqrt{\frac{\tau_0}{\rho}} \partial u^+ \quad , \quad \partial y = \frac{\nu \partial y^+}{\sqrt{\tau_0/\rho}}$$

Back substitute into ①

$$\frac{\tau_0}{\rho} = K y \sqrt{\frac{\tau_0}{\rho}} \cdot \frac{\partial u^+}{\partial y^+} \cdot \sqrt{\frac{\tau_0}{\rho}} \cdot \sqrt{\frac{\tau_0}{\rho}} \cdot \frac{1}{\nu}$$

$$1 = \underbrace{\frac{y}{\nu} \cdot \sqrt{\frac{\tau_0}{\rho}}}_{y^+} \cdot K \frac{\partial u^+}{\partial y^+}$$

$$K y^+ \frac{\partial u^+}{\partial y^+} = 1$$

$$\int \partial u^+ = \frac{1}{K} \int \frac{\partial y^+}{y^+}$$

$$\boxed{u^+ = \frac{1}{K} \ln y^+ + C} \Rightarrow \text{The log layer (away from the wall)}$$

Note, very near the wall, we can say that the turbulent eddies are very small since $l = Ky$. So:

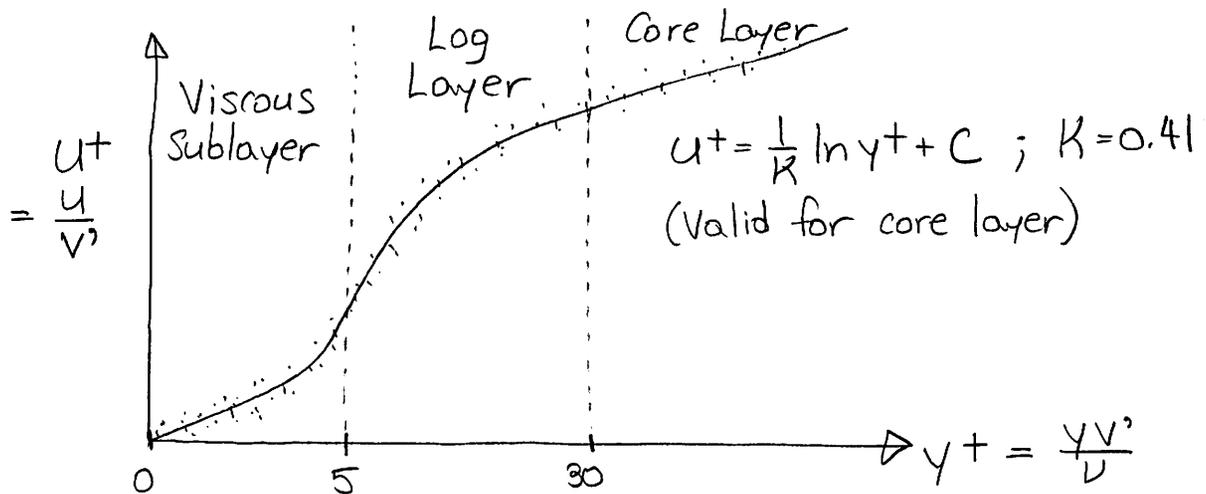
$$l \rightarrow 0, \text{ and } \varepsilon = v' l \rightarrow 0 \Rightarrow \text{Viscosity dominates} \\ \text{Turbulent viscosity} \rightarrow 0$$

$$\tau_0 = \mu \frac{\partial u}{\partial y} \approx \mu \frac{u}{y}$$

$$\approx \rho \nu \frac{u}{y} \Rightarrow \frac{\tau_0}{\rho} = \nu \frac{u}{y} \Rightarrow \sqrt{\frac{\tau_0}{\rho}} \cdot \frac{y}{\nu} = \frac{u}{\sqrt{\tau_0/\rho}}$$

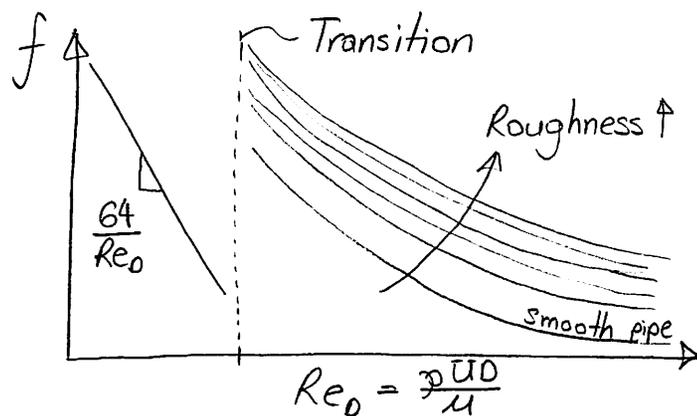
$$\boxed{u^+ = y^+} \Rightarrow \text{The Viscous sublayer, } y^+ < 5. \quad (148)$$

Note people have plotted and measured these results experimentally:



$0 < y^+ < 5$	$U^+ = y^+$	⇒ Experimentally Determined. (Note, core layer $K=0.41$)
$5 < y^+ < 30$	$U^+ = 5.0 \ln y^+ - 3.05$	
$y^+ > 30$	$U^+ = 2.5 \ln y^+ + 5.5$	

But we as engineers care about pressure drop. For this, we cannot solve analytically due to turbulence, but we can solve experimentally. (Moody chart)



See page 368 of Mills

$4.0 \times 10^4 < Re_0 < 10^5$

Good correlations to remember are: $f = 0.184 Re_0^{-0.2}$

$f = (0.790 \ln Re_0 - 1.64)^{-2} \Rightarrow 10^4 < Re_0 < 10^6$

$f = \frac{\Delta P}{(L/D) \frac{1}{2} \rho u^2}$, $C_f = \frac{\tau_w}{\frac{1}{2} \rho u^2}$, $C_f = \frac{f}{4}$ (for smooth pipes)

END OF LECTURE 17

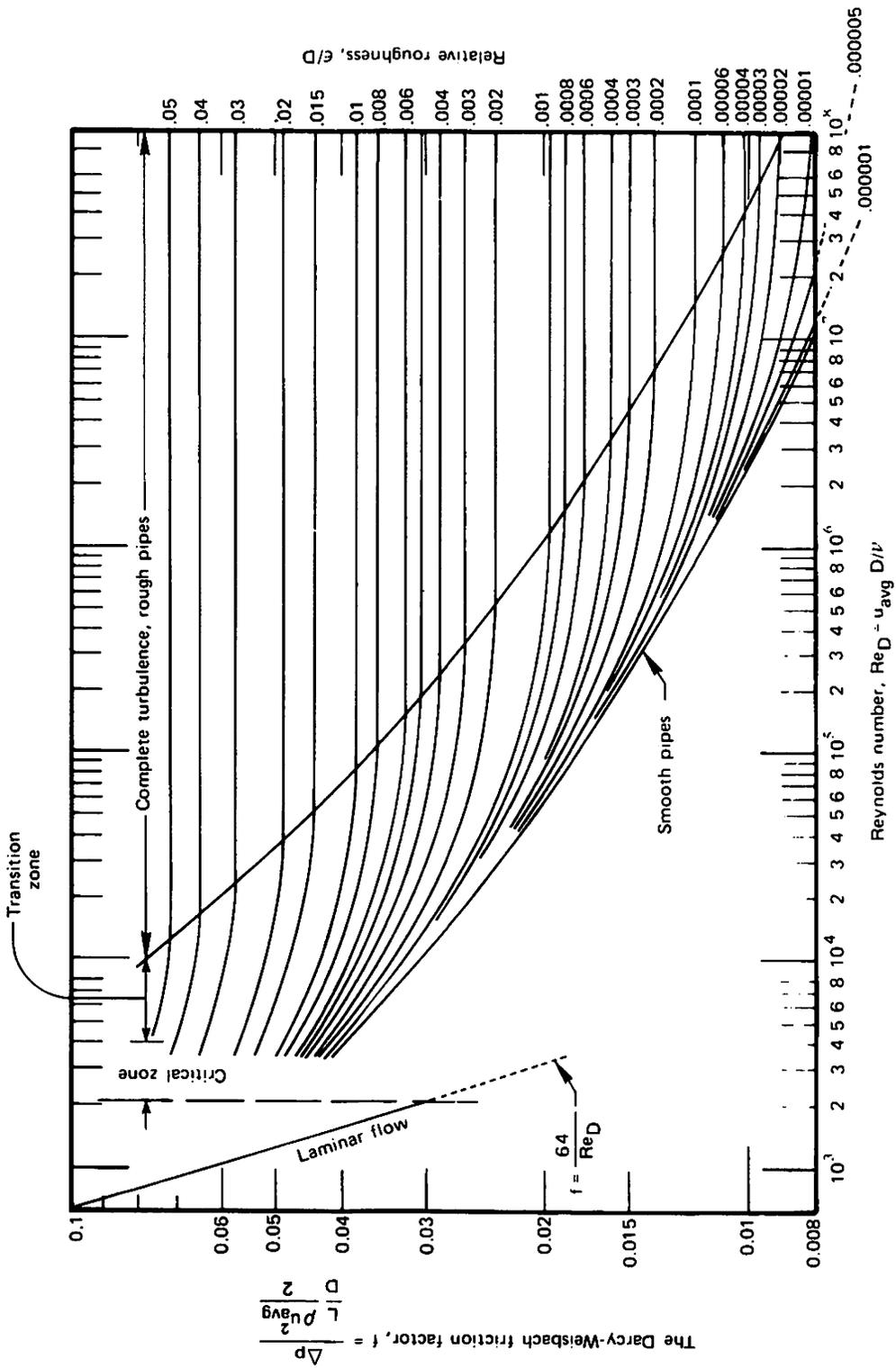
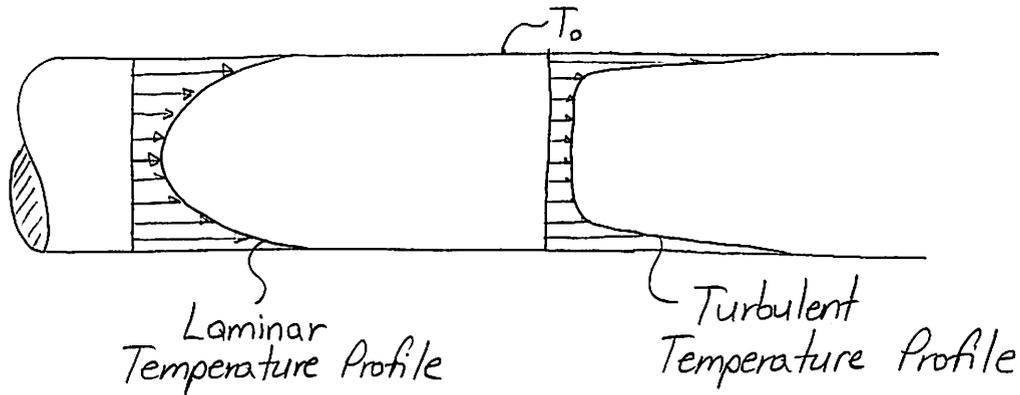


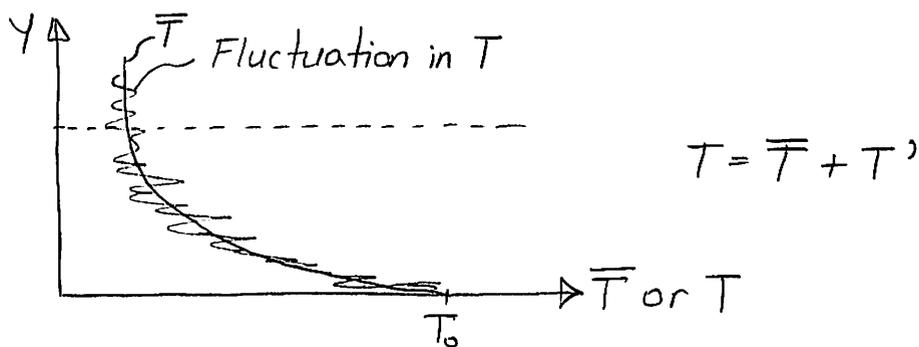
Figure 7.6 Pipe friction factors.

* Adapted from Lienhard & Lienhard, "A Heat Transfer Textbook"

Note however we want to solve for heat transfer



We can follow the exact same procedure as we did with the momentum balance, but now with energy



$$\bar{q}'' = -k \frac{\partial \bar{T}}{\partial y} + \overline{v'T} \rho c_p \Rightarrow \text{Energy integral formulation}$$

$$\bar{q}'' = -k \frac{\partial \bar{T}}{\partial y} + \overline{v'T'} \cdot \rho c_p$$

$$\overline{v'T'} = -\epsilon_T \frac{\partial \bar{T}}{\partial y}$$

$$Pr_{\pm} = \frac{\epsilon}{\epsilon_{\pm}} \rightarrow \text{Turbulent Prandtl Number}$$

Now we can do the following:
So our heat transfer becomes:

$\bar{q}'' = -\rho c_p (\alpha + \epsilon_T) \frac{\partial \bar{T}}{\partial y}$
$\bar{\tau} = \rho (\nu + \epsilon) \frac{\partial \bar{u}}{\partial y}$

We can now make a useful analogy
(Turbulent Colburn analogy)

$$\frac{\bar{q}''/\Delta T}{\rho C_p U} = \frac{1}{2} \left(\frac{\bar{C}_o}{\frac{1}{2} \rho U^2} \right) \frac{1}{Pr^{2/3}} = \frac{C_f}{2 Pr^{2/3}} = \frac{f}{8 Pr^{2/3}}$$

$$St = \frac{C_f}{2} \cdot \frac{1}{Pr^{2/3}} = \frac{f}{8} \cdot \frac{1}{Pr^{2/3}} \Rightarrow \text{Stanton number}$$

$$St = \frac{Nu_D}{Re_D \cdot Pr} = \frac{\bar{h}}{\rho C_p U} \Rightarrow \text{Note most results are plotted as } St \text{ vs } Re_D, \text{ for experiments.}$$

Aside: Comes from:

$$\frac{Nu_D}{Re_D \cdot Pr} = \frac{\frac{\bar{h} D}{k}}{\left(\frac{\rho U D}{\mu} \right) \left(\frac{\rho \nu}{k} \right)}$$

$$= \frac{\bar{h}}{\rho C_p U} = \frac{\bar{q}''/\Delta T}{\rho C_p U}$$

Note, this is the same as we had before in laminar flow over a flat plate called the Colburn analogy. See page 90 of the class notes.

$$St \cdot Pr^{2/3} = \frac{C_f}{2} = \frac{f}{8} \quad (\text{For smooth pipes})$$

$$\frac{Nu_D}{Re_D \cdot Pr} \cdot Pr^{2/3} = \frac{f}{8} \Rightarrow \text{from page 149, } f = 0.184 Re_D^{-0.2}$$

$$\frac{Nu_D}{Re_D \cdot Pr^{1/3}} = \frac{0.184}{8} Re_D^{-0.2}$$

$$Nu_D = 0.023 Re^{0.8} Pr^{1/3} \Rightarrow \text{Turbulent pipe flow Nusselt \# Analytical Result}$$

Valid for: $4 \times 10^4 \leq Re_D \leq 10^5$

Note, the experimental result is:

$$Nu_{D, \text{exp}} = 0.023 Re^{0.8} Pr^{0.4} \Rightarrow \text{Dittus-Boelter equation}$$

$\Rightarrow Re_D > 10000$.

Most accurate correlation to use is Gnielinski correlation for Pipe flow

$$Nu_D = \frac{(f/8)(Re_D - 1000)Pr}{1 + 12.7(f/8)^{1/2}(Pr^{2/3} - 1)} \Rightarrow 3000 \leq Re_D \leq 5 \times 10^6$$

Turbulent Heat Transfer over a Flat Plate

For Laminar flow, we derived:

$$C_{f,x} = \frac{\bar{\tau}_{0,x}}{\frac{1}{2} \rho V_\infty^2} = \frac{0.664}{Re_x^{1/2}}$$

$C_{f,x}$ = skin friction coefficient at location x along plate

$\bar{\tau}_{0,x}$ = shear stress at plate surface at location x along plate

$$C_{f,av} = \frac{1.328}{Re_L^{1/2}}$$

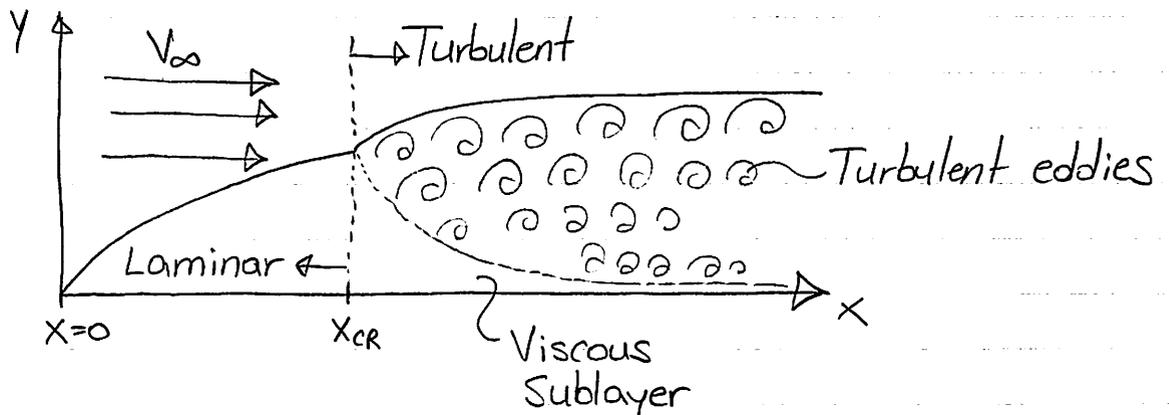
$C_{f,av}$ = average friction coefficient over the whole plate.

$$\frac{\delta}{x} = \frac{4.92}{Re_x^{1/2}}$$

$$Nu_x = 0.332 Re_x^{1/2} Pr^{1/3} \Rightarrow \overline{Nu}_L = \frac{\bar{h}L}{k} = 0.664 Re_L^{1/2} Pr^{1/3}$$

$$St_x \cdot Pr^{2/3} = \frac{C_{f,x}}{2}$$

Now we can examine the turbulent flow regime



$$Re_{CR} = \frac{V_\infty x_{CR}}{\nu} = 50,000 - 500,000$$

Due to the difficulty in analytically modeling the turbulent boundary layer on a flat plate, many of the following expressions stem from experimental correlations.

$$C_{f,x} = 0.0592 Re_x^{-1/5}$$

$$10^5 \leq Re_x \leq 10^7$$

$$C_{f,x} = 0.026 Re_x^{-1/7}$$

$$10^7 \leq Re_x \leq 10^9$$

\Rightarrow Turbulent flow skin friction coefficient.

For even greater accuracy: (Developed by White)

$$\boxed{C_{f,x} = \frac{0.455}{(\ln 0.06 Re_x)^2} ; 10^5 \leq Re_x \leq 10^9} \Rightarrow \text{Turbulent flow over a flat plate.}$$

Note, x is still the distance from the beginning of the flat plate, not from x_{CR} .

Typically, we want the average skin friction coefficient for the whole plate

$$\begin{aligned} \overline{C_{o,x}} &= \frac{1}{2} \rho V_\infty^2 C_{f,x} = \frac{1}{2} \rho V_\infty^2 \cdot 0.0592 Re_x^{-0.2} \\ &= \frac{1}{2} \rho V_\infty^2 \cdot 0.0592 \left(\frac{\rho V_\infty x}{\mu} \right)^{-0.2} \\ &= C x^{-0.2} \quad \text{where } C = \frac{1}{2} \rho V_\infty^2 \cdot 0.0592 \left(\frac{\rho V_\infty}{\mu} \right)^{-0.2} \end{aligned}$$

$$\overline{C_{o,L}} = \frac{C}{L} \int_0^L \frac{dx}{x^{0.2}} = \frac{C x^{0.8}}{0.8L} = \frac{C}{0.8L^{0.2}} = \frac{CL^{-0.2}}{0.8} = \frac{\overline{C_{o,L}}}{0.8}$$

We can now solve for $C_{f,av}$

$$C_{f,av} = \frac{C_{f,L}}{0.8} = \frac{0.0592 Re_L^{-1/5}}{0.8}$$

$$\boxed{C_{f,av} = 0.074 Re_L^{-1/5}}$$

Applying our Colburn analogy: $St_x = \frac{Nu_x}{Re_x Pr}$ (Stanton number)

We know for a flat plate: $St_x Pr^{2/3} = \frac{C_{f,x}}{2}$

$$\frac{Nu_x}{Re_x Pr} \cdot Pr^{2/3} = \frac{C_{f,x}}{2}$$

$$Nu_x = \frac{C_{f,x}}{2} Re_x Pr^{1/3}$$

Using $C_{f,x} = 0.0592 Re_x^{-1/5}$

$$Nu_x = \frac{0.0592}{2} Re_x^{-1/5} \cdot Re_x Pr^{1/3}$$

$$Nu_x = 0.029 Re_x^{0.8} Pr^{1/3} \Rightarrow \text{Colburn correlation (from analogy)}$$

$$Nu_x = 0.029 Re_x^{0.8} Pr^{0.43} \Rightarrow \text{Whitaker correlation}$$

$$\overline{Nu}_L = \frac{\bar{h}L}{k_f} = 0.036 Re_L^{0.8} Pr^{0.43} \Rightarrow \begin{cases} 0.7 < Pr < 400 \\ 5 \times 10^5 < Re_x < 3 \times 10^7 \end{cases}$$

For a more accurate correlation

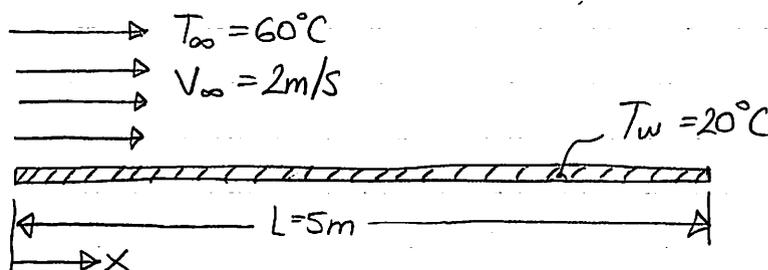
$$Nu_x = \frac{\left(\frac{C_{f,x}}{2}\right) \cdot Re_x Pr}{1 + 12.7 \left(\frac{C_{f,x}}{x}\right)^{1/2} (Pr^{2/3} - 1)} \Rightarrow \text{White correlation}$$

$$\begin{matrix} 0.5 < Pr < 2000 \\ 5 \times 10^5 < Re_x < 10^7 \end{matrix}$$

Note, the closest we got to an analytical result was with the Colburn analogy, but even then we had to use the local skin friction coefficient ($C_{f,x}$) based on experiments.

For more reading, see page 312 in Mills textbook.

Example | Engine oil at 60°C flows over a 5m long flat plate whose temperature is 20°C with a velocity of 2 m/s. Determine the total drag force and heat transfer per unit width of the entire plate.



Assume : 1) $T_w = \text{constant} = f(x) = 20^\circ\text{C}$
 2) Constant properties, steady flow.

$$T_f = \bar{T} = \frac{T_w + T_\infty}{2} = 40^\circ\text{C}$$

$$\rho_{\text{oil}} = 876 \text{ kg/m}^3$$

$$k_{\text{oil}} = 0.144 \text{ W/m}\cdot\text{K}$$

$$Pr = 2870 \text{ (Note how large this Prandtl number is!)}$$

$$U_{\text{oil}} = 242 \times 10^{-6} \text{ m}^2/\text{s}$$

$$Pr_{\text{water}} \approx 7$$

$$Pr_{\text{air}} \approx 1$$

First we check if we are in the turbulent or laminar regime

$$Re_L = \frac{V_\infty L}{\nu_{\text{oil}}} = 4.13 \times 10^4 < 5.0 \times 10^5 \text{ (Laminar flow)}$$

We know from our previous derivations (page 85 of notes)

$$C_{f,av} = \frac{\bar{C}_{o,av}}{\frac{1}{2} \rho V_\infty^2} = 1.328 Re_L^{-0.5} = 0.00653$$

$$F_D = C_{f,av} \cdot A \cdot \frac{\rho V_\infty^2}{2} = (0.00653)(5\text{m} \cdot 1\text{m}) \frac{(876 \text{ kg/m}^3)(2\text{m/s})^2}{2}$$

$$\boxed{F_D = 57.2 \text{ N}}$$

For heat transfer:

$$Nu_L = \frac{\bar{h} L}{k_{\text{oil}}} = 0.664 Re_L^{1/2} Pr^{1/3} = 1918$$

$$\bar{h} = 55.2 \text{ W/m}^2 \cdot \text{K}$$

$$\boxed{Q = hA(T_\infty - T_w) = 11.04 \text{ kW}}$$

Now what if we 15x the speed of our flow? Will our friction and heat transfer 15x as well?

$$Re_L = \frac{V_\infty L}{\nu_{oil}} = 6.2 \times 10^5 > Re_{crit} \text{ (Turbulent flow)}$$

$$C_{f,av} = 0.074 Re_L^{-0.2} = 0.00514 \text{ (page 153 of notes)}$$

$$F_D = C_{f,av} \cdot A \cdot \frac{\rho V_\infty^2}{2} = (0.00514)(5m^2) \frac{(876 kg/m^3)(90m/s)^2}{2}$$

$$F_D = 10.13 kN$$

For heat transfer:

$$\overline{Nu}_L = \frac{\bar{h}L}{k_{oil}} = 0.036 Re_L^{0.8} Pr^{0.43} \text{ (Estimate only since } Pr > 400)$$

$$= 0.036 (6.2 \times 10^5)^{0.8} (2870)^{0.43} = 47545 \text{ } \rightarrow \text{Could have used White correlation}$$

$$\bar{h} = \frac{\overline{Nu}_L \cdot k_{oil}}{L} = \frac{(47545)(0.144 W/m \cdot K)}{5m}$$

$$\bar{h} = 1369.3 W/m^2 \cdot K$$

$$Q = \bar{h}A(T_\infty - T_w) = 273.9 kW$$

Now lets compare:

$$\frac{V_{\infty,2}}{V_{\infty,1}} = \frac{15}{1} = 15$$

$$\frac{F_{D,2}}{F_{D,1}} = \frac{10130N}{57.2N} = 177.1 \text{ (Drag increases dramatically)}$$

$$\frac{Q_2}{Q_1} = \frac{273.9 kW}{11.04 kW} = 24.8 \text{ (Heat transfer increases modestly)}$$

Note, this is where you strike a balance in heat exchanger design in terms of heat transfer and pressure drop.

Note also, another important factor to keep in mind is the working fluid. In this case, oil is used with a high Prandtl number and high viscosity. If we had chosen water or air, how would our results change:

Water:

$$\rho_w = 1000 \text{ kg/m}^3$$

$$k_w = 0.6 \text{ W/m}\cdot\text{K}$$

$$Pr = 7$$

$$\nu_w = 0.658 \times 10^{-6} \text{ m}^2/\text{s}$$

$$Re_L = \frac{V_\infty L}{\nu_w} = \frac{(2 \text{ m/s})(5 \text{ m})}{0.658 \times 10^{-6} \text{ m}^2/\text{s}} = 1.52 \times 10^7 \text{ (Turbulent flow)}$$

$$C_{f,av} = 0.074 Re_L^{-0.2} = 0.0027$$

$$F_{D,w} = C_{f,av} \cdot A \cdot \frac{\rho V_\infty^2}{2} = (0.0027)(5 \text{ m}^2) \frac{(1000 \text{ kg/m}^3)(2 \text{ m/s})^2}{2}$$

$$F_{D,w} = 27.1 \text{ N}$$

For heat transfer:

$$\overline{Nu}_L = \frac{\bar{h} L}{k_w} = 0.036 Re_L^{0.8} Pr^{0.43}$$

$$= 0.036 (1.52 \times 10^7)^{0.8} (7)^{0.43} = 46256.2$$

$$\bar{h} = \frac{\overline{Nu}_L \cdot k_w}{L} = \frac{(46256.2)(0.6 \text{ W/m}\cdot\text{K})}{5 \text{ m}}$$

$$\bar{h} = 5550.8 \text{ W/m}^2 \cdot \text{K}$$

$$Q_w = \bar{h} A (T_\infty - T_w) = 1.11 \text{ MW}$$

$$\frac{F_{D,oil}}{F_{D,w}} = 2.11 \quad \frac{Q_{oil}}{Q_w} = 0.01 \text{ or } 1\% \text{ only.}$$

This simple analysis shows the importance of selecting the proper working fluid. You may be asking yourself, why would anybody use a high Pr oil as a working fluid in a thermal application? Isn't water always better?

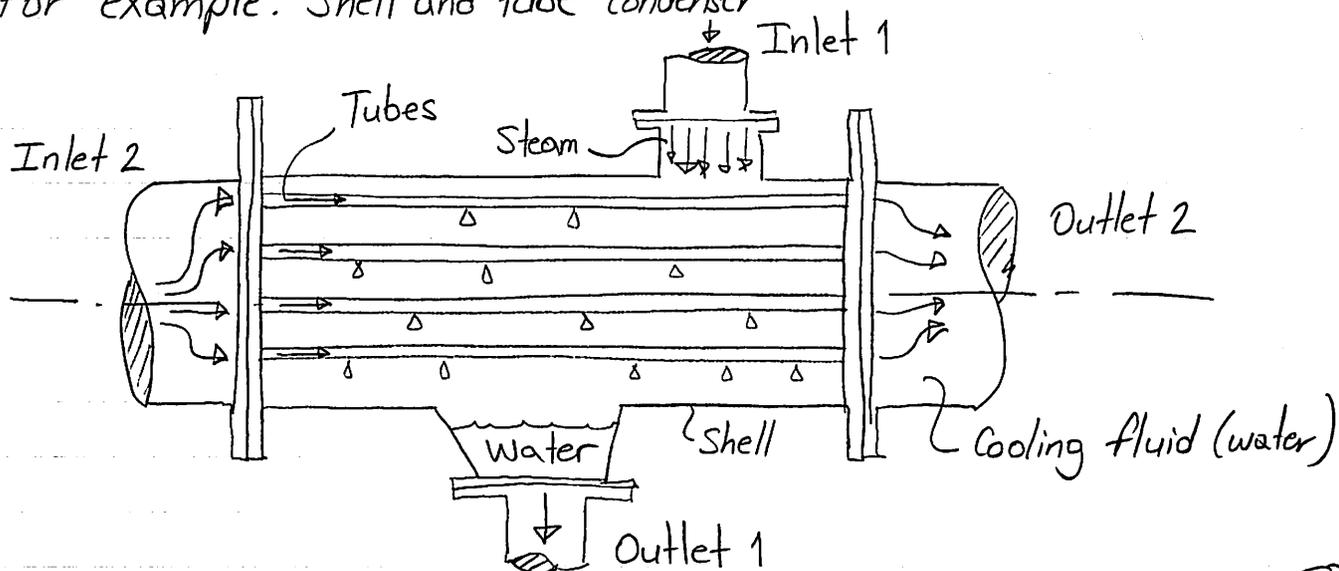
The answer is NO! Oils have 2 main advantages over water.

- 1) Temperature range (Typically -40 to 300°C). Water will freeze or boil unless its pressurized which makes the advantages of water not worth it (pressurization is tricky and requires a lot of design, maintenance, and for water, high pressures).
- 2) Low volatility. Water has a relatively high vapor pressure meaning it evaporates and coolant can be lost and needs to be replaced. Oil's vapor pressure at $T = 0-60^{\circ}\text{C}$ is usually 0 Pa , meaning no evaporative mass loss.
- 3) Corrosion resistance and enhanced lubrication.

Cylinders and Spheres

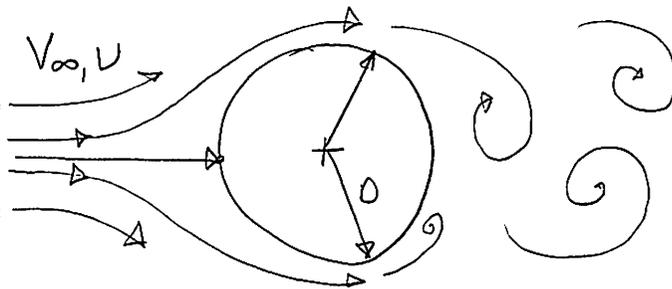
So far, we've only looked at internal flow in a tube, and external flow on a flat plate. Most heat exchangers are not in this configuration.

For example: Shell and tube condenser

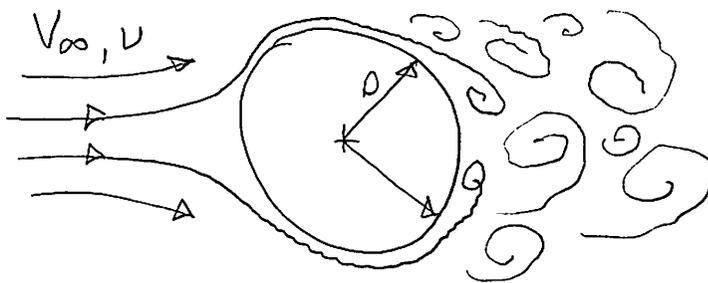


Here we have a tube geometry with external heat transfer.

Since external flow, the turbulence criteria still remains, i.e. $Re_{tr} \approx 3 \times 10^5$.

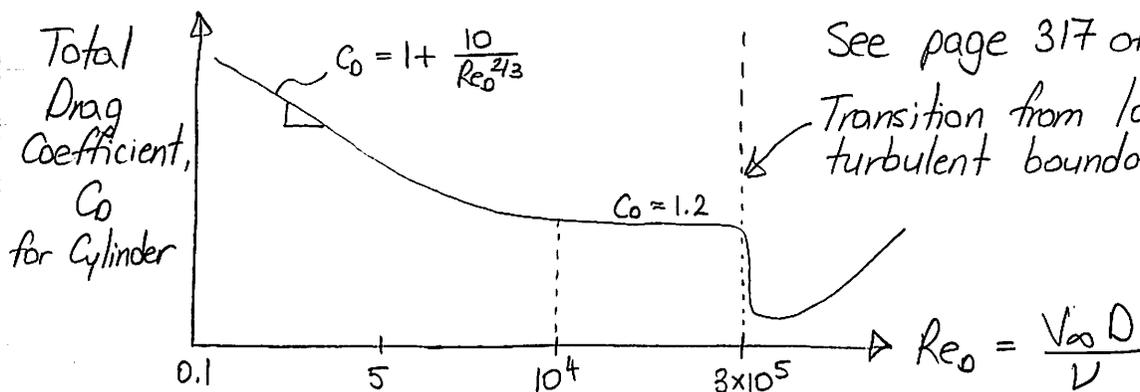


Note, Laminar flow with separation of the laminar boundary layer.
 $40 \leq Re_0 \leq 150$



Turbulent boundary layer with separation.
 $Re_0 \geq 3.0 \times 10^5$

In these flows, more criteria is required to describe the flow: laminar or turbulent b.l., separated or intact flow, type of wake. In general:



See page 317 of Mills.
 Transition from laminar to turbulent boundary layer.

Note, the turbulent boundary layer, although creating more skin friction drag ($C_{f,x}$) decreases the form, pressure, or wake by keeping the boundary layer from separating, due to its higher efficiency mixing and larger momentum.

$$C_D = \frac{F}{\frac{1}{2} \rho V_\infty^2 A_f}$$

A_f = area of cylinder normal to the flow
 = $D \cdot L$ (Diameter times length).
 F = total drag force due to form and skinfrict.

At low Reynolds numbers, $Re_0 < 10^4$ (viscous drag dominates)

$$C_D = 1 + \frac{10}{Re_0^{2/3}} ; 1 < Re_0 < 10^4 \text{ (Laminar flow)}$$

$$C_D = 1.2 ; Re_0 > 10^4 \text{ (Laminar flow, separated b.l., where wake drag now dominates)}$$

Due to the difficulty in modelling of this flow, most results are from experimental correlations:

$$\overline{Nu}_D = \frac{1}{0.8237 - \ln(Re_0 Pr)^{1/2}} ; Re_0 Pr < 0.2$$

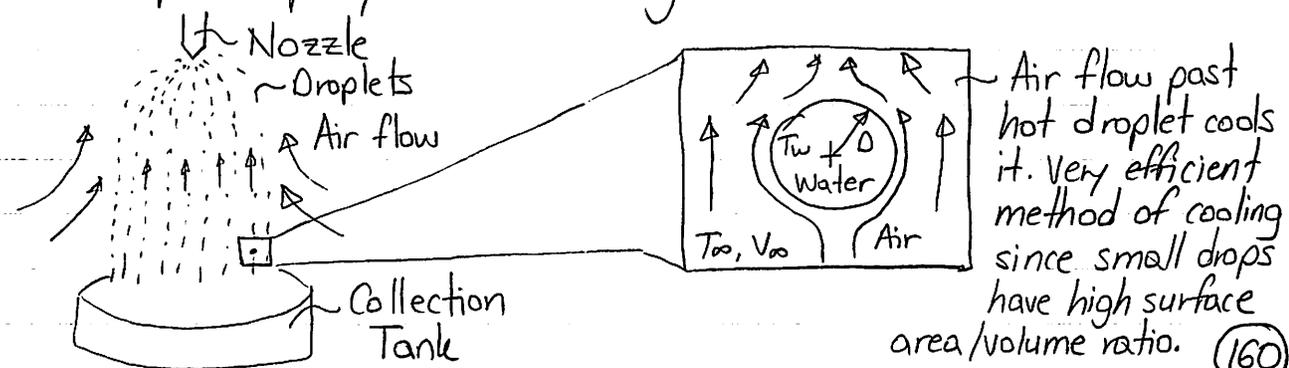
$$\overline{Nu}_D = 0.3 + \frac{0.62 Re_0^{1/2} Pr^{1/3}}{[1 + (0.4/Pr)^{2/3}]^{1/4}} ; Re_0 < 10^4$$

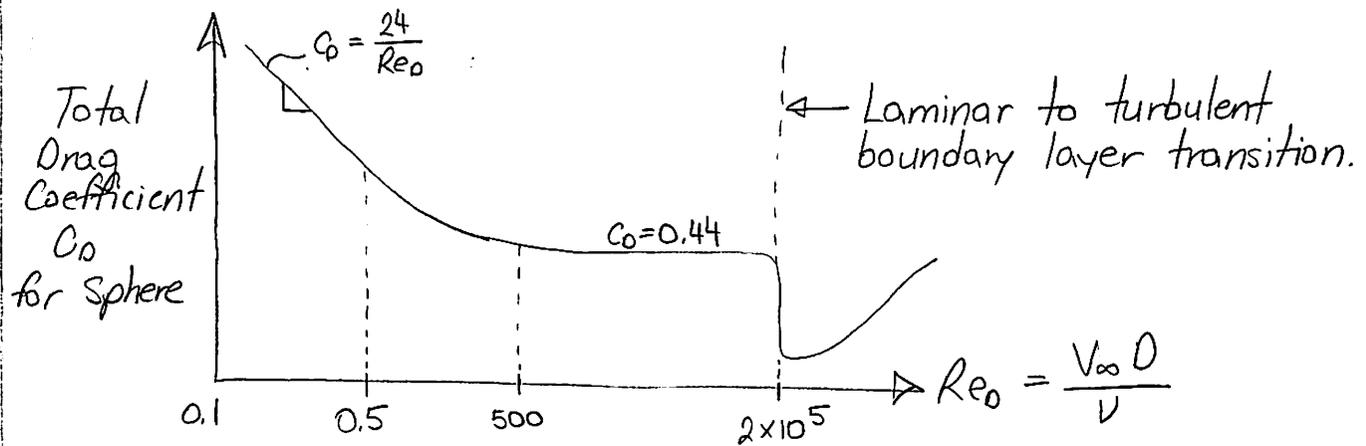
See pg. 319, equations 4.71b and 4.71c for rest of the regimes, in Mills.

Flow over a Sphere

The spherical geometry is not a typical heat exchanger geometry per say, however it is vitally important in flows involving droplets and bubbles.

For example: Spray down cooling in a air cooled condenser





In general, for a sphere:

$C_D = \frac{24}{Re_D}$	$Re_D < 0.5$ (Laminar flow, Stokes flow)	} Experimental Correlations
$F_D = 3\pi\mu V_\infty D$	\Rightarrow Stokes drag	
$C_D \approx \frac{24}{Re_D} \left(1 + \frac{Re_D^{2/3}}{6}\right)$	$2 < Re_D < 500$	
$C_D \approx 0.44$	$500 < Re_D < 2 \times 10^5$	

↳ Can solve for this analytically using our tools.

For heat transfer:

$$\overline{Nu}_D = 2 + (0.4 Re_D^{1/2} + 0.06 Re_D^{2/3}) Pr^{0.4}; \quad 3.5 < Re_D < 8 \times 10^4$$

$$0.7 < Pr < 380$$

Note, as $Re_D \rightarrow 0$, $\overline{Nu}_D = 2$ which corresponds to a sphere conducting heat to an infinite medium:

$$Q_{sph} = \frac{4\pi k (T_1 - T_2)}{\frac{1}{R_1} - \frac{1}{R_2}} = \frac{\Delta T}{R_{thermal}}$$

$$D = 2R_1, \quad R_2 \rightarrow \infty$$

$$Q_{sph} = \frac{24\pi k \Delta T D}{2} = 2\pi k D \Delta T$$

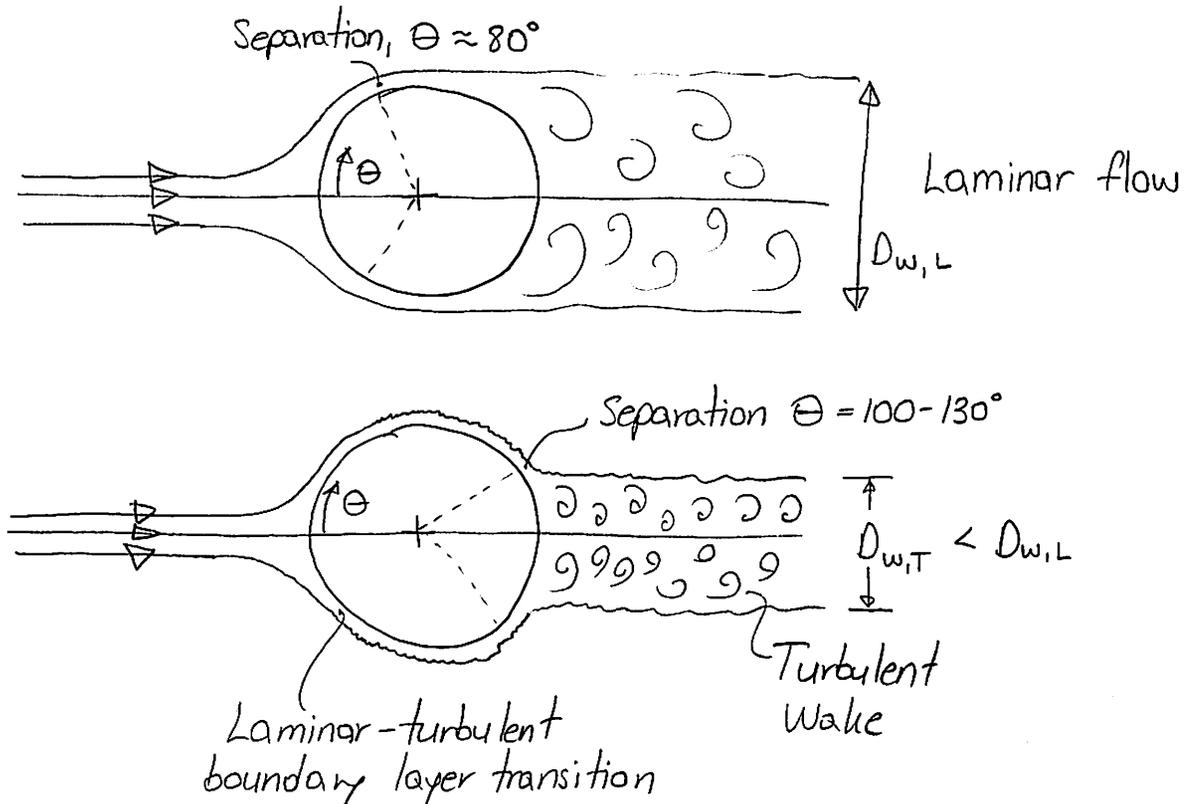
$$\frac{1}{A_{sph}} \left(\frac{Q_{sph}}{\Delta T} \right) \cdot \frac{D}{k} = \frac{1}{24\pi R^2} \cdot 2\pi k D \cdot \frac{D}{k} = \frac{4R^2}{2R^2} = 2 \Rightarrow \boxed{Nu_D = 2} \Rightarrow \text{Makes sense.}$$

Note, the drag on a bluff body can be decomposed into two components:

$$F_D = F_{SF} + F_{FD} \leftarrow \text{Form drag due to pressure difference or wake drag.}$$

↳ Skin Friction Drag, $C_{f,x}$ or Viscous drag.

For flow past a sphere or cylinder, turbulence inside the boundary layer plays a very important effect on drag as we've already seen in the C_D plots:

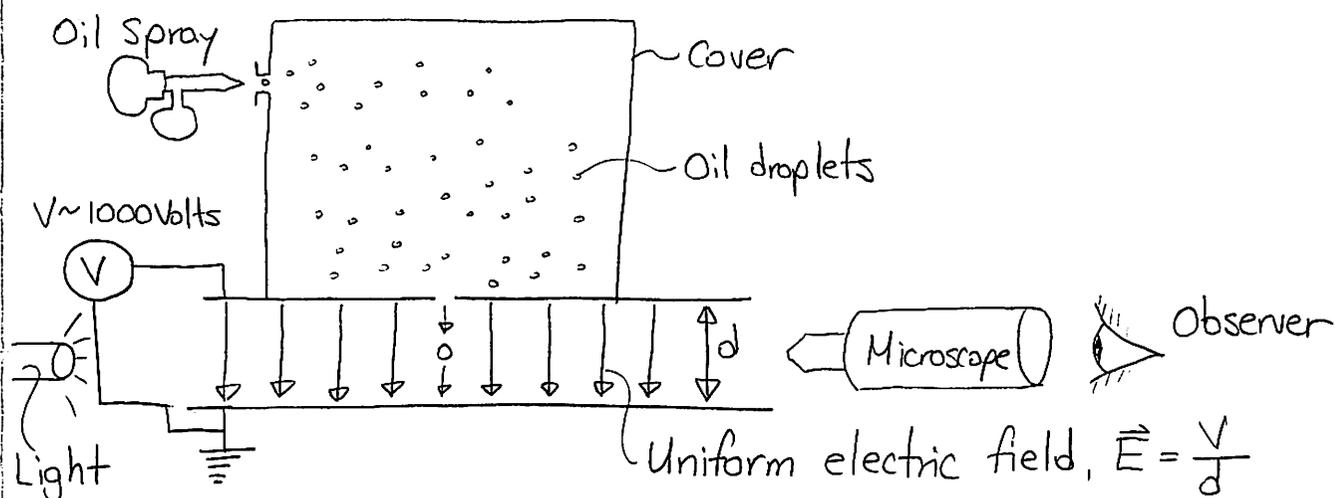


The creation of a turbulent boundary layer tends to enhance mixing, increase momentum in opposition to adverse pressure gradients, and keep the boundary layer attached for longer, reducing the form drag (F_{FD}) more than increasing the skin friction drag (F_{SF}). Note, this is why golf balls are dimpled!

Example | Stokes flow and charge of an electron

In the early 1900's, scientists thought that electrical charge wasn't quantized and was continuous. Robert A. Millikan and his graduate student Fletcher wanted to check.

The experiment:



Millikan and Fletcher would spray oil droplets and let one settle into the gap between the plates.

They used oil due to its low vapor pressure (low evaporation) and because it would gain some charge when sprayed. (tribo-electrification)

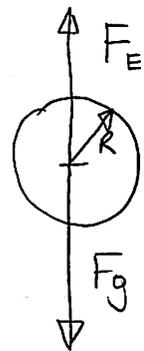
The analysis:

Millikan would first let the droplet fall with no applied electric field and measure its terminal velocity V_t .

Assuming that $Re_0 \ll 1$, he used Stokes drag to calculate the droplet size:

$$F_0 = 6\pi\mu V_t R = F_g = \frac{4}{3}\pi R^3 g (\rho_{oil} - \rho_{air}) \Rightarrow R = \sqrt{\frac{9\mu_{oil} V_t}{2g(\rho_{oil} - \rho_{air})}}$$

Next he would turn the electric field on just enough so that the electric field force was just high enough to balance the gravitational force. So the droplet would remain perfectly still in the microscope:

$$F_E = q \cdot E = q \cdot \frac{V}{d}$$


\Rightarrow Single droplet that he focused on between the parallel plates.

$$F_g = \frac{4}{3} \pi R^3 g (\rho_{oil} - \rho_{air})$$

$$q \cdot \frac{V}{d} = \frac{4}{3} \pi R^3 g (\rho_{oil} - \rho_{air})$$

$$q = \frac{4}{3} \frac{d}{V} \pi R^3 g (\rho_{oil} - \rho_{air})$$

$$= \frac{4}{3} \frac{d}{V} \cdot \pi g (\rho_{oil} - \rho_{air}) \left(\frac{q \mu_{oil} V_E}{2g (\rho_{oil} - \rho_{air})} \right)^{3/2}$$

$$q = 12.7279 \pi \frac{(\mu_{oil} V_E)^{3/2}}{(g (\rho_{oil} - \rho_{air}))^{1/2}} \cdot \frac{d}{V} \Rightarrow \text{electric charge on the individual droplet.}$$

\hookrightarrow Measured parameters: V_E , V , d , ρ_{oil} , ρ_{air} , μ_{oil}
 He then measured the charge of many droplets and tabulated them.
 From his results, Millikan showed a fascinating trend.

$$q_{\text{droplet}} = n \cdot q_{e^-}, \text{ where}$$

$$q_{e^-} = 1.5924 \times 10^{-19} \text{ Coulombs}$$

Today's accepted value is:

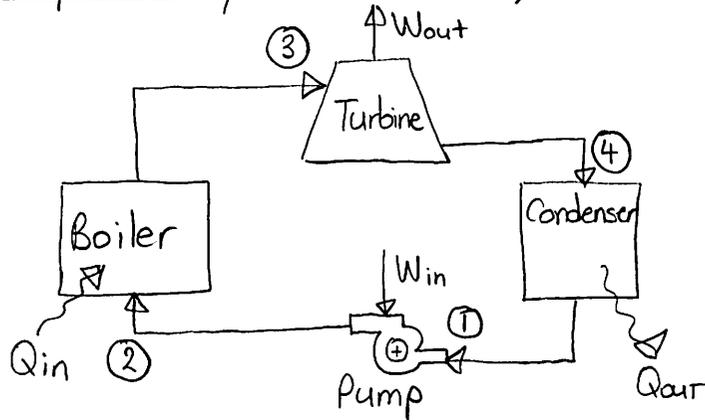
$$q_{e^-} = 1.602176 \times 10^{-19} \text{ Coulombs}$$

Millikan showed definitively that electric charge was indeed quantized and won the Nobel prize in 1923 for it. The main tool he used was Stokes drag!

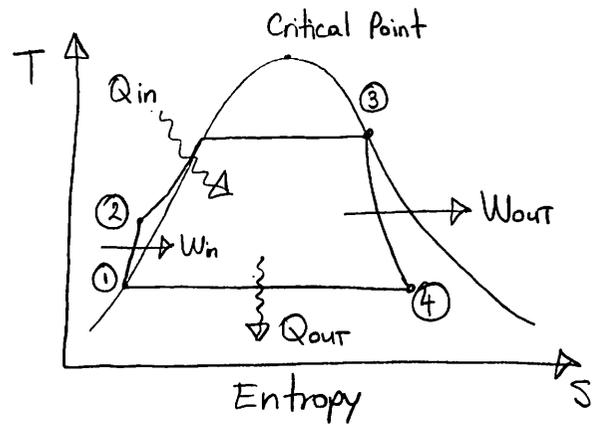
Condensation Heat Transfer

Why do we care? Well 90% of our electricity generated comes from steam power plants. In these plants, the Rankine cycle is used which requires a condenser. Increasing this condensation heat transfer efficiency can increase the overall power plant efficiency and help reduce CO₂ emissions, fuel consumption and cost.

Physical System: (Rankine)

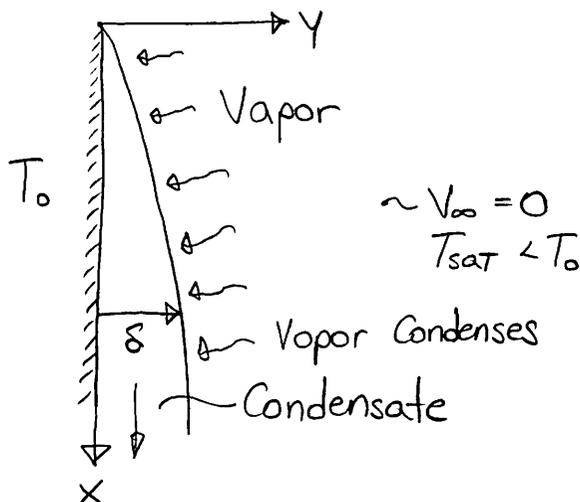


Thermodynamic States:



The condensation and boiling processes are very important for the efficient operation of our most common means of energy production.

Condensation on a Vertical Flat Plate



Assumptions:

- 1) Pure vapor
- 2) Vapor is stationary ($V_{\infty} = 0$)
- 3) $\mathcal{C} = 0$ at the l-v interface
- 4) Inertia can be neglected or Laminar film condensation.
- 5) Constant properties
- 6) $\frac{\partial p}{\partial y} = 0$ (pressure gradients in the y-direction are negligible)

Note assumption #4 is valid since the latent heat of fluids is typically much larger than their specific heat. Therefore we will have very thin films during filmwise condensation. These films move very slowly (due to high shear at the wall) and inertia can be neglected when compared to viscous or gravity forces.

x-momentum: (in the condensate)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho_f} \frac{\partial p}{\partial x} + \nu_f \frac{\partial^2 u}{\partial y^2} + g$$

$$\frac{1}{\rho_f} \frac{\partial p}{\partial x} = \nu_f \frac{\partial^2 u}{\partial y^2} + g \quad (1)$$

Aside:

Another way to show inertia ≈ 0 is conservat of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$v=0, \frac{\partial v}{\partial y} = 0$$

$$\text{So } \frac{\partial u}{\partial x} = 0$$

x-momentum: (in the vapor)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho_g} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + g$$

Since $v_{\infty} = u = v = 0$

$0 (\nu = 0)$

$$\frac{\partial p}{\partial x} = \rho_g g \quad (2) \Rightarrow \text{Hydrostatic pressure (something similar on pg. 131 of notes)}$$

Substituting (2) into (1), we obtain:

$$\frac{1}{\rho_f} \cdot \rho_g g = \nu_f \frac{\partial^2 u}{\partial y^2} + g$$

$$-g(\rho_f - \rho_g) = \nu_f \frac{\partial^2 u}{\partial y^2} \Rightarrow \text{Multiply both sides by } \frac{1}{\rho_f}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho_f - \rho_g}{\rho_f} \cdot \frac{g}{\nu_f} \quad (3)$$

Now we can look at heat transfer. Since inertia is negligible, we can neglect the convective terms in the energy equation. We are left with simple conduction across the film.

$$\underbrace{u \frac{\partial T}{\partial x}} + \underbrace{v \frac{\partial T}{\partial y}}_{v=0} = \alpha \frac{\partial^2 T}{\partial y^2} \Rightarrow \underbrace{\alpha \frac{\partial^2 T}{\partial y^2}} = 0 \quad (4)$$

O since inertia is negligible

Only in y-dir since gradient is in y.

Now we can apply our B.C.'s and solve equation (3)

$$u(y=0) = 0 \quad (\text{No slip at the wall})$$

$$\left. \frac{\partial u}{\partial y} \right|_{y=\delta} = 0 \quad (\text{No shear stress at the l-v interface since } \frac{\mu_g}{\mu_f} \ll 1)$$

Solving, we obtain:

$$u = \frac{(\rho_f - \rho_g) g \delta^2}{2\mu_f} \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \right] \quad (5)$$

Applying our B.C.'s to our energy equation (4)

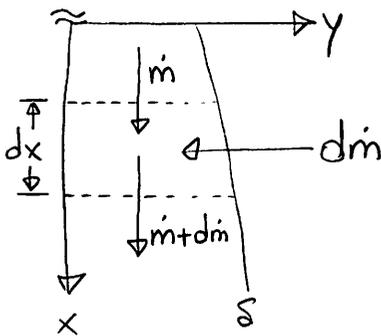
$$T(y=0) = T_0$$

$$T(y=\delta) = T_{\text{sat}}$$

Solving, we obtain:

$$T = T_0 + (T_{\text{sat}} - T_0) \frac{y}{\delta} \quad (6)$$

Note that both (5) and (6) are in terms of δ . We need to solve for δ if we want to move on. Looking closely at our condensate



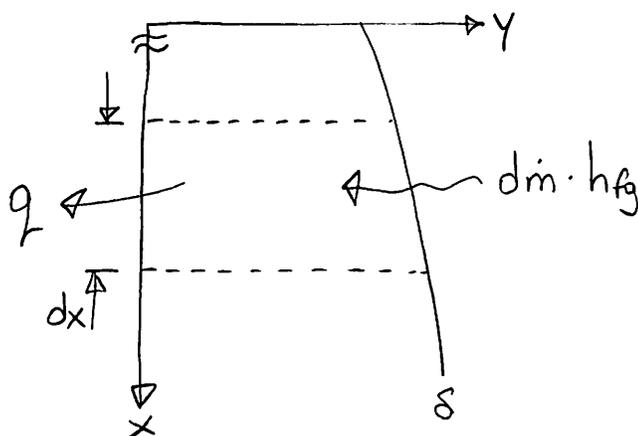
$$\dot{m} = \int_0^{\delta} \rho_f u dy = \int_0^{\delta} \frac{\rho_f (\rho_f - \rho_g) g \delta^2}{2\mu_f} \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \right] dy$$

$$= \frac{\rho_f (\rho_f - \rho_g) g \delta^2}{2 \mu_f} \left[\frac{y^2}{\delta} - \frac{y^3}{3\delta^2} \right] \Big|_0^\delta$$

$$\frac{\delta^2}{\delta} - \frac{\delta^2}{3\delta^2} = \frac{2}{3} \delta$$

$$\dot{m} = \frac{\rho_f (\rho_f - \rho_g) g \delta^3}{3 \mu_f} \quad (7) \Rightarrow \text{Per unit width of plate } \left[\frac{\text{kg}}{\text{m}\cdot\text{s}} \right]$$

We can do the exact same procedure in terms of energy:



h_{fg} = latent heat of phase change [J/kg]
 (How much heat is released from converting 1 kg of vapor into liquid).

Doing an energy balance

$$q'' = k \frac{\partial T}{\partial y} \Big|_{y=0} = \dot{m} \cdot h_{fg} \left(\frac{1}{dx} \right)$$

need to divide out by dx since q'' is a flux, and $\dot{m} \cdot h_{fg}$ is not.

From equation (6), $\frac{\partial T}{\partial y} = \frac{T_{\text{sat}} - T_0}{\delta}$

$$k \frac{T_{\text{sat}} - T_0}{\delta} = h_{fg} \frac{d\dot{m}}{dx} \quad (8)$$

We know from (7) to solve for $\frac{d\dot{m}}{dx}$:

$$k \frac{T_{\text{sat}} - T_0}{\delta} = \frac{h_{fg} \rho_f (\rho_f - \rho_g)}{\mu} g \delta^2 \frac{\partial \delta}{\partial x}$$

$$\int_0^x \frac{k (T_{\text{sat}} - T_0) \mu}{h_{fg} \rho_f (\rho_f - \rho_g) g} dx = \int_0^{\delta(x)} \delta^3 d\delta = \frac{\delta^4}{4} \quad \left. \begin{array}{l} \delta(x=0) = 0 \\ \delta(x) = \delta \end{array} \right\}$$

$$\delta = \left[\frac{4k_f (T_{\text{sat}} - T_0) \mu_f x}{\rho_f (\rho_f - \rho_g) g h_{fg}} \right]^{1/4}$$

Now we can solve for our Nusselt number:

$$h = \frac{q''}{T_{\text{sat}} - T_0} = \frac{1}{T_{\text{sat}} - T_0} \left[\frac{k_f (T_{\text{sat}} - T_0)}{\delta} \right] = \frac{k_f}{\delta}$$

$$Nu_x = \frac{hx}{k_f} = \frac{k_f}{\delta} \cdot \frac{x}{k_f} = \frac{x}{\delta}$$

$$Nu_x = 0.707 \left[\frac{\rho_f (\rho_f - \rho_g) g h_{fg} x^3}{\mu_f k_f (T_{\text{sat}} - T_0)} \right]^{1/4} \Rightarrow \text{Laminar filmwise condensation Nusselt \#}$$

Note, this analysis was first done by Nusselt in 1916 and is still widely used and accurately validated.

For average Nusselt number over the whole plate:

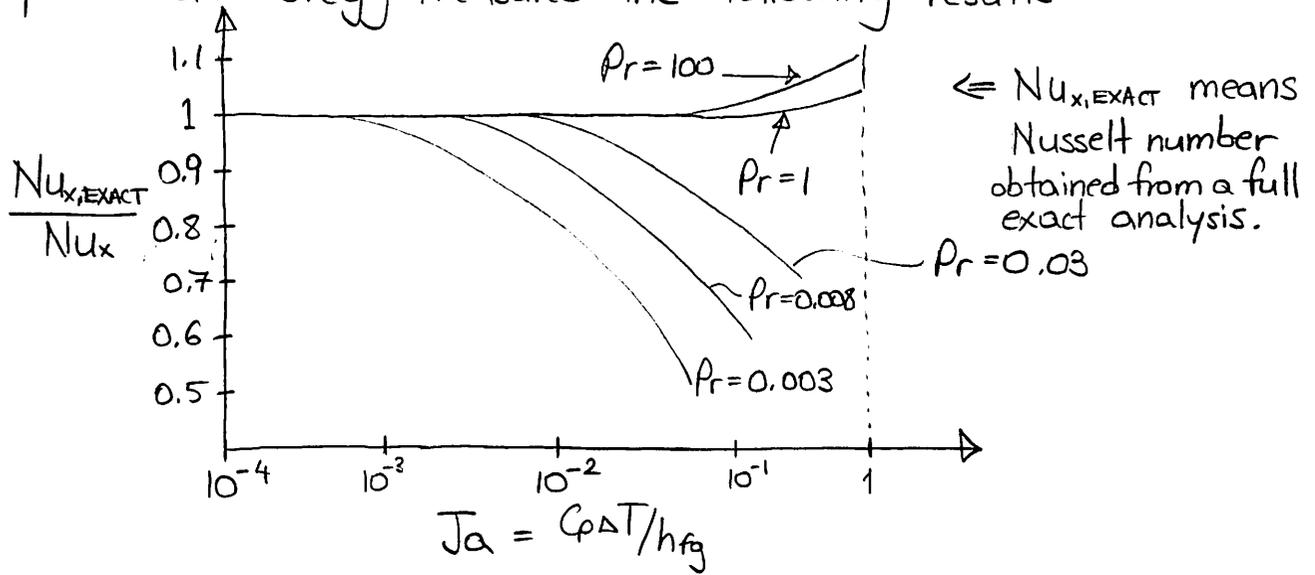
$$\bar{h} = \frac{1}{L} \int_0^L h(x) dx = \frac{4}{3} h(L) \Rightarrow \overline{Nu}_L = \frac{\bar{h} L}{k_f}$$

$$\overline{Nu}_L = 0.943 \left[\frac{\rho_f (\rho_f - \rho_g) g h_{fg} L^3}{\mu_f k_f (T_{\text{sat}} - T_0)} \right]^{1/4}$$

Some Notes:

- 1) Liquid condensate properties (μ_f, ρ_f, k_f) are evaluated at the film temperature: $\bar{T} = (T_{\text{sat}} + T_0)/2$.
- 2) ρ_g and h_{fg} are evaluated at T_{sat} .
- 3) In reality, film condensation depends on the specific or sensible heat absorbed by the condensate in addition to the latent heat (which we've considered)

Sparrow and Gregg measured the following results:

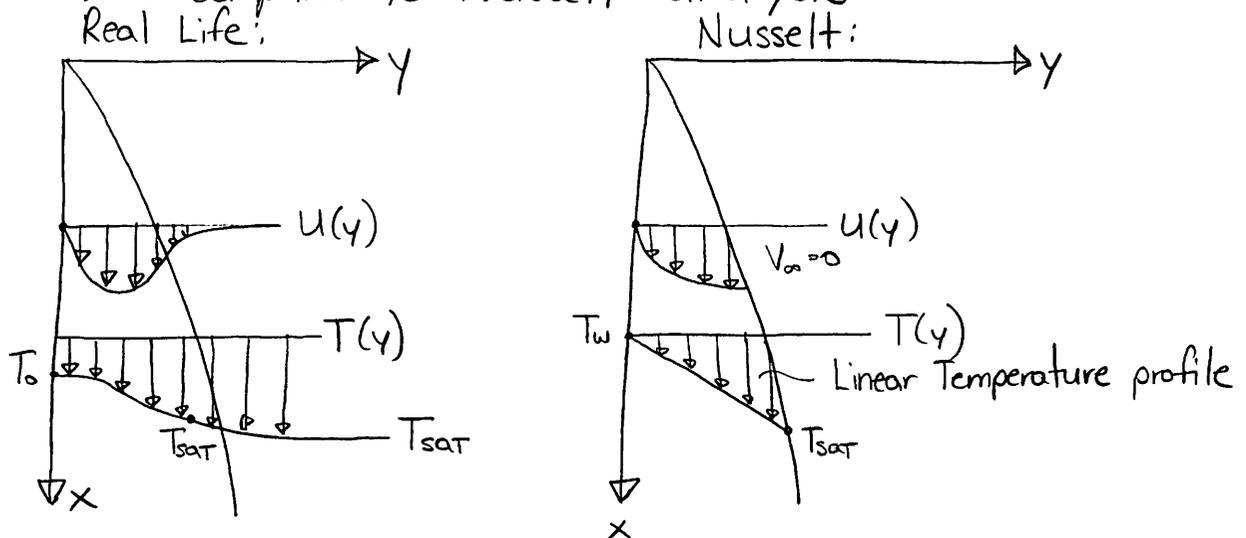


To account for the effect of Pr ,

$$h_{fg}' = h_{fg} \left[1 + (0.68 - 0.228/Pr) Ja \right] \quad \text{for } Pr \geq 0.6$$

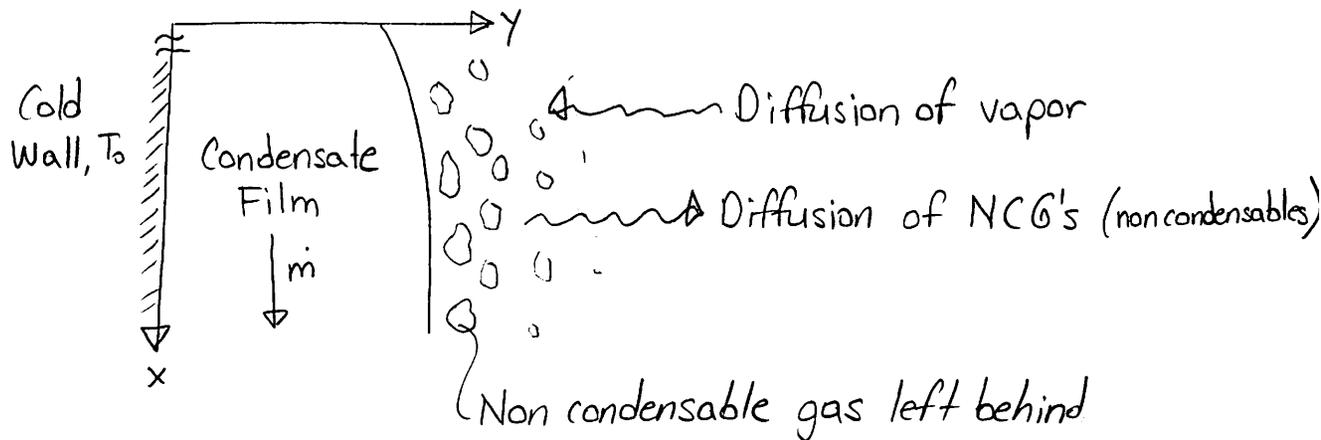
END OF LECTURE 20

5) Real Life compared to Nusselt analysis



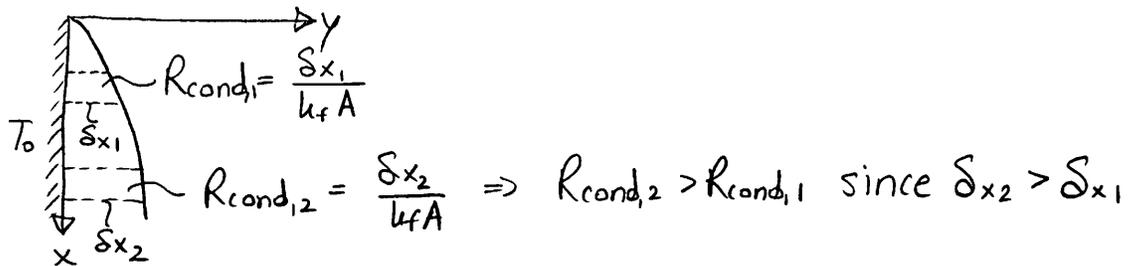
6) The presence of non-condensable gases acts to severely degrade heat transfer. When the condensing vapor is mixed with noncondensable air, uncondensed air must constantly diffuse away from the condensing film and vapor must diffuse inward toward the film. This coupled diffusion process

can considerably slow condensation. The resulting h can be cut by a factor of 5 if there is as little as 5% by mass of air mixed into the steam.



7) We see that $h_x = \frac{k_f}{\delta} \sim \frac{1}{x^{1/4}}$

So $h \downarrow$ as $x \uparrow$. This is just like laminar boundary layer theory and indicates that thicker condensate films add conduction resistance.



So having shorter surfaces and many of them is beneficial.

Example Water at 1 atm condenses on a strip 30cm high that is held at 90°C. Calculate Q /meter, δ ($x=30\text{cm}$), and m /meter.

$$\delta = \left[\frac{4k(T_{\text{sat}} - T_0)\mu_f x}{\rho_f(\rho_f - \rho_g)gh_{fg}'} \right]^{1/4}$$

$$h_{fg}' = 2257 \left(1 + \left(0.68 - \frac{0.228}{1.86} \right) \frac{4.211(10)}{2257} \right) = 2281 \text{ kJ/kg}$$

$$\text{So: } \delta = \left[\frac{4(0.677)(10)(2.99 \times 10^{-4}) \times}{961.9(961.9 - 0.6)(9.81)(2281 \times 10^3)} \right]^{1/4} = 0.000141 \times^{1/4}$$

$$\delta(L) = 0.104 \text{ mm} = 104 \mu\text{m} \Rightarrow \text{Thickness of your hair is } 100 \mu\text{m!}$$

$$\overline{Nu}_L = \frac{4}{3} \frac{L}{\delta} = \frac{4(0.3 \text{ m})}{3(0.000104 \text{ m})} = 3846$$

$$q'' = \frac{\overline{Nu}_L k \Delta T}{L} = 86.8 \text{ kW/m}^2 \quad \text{or} \quad \boxed{\overline{h}_L = 8.68 \text{ kW/m}^2 \cdot \text{K}}$$

$$Q_{\text{TOT}} = (86.8 \text{ kW/m}^2)(0.3 \text{ m}) = \boxed{26.0 \text{ kW/m}}$$

For condensate flow rate per meter:

$$\dot{m} = \frac{Q}{h_{fg}} = \frac{26.0}{2281} = 0.0114 \text{ kg/m} \cdot \text{s}$$

Note how high \overline{h}_L is for condensation phase change:

$$\boxed{\overline{h}_L = 8.68 \text{ kW/m}^2 \cdot \text{K} \gg \overline{h}_{L, \text{convection}}}$$

Typically, phase change heat transfer processes are much higher in \overline{h} due to the enormous amount of energy of latent heats of phase change.

Turbulent Film Condensation

We can define a Reynolds number for condensation as:

$$Re_s = \frac{\rho_f V_f D_h}{\mu_f} = \frac{V_f D_h}{\nu_f} = \frac{Q D_h}{\mu_f A}, \quad \text{Where } Q = \text{volumetric flow rate}$$

$A = \text{area cross section}$

$$D_h = \frac{4A}{P}, \quad P = w \text{ (wetted perimeter, only the wall width, } w)$$

$$Q = \frac{\dot{m}}{\rho_f} \Rightarrow Re_s = \frac{\dot{m}}{\rho_f} \frac{\rho_f}{\mu_f} \frac{4A}{PA} \Rightarrow \boxed{Re_s = \frac{4\dot{m}}{\mu_f w}} \quad \begin{array}{l} w = \text{width} \\ \text{of condensing} \\ \text{plate.} \end{array}$$

Now we can solve for Re_s explicitly since we know \dot{m} .

$$\dot{m} = \frac{w \rho_f (\rho_f - \rho_g) g \delta^3}{3 \mu_f}$$

$$Re_s = \frac{4g\delta^3}{3\nu_f^2} \Rightarrow$$

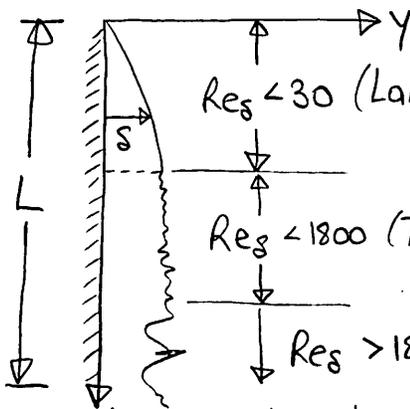
$$L_c = \text{condensation length scale} = \left(\frac{\nu_f^2}{g}\right)^{1/3}$$

Experimental correlations for Re_s :

$$Re_s = 3.78 \left[\frac{k_f L \Delta T}{\mu_f h_{fg} (\nu_f^2/g)^{1/3}} \right]^{3/4}; Re_s < 30$$

$$Re_s = \left[\frac{3.7 k_f L \Delta T}{\mu_f h_{fg} (\nu_f^2/g)^{1/3}} + 4.8 \right]^{0.82}; 30 \leq Re_s \leq 1800$$

$$Re_s = \left[\frac{0.069 k_f L \Delta T}{\mu_f h_{fg} (\nu_f^2/g)^{1/3}} Pr_f^{0.5} - 151 Pr_f^{0.5} + 253 \right]^{4/3}; Re_s > 1800$$



$$Re_s < 30 \text{ (Laminar)}; \bar{Nu} = \left(\frac{3}{4} Re_s\right)^{-1/3}$$

$$30 < Re_s < 1800 \text{ (Transition)}; \bar{Nu} = 0.822 Re_s^{-0.22}$$

$$Re_s > 1800 \text{ (Turbulent)}; \bar{Nu} = 3.8 \times 10^{-3} Re_s^{0.4} Pr_f^{0.65}$$

*May need iteration to solve.

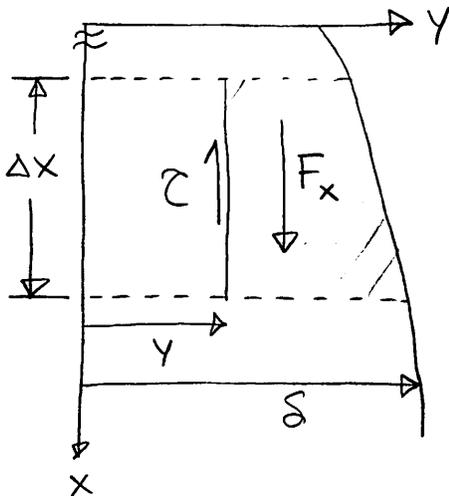
Where

$$\bar{Nu} = \frac{\bar{h} L_c}{k_f}; L_c = \left(\frac{\nu^2}{g}\right)^{1/3}$$

Film Condensation on Arbitrary Shapes

We can generalize our previous approach to any body undergoing condensation.

Looking at our condensate more closely



F_x = body force per unit volume
 For example, a vertical flat plate would have: $F_x = \underbrace{g(\rho_f - \rho_g)}_{\text{gravity}} \underbrace{1}_{\text{Boyanxy}}$

Applying a general force balance on the shaded area, we obtain:

$$\tau = \mu \frac{\partial u}{\partial y} \Delta x$$

$$\mu \frac{\partial u}{\partial y} \Delta x = F_x (\delta - y) \Delta x$$

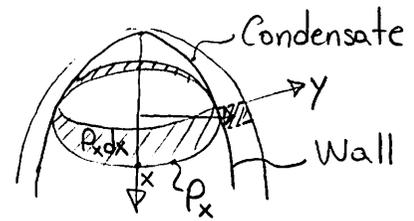
$$\mu \partial u = F_x (\delta - y) dy$$

$$u = \frac{F_x}{\mu} \left(\delta y - \frac{y^2}{2} \right)$$

The mass flow rate is: (P_x = perimeter of body at location x or wetted perimeter)

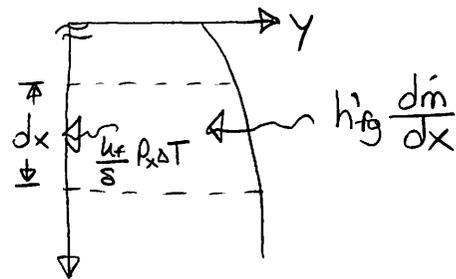
$$\dot{m} = \rho_f \int_0^\delta u P_x dy = P_x F_x \left(\frac{\rho_f}{3\mu} \right) \delta^3 \quad (1)$$

Note, $P_x dy$ = Area cross section



Again using the energy equation we developed previously

$$\underbrace{\frac{k_f}{\delta} P_x \Delta T}_{\text{Conduction at the wall}} = \underbrace{h'_{fg} \frac{d\dot{m}}{dx}}_{\text{Latent heat}} \quad (2)$$



Now we can re-arrange eq. (1)

$$\delta = \dot{m}^{1/3} \left[\frac{3\mu_f}{\rho_f P_x F_x} \right]^{1/3} \quad (3)$$

Substituting (3) into (2) and integrating yields the total condensate flow rate:

$$\dot{m} = \frac{4^{3/4}}{3} \left[\frac{\rho_f k_f^3 \Delta T^3}{\mu_f h_{fg}^3} \right]^{1/4} \left[\int_0^L P_x^{4/3} F_x^{1/3} dx \right]^{3/4}$$

Finally, since:

$$\dot{m} h_{fg} = \bar{h} \left(\int_0^L P_x dx \right) \Delta T = \bar{h} A_{cond} \Delta T$$

We can re-write the average heat transfer coefficient as:

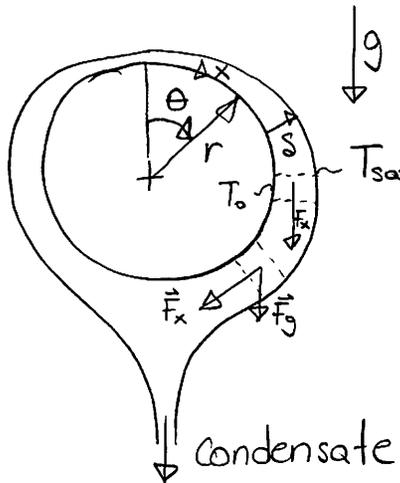
$$\bar{h} = \frac{4^{3/4}}{3} \cdot \frac{1}{A_{\text{cond}}} \left[\frac{\rho_f k_f^3 h_{fg}'}{\mu_f \Delta T} \right]^{1/4} \left[\int_0^L \rho_x^{4/3} F_x^{1/3} dx \right]^{3/4}$$

$$A_{\text{cond}} = \int_0^L \rho_x dx$$

⇒ Condensation Area

↳ Condensation \bar{h} for any arbitrary body undergoing filmwise laminar cond.

Example Find \bar{h} for a horizontal tube.



$$dx = r d\theta$$

$$F_x = g(\rho_f - \rho_g) \sin \theta$$

$$\rho_x = 1$$

$$A_{\text{cond}} = \int_0^\pi 1 r d\theta = \pi r$$

↳ Note didn't integrate to 2π due to symmetry.

Now we can solve:

$$\bar{h} = \frac{4^{3/4}}{3} \cdot \frac{1}{\pi r} \left[\frac{\rho_f k_f^3 h_{fg}'}{\mu_f \Delta T} \right]^{1/4} \left[\int_0^\pi \sin^{1/3} \theta d\theta \right]^{3/4} \cdot \underbrace{r^{3/4} [g(\rho_f - \rho_g)]^{1/4}}_{\text{From } dx=r d\theta}$$

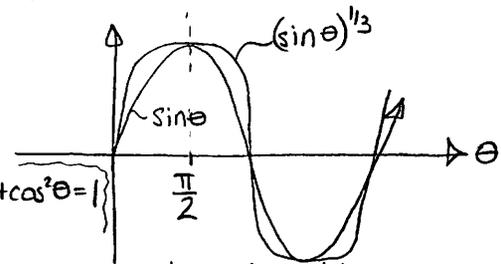
The integral here is not trivial to solve, so we can use a trick:

$$\int_0^\pi (\sin \theta)^{1/3} d\theta = 2 \int_0^{\pi/2} (\sin \theta)^{1/3} d\theta$$

Let $s = \sin^2 \theta$, $ds = 2 \sin \theta \cos \theta d\theta$, $\sin^2 \theta + \cos^2 \theta = 1$

$$2 \int_0^{\pi/2} (\sin \theta)^{1/3} d\theta = \int_0^1 s^{2/3-1} (1-s)^{1/2-1} ds \Rightarrow \text{Looking up in integral table or see page 129 of our notes.}$$

$$= \frac{\Gamma(2/3) \Gamma(1/2)}{\Gamma(2/3 + 1/2)} = 2.5871, \quad \Gamma = \text{Gamma function}$$



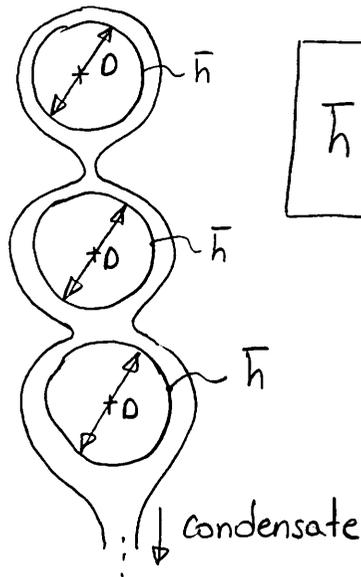
Now we can solve for \bar{h}

$$\bar{h} = \frac{4^{3/4}}{3} \frac{2^{1/4}}{\pi} (2.5871)^{3/4} \left[\frac{\rho_f (\rho_f - \rho_g) g h_f^3 k_f'}{D \mu_f \Delta T} \right]^{1/4}, \quad D = \text{cylinder diameter}$$

$$\bar{h} = 0.728 \left[\frac{\rho_f (\rho_f - \rho_g) g h_f^3 k_f'}{D \mu_f \Delta T} \right]^{1/4}$$

⇒ heat transfer coefficient for laminar film condensation on the outside of horizontal cylinder.

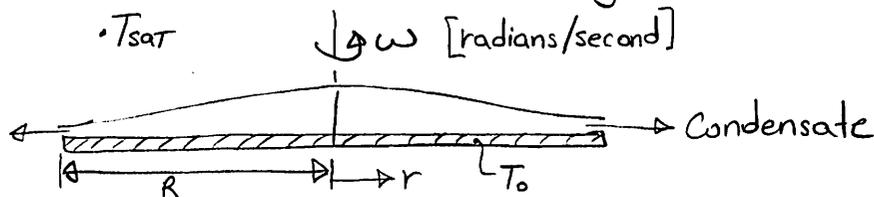
Note, if we had a tube bundle, with a continuous liquid condensate film and n tubes



$$\bar{h} = 0.728 \left[\frac{\rho_f (\rho_f - \rho_g) g h_f^3 k_f'}{n D \mu_f \Delta T} \right]^{1/4}$$

↳ n = number of tubes in series.

Example | Condensation on a rotating disc in zero gravity



For this case; $A_{\text{cond}} = \pi R^2$ (Total condensation area)

$P_x = 2\pi r$ (Perimeter of wetted area at location x)

$F_x = \rho_f r \omega^2$ (Body force per unit volume)

$dx = dr$

We can first evaluate our integral in our generalized equation for \bar{h} :

$$\int_0^L \rho_x^{4/3} F_x^{1/3} dx = (2\pi)^{4/3} (\rho_f \omega^2)^{1/3} \int_0^R r^{4/3} r^{1/3} dr$$

$$= (2\pi)^{4/3} (\rho_f \omega^2)^{1/3} (3/8) R^{8/3}$$

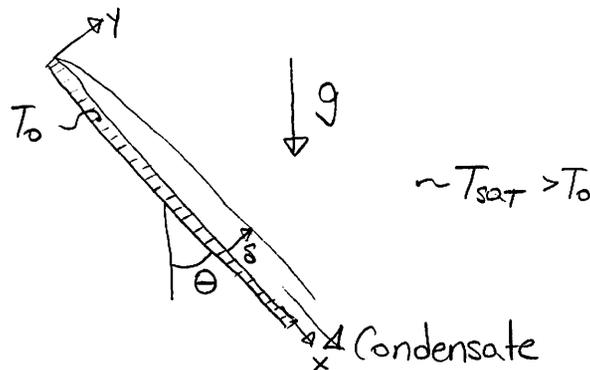
$$\frac{1}{A_{\text{cond}}} \left[\int_0^L \rho_x^{4/3} F_x^{1/3} dx \right]^{3/4} = \frac{2\pi}{\pi R^2} \left(\frac{3}{8} \right)^{3/4} R^2 (\rho_f \omega^2)^{1/4} = 2 \left(\frac{3}{8} \right)^{3/4} (\rho_f \omega^2)^{1/4}$$

Back substituting, we obtain:

$$\bar{h} = \left[\frac{2 h_f^3 (\rho_f \omega)^2 h'_{fg}}{3 \mu_f \Delta T} \right]^{1/4} \Rightarrow \text{Average heat transfer coefficient over the entire surface.}$$

END OF LECTURE 21

Example | Inclined surface at angle θ from the vertical



$$A_{\text{COND}} = (1)L = L$$

$$\rho_x = 1$$

$$F_x = (\rho_f - \rho_g) g \cos \theta$$



Solving our integral term:

$$\int_0^L \rho_x^{4/3} F_x^{1/3} dx = [(\rho_f - \rho_g) g \cos \theta]^{1/3} \int_0^L dx = ((\rho_f - \rho_g) g \cos \theta)^{1/3} L$$

$$\bar{h} = \frac{4^{3/4}}{3} \frac{1}{L} \left[\frac{\rho_f h_f^3 h'_{fg}}{\mu_f \Delta T} \right]^{1/4} [L ((\rho_f - \rho_g) g \cos \theta)^{1/3} L]^{3/4}$$

$$\bar{h} = 0.942 \left[\frac{\rho_f (\rho_f - \rho_g) g \cos \theta k_f^3 h'_{fg}}{L \mu_f \Delta T} \right]$$

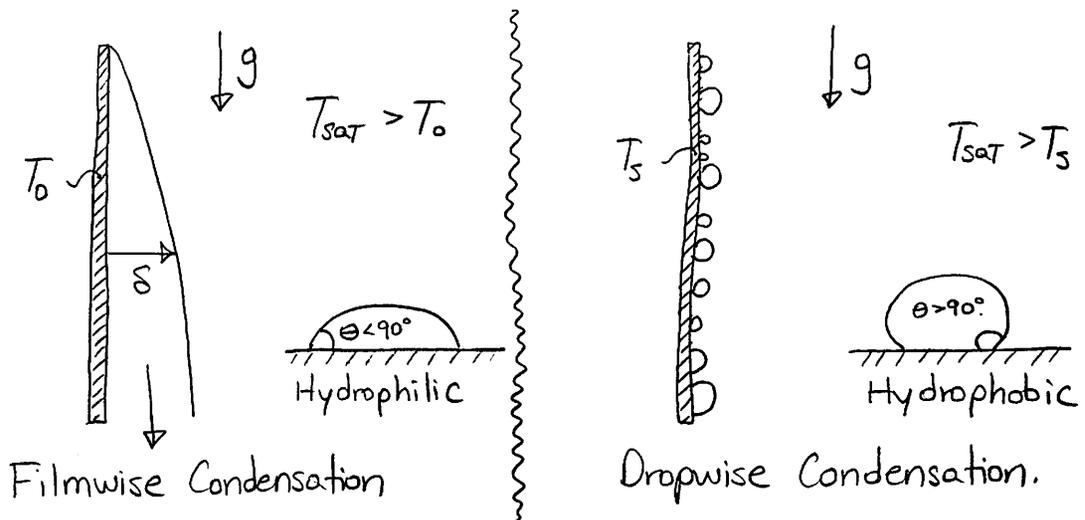
⇒ Average condensation heat transfer coefficient for an inclined plate.

Note, this result is exactly the same as the vertical plate result but with g replaced with $g \cos \theta$.

We can see from this general analysis that we can now solve for more complex shapes such as spheres, cones, etc...

Dropwise Condensation

So far, we have only dealt with filmwise condensation which occurs on surfaces where the developed condensate film wets the surface. If however the surface is non-wetting discrete droplets will form and grow.



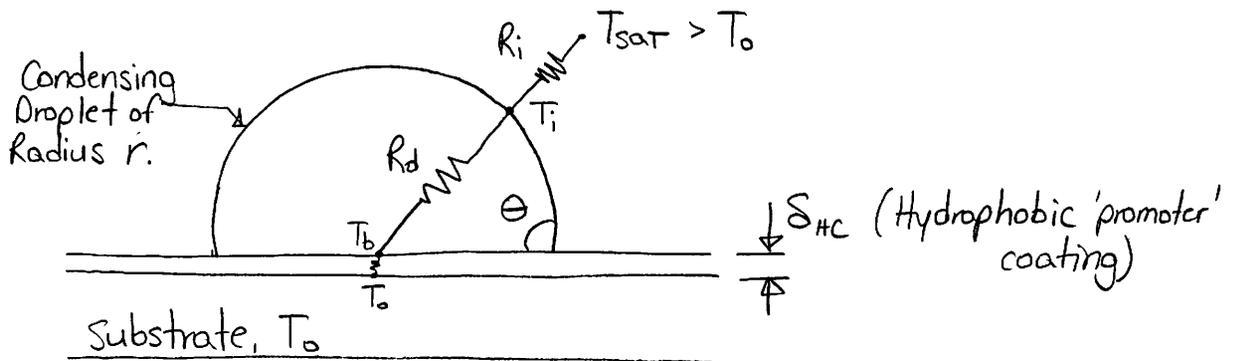
Condensate drainage in dropwise condensation is governed by gravity and droplet coalescence and sliding down the surface. This sliding motion cleans the surface for re-nucleation making this mode of condensation highly advantageous.

Filmwise	Dropwise
Occurs on wetting substrates; metal oxides, glass, clean substrates	Occurs on non-wetting substances; fatty acids, polymers, oils, contaminated surfaces.

- | | |
|--|---|
| <p>2) Most heat exchangers undergo filmwise condensation</p> <p>3) Relatively high \bar{h} ($\sim 10000 \text{ W/m}^2\text{K}$)</p> <p>4) Durable</p> <p>5) Not as sensitive to non-condensable gases.</p> | <p>2) Surfaces must be treated with ultra thin coatings of 'promoter' coating to undergo dropwise.</p> <p>3) Ultra high \bar{h} ($\sim 100000 \text{ W/m}^2\text{K}$)</p> <p>4) Not-robust, coatings degrade overtime</p> <p>5) Very sensitive to non-condensables due to rapid accumulation at surface</p> |
|--|---|

The Heat Transfer Mechanism (a model)

To model dropwise condensation, we must first understand the individual droplet heat transfer:



We can model the individual droplet heat transfer in terms of a simplified resistance network:

$$R_i = \frac{1}{h_i A_i} = \frac{1}{h_i \underbrace{2\pi r^2(1-\cos\theta)}_{A_i}}$$

(for h_i , see pg. 721 of Mills)
 (Convective interfacial resistance, where h_i is the interfacial heat transfer coefficient, A_i is the interface area between liquid & vapor)

$$R_d = \frac{\theta}{4\pi k_f r \sin\theta} \approx \frac{L}{k_f A_d}$$

(Droplet conduction resistance, where A_d = effective droplet conduction area. $R_d \approx$ effective conduction factor).

$$R_{hc} = \frac{S_{hc}}{k_{hc} A_{hc}} = \frac{S_{hc}}{k_{hc} \pi r^2 \sin^2\theta}$$

(Coating conduction thermal resistance, where A_{hc} = area of the droplet base, and k_{hc} = hydrophobic coating thermal conductivity)

We can now put our thermal resistances together and solve for the individual droplet heat transfer rate, q .

$$q(r, \theta) = \frac{\Delta T}{R_i + R_d + R_{hc}}$$

$$= \frac{\Delta T}{\frac{1}{h_i 2\pi r^2 (1 - \cos \theta)} + \frac{\theta}{4\pi k_f r \sin \theta} + \frac{\delta_{hc}}{k_{hc} \pi r^2 \sin^2 \theta}}$$

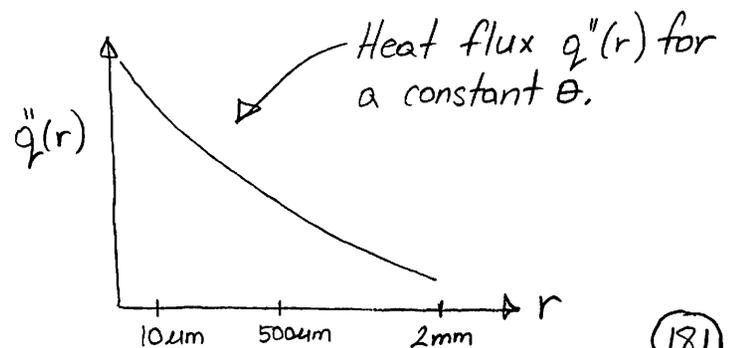
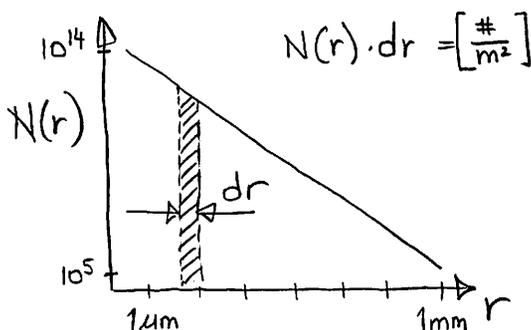
$$q(r, \theta) = \frac{\pi r^2 \Delta T}{\frac{1}{2h_i(1 - \cos \theta)} + \frac{r\theta}{4k_f \sin \theta} + \frac{\delta_{hc}}{k_{hc} \sin^2 \theta}} \quad (1) \Rightarrow \text{Individual Droplet Heat Transfer [W]}$$

Now we need to know the distribution of droplet sizes on the condensing surface. Assuming $\theta = \text{constant}$ for all growing droplets, (safe assumption), Rose and LeFevre developed an experimental distribution:

$$N(r) = \frac{1}{3\pi r^2 \hat{r}} \left(\frac{r}{\hat{r}}\right)^{-2/3} \quad (2); \hat{r} = \text{maximum size of droplet on the surface just before gravity removes it } (\approx 1\text{mm}), \hat{r} = f(\theta)$$

Note, the units of $N(r)$ are # droplets / m^3 . Physically, this distribution represents the number of droplets of radius r in a size range from r to $r + dr$.

Plotting our two results separately;



Putting our equations ① and ② together, we can calculate an overall surface heat flux q'' .

$$q'' = \int_0^{\hat{r}} q_2(r, \theta) \cdot N(r) dr \quad \text{③} \Rightarrow \text{Overall surface heat flux for dropwise condensation.}$$

Equation ③ forms the basis for all dropwise condensation models and can predict experimental results very accurately ($\pm 15\%$). Many more models exist that have extended the previous model however all have used the same approach.

Equation ③ must be solved numerically, hence it's not as useful as an analytical result. In general, experiments of steam condensation show:

$$q'' = (T_{\text{sat}} - 273.15)^{0.8} [5\Delta T + 0.3(\Delta T)^2] \Rightarrow \text{Rose correlation (experimental, see paper online)}$$

$\hookrightarrow T_{\text{sat}}$ is the saturation temperature in Kelvin

$$\Delta T = T_{\text{sat}} - T_0$$

q'' = condensation heat flux in $[\text{kW}/\text{m}^2]$.

Example | Using the Rose correlation, compare the previous example having filmwise condensation to the \bar{h} with dropwise.

Last example on page 172, we had:

$$\Delta T = 10^\circ\text{K}$$

$$P_{\text{sat}} = 1\text{ATM}, T_{\text{sat}} = 100^\circ\text{C} = 373.15^\circ\text{K}$$

$$\bar{h}_L = 8.68 \text{ kW}/\text{m}^2 \cdot \text{K}$$

Now with dropwise condensation:

$$q'' = (373.15 - 273.15)^{0.8} [5(10) + 0.3(10)^2] = 3.184 \text{ MW}/\text{m}^2$$

$$\bar{h}_{L, \text{DWC}} = \frac{q''}{\Delta T} = 318.5 \text{ kW}/\text{m}^2 \cdot \text{K} \Rightarrow 36.7 \times \text{enhancement over filmwise!}$$

END OF LECTURE 22

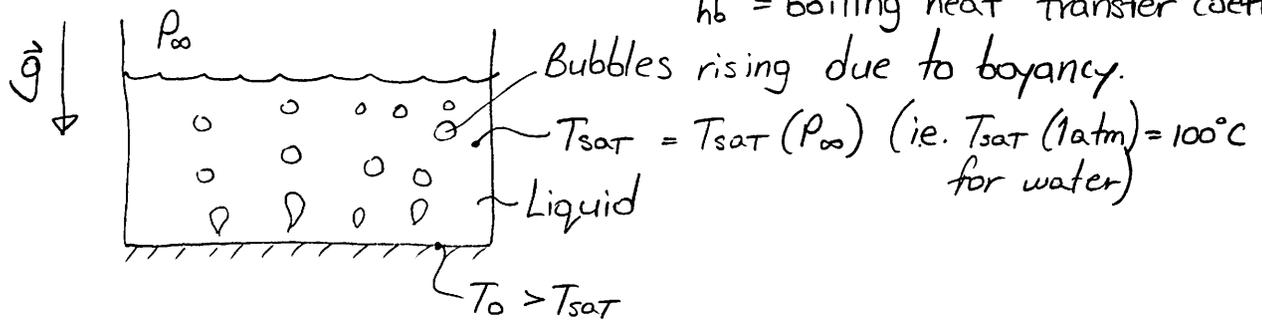
Boiling Heat Transfer

Very important in power generation systems (Rankine cycle). Opposite of condensation since heat is added to a liquid which changes phase to a vapor. More common than condensation due to its daily use in the kitchen for cooking.

In terms of heat transfer, we characterize boiling as:

$$q''_b = h_b (T_o - T_{sat}) ; \quad \begin{array}{l} T_o = \text{surface temperature} \\ T_{sat} = \text{fluid saturation temperature} \\ \text{at that pressure.} \\ h_b = \text{boiling heat transfer coeff.} \end{array}$$

Diagrammatically:

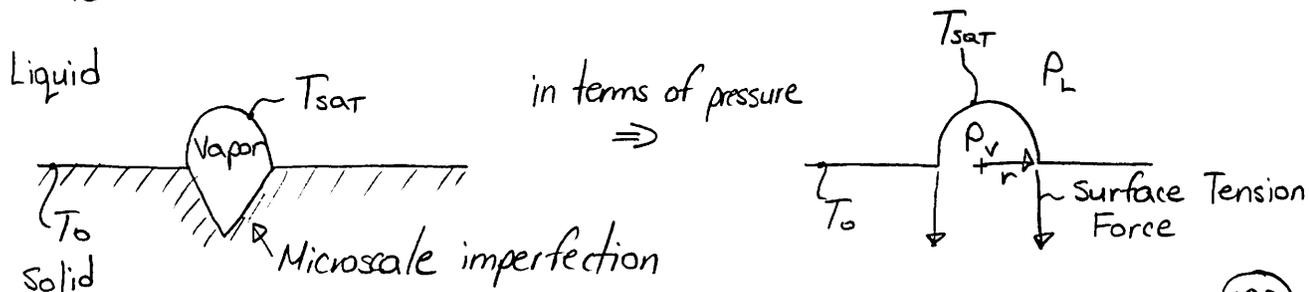


In general, boiling is characterized by the vapor bubbles that grow and detach from the surface. This depends on:

- 1) The temperature difference, $T_o - T_{sat}$
- 2) The nature of the surface (i.e. rough, smooth, pitted, etc...)
- 3) Thermophysical properties of the fluid (i.e. surface tension).

Initiation of Boiling, Bubble Growth

Boiling initiates by the formation of vapor bubbles at imperfections at the surface.



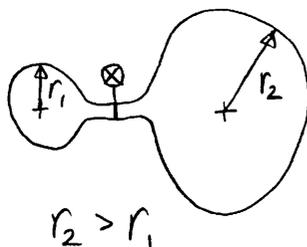
If we apply a force balance on our bubble:

$$\underbrace{(P_v - P_L)\pi r^2}_{\text{Pressure Force}} = \underbrace{2\pi r \sigma}_{\text{Surface Tension Force}} \quad ; \quad \sigma = \text{fluid surface tension [N/m]}$$

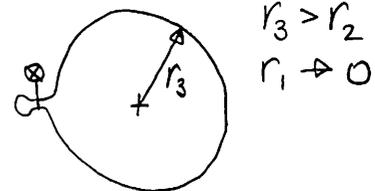
$$\boxed{\Delta P = \frac{2\sigma}{r}} \Rightarrow \text{Pressure difference across the liquid-vapor interface.}$$

Note, as $r \rightarrow 0$, $\Delta P \rightarrow \infty$, meaning that small bubbles have a much larger internal pressure.

Aside: If I connect 2 balloons with r_1 & r_2 together, & $r_1 < r_2$, what will happen:



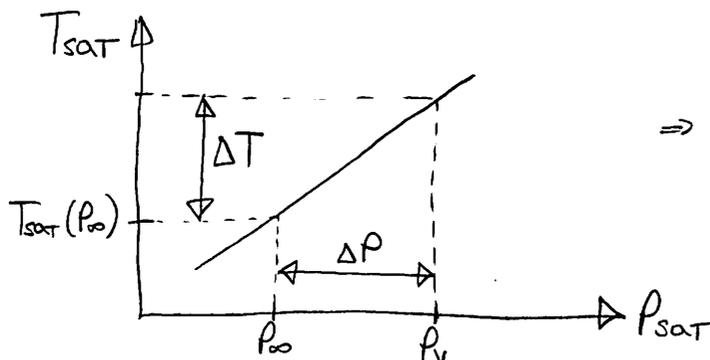
Answer
 \Rightarrow



Since $P_{v,1} > P_{v,2}$, the small balloon will drain into the big one.

The ΔP we've calculated is needed to create a vapor bubble of that given size, and overcome the bottleneck. So how can we relate this to boiling and surface temperature (T_0)?

Fluid Saturation curve:



\Rightarrow The ΔT you need to create a bubble nucleation site corresponding to ΔP .

So as you increase your ΔT , you create more nucleation sites with smaller radii and increase bubble creation frequency from existing sites.

We can relate our ΔP to ΔT through the Clausius-Clapeyron relation:

$$\left(\frac{dT}{dP}\right)_{\text{SAT}} = \frac{\Delta U T_{\text{SAT}}}{h_{fg}} \approx \frac{T_{\text{SAT}}}{\rho_v h_{fg}}$$

ΔU = specific volume change
 (since $\Delta U = U_g - U_f \approx U_g$ since $U_f \ll U_g$)

Aside: Derrivation of the Clausius-Clapeyron relation
 From thermodynamics of a state in equilibrium (state postulate)

$$ds = \left(\frac{\partial s}{\partial u}\right)_T du + \left(\frac{\partial s}{\partial T}\right)_u dT$$

s = specific entropy
 u = specific volume
 T = temperature

For phase change, $dT = 0$

$$ds = \left(\frac{\partial s}{\partial u}\right)_T du$$

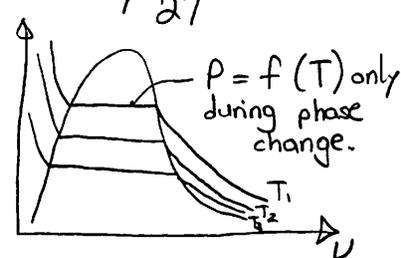
Using the appropriate Maxwell relation: $\left(\frac{\partial s}{\partial u}\right)_T = \left(\frac{\partial P}{\partial T}\right)_u$

$$ds = \left(\frac{\partial P}{\partial T}\right)_u du$$

Since $P, T = \text{constant}$ during a phase transition, $\frac{\partial P}{\partial T} = \text{constant}$ for different u . So now we can say:

$$\int ds = \int \left(\frac{dP}{dT}\right)_{\text{SAT}} du$$

$$\Delta S = \left(\frac{dP}{dT}\right)_{\text{SAT}} \Delta U, \quad \Delta S = S_g - S_f, \quad \Delta U = U_g - U_f$$



For a closed system undergoing an internally reversible process:
 $du = \delta q - \delta w = Tds - Pdu$, u = specific energy

We know specific enthalpy, h , as: $dh = du + Pdu$

$$dh = Tds \Rightarrow \int_f^g ds = \int_f^g \frac{dh}{T} \Rightarrow \Delta S = \frac{\Delta h}{T}, \quad \begin{array}{l} \Delta S = S_g - S_f \\ \Delta h = h_g - h_f \end{array}$$

For a phase change process: $h_g - h_f = h_{fg}$

$$\Delta S = \frac{h_{fg}}{T_{sat}} = \left(\frac{dT}{dP} \right)_{sat} \Delta U$$

So: $\boxed{\left(\frac{dT}{dP} \right)_{sat} = \frac{\Delta U T_{sat}}{h_{fg}}} \Rightarrow$ Clausius - Clapeyron relation.

So back to our boiling derivation

$$\Delta T = \frac{\Delta P T_{sat}}{p_v h_{fg}}$$

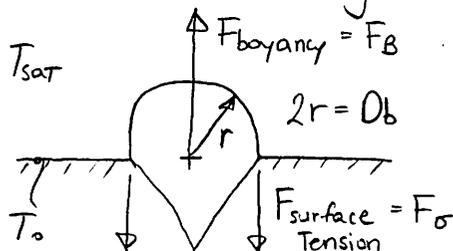
We know for a bubble that $\Delta P = \frac{2\sigma}{r}$, so

$$\boxed{\Delta T = \frac{2\sigma T_{sat}}{r p_v h_{fg}}}, \quad \Delta T = T_0 - T_{sat} \Rightarrow \text{Temperature difference required to nucleate a bubble of radius } r.$$

Note, often ΔT is referred to as "wall superheat".

Bubble Departure Dynamics

For a bubble sitting on a surface, we can do a force balance

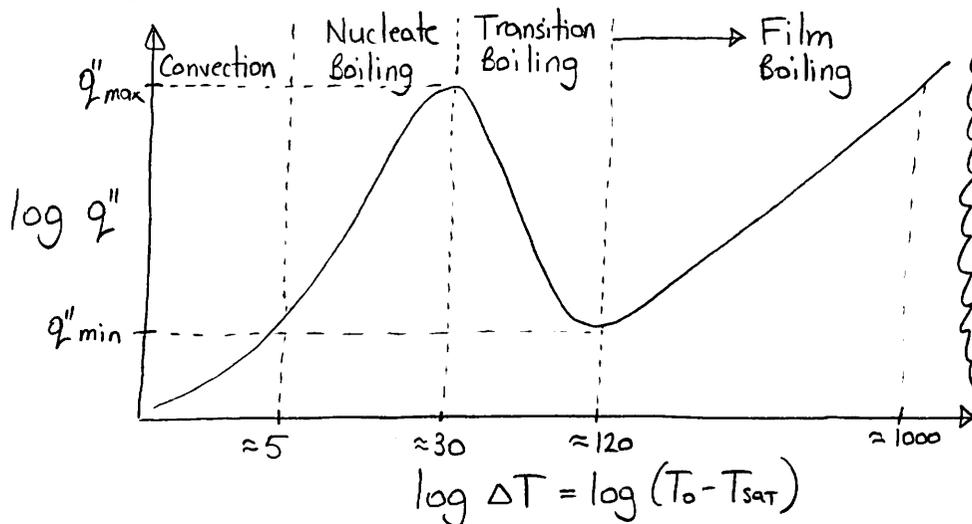


$$\begin{aligned} F_B &\sim D_b^3 (\rho_f - \rho_g) g \\ F_\sigma &\sim \Delta P A_{cs} \sim \frac{2\sigma}{D_b/2} \cdot \frac{D_b^2}{4} \sim D_b \sigma \end{aligned}$$

$$F_B = F_\sigma \Rightarrow \boxed{D_b \sim \left[\frac{\sigma}{g(\rho_f - \rho_g)} \right]^{1/2}}$$

D_b = bubble departure diameter in boiling heat transfer.

Heat Transfer: Now we can say something about the heat transfer taking place from the wall to the fluid. Experimentally, researchers have discovered different regimes of boiling: (note, values of ΔT shown are for water.)



Aside:

Note, in the convection regime, boiling does not occur since ΔT is too low to create bubbles and natural convection removes the heat from the hot plate.

We will discuss each regime in detail.

Nucleate Boiling:

For a hemispherical bubble ($\theta = 90^\circ$), a force balance reveals:

$$2\pi\sigma R_b = \frac{2}{3}\pi R_b^3(\rho_f - \rho_g)g ; R_b = \frac{D_b}{2}$$

$$R_b = \left[\frac{3\sigma}{(\rho_f - \rho_g)g} \right]^{1/2} \Rightarrow \text{Bubble departure size}$$

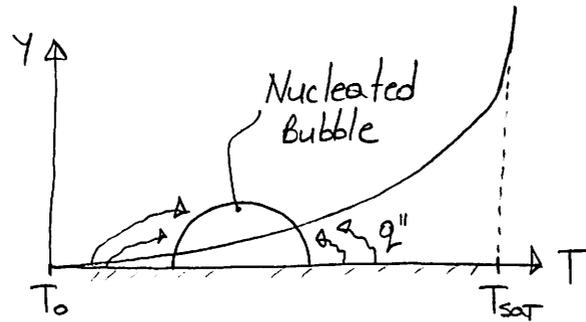
For our analysis, the characteristic length for boiling will be taken as:

$$L_c = \left[\frac{\sigma}{(\rho_f - \rho_g)g} \right]^{1/2} \Rightarrow \text{Boiling characteristic length}$$

$$Nu = \frac{hL_c}{k_f} \Rightarrow \text{Boiling Nusselt number}$$

So the rate of bubble growth depends on the convective heat transfer through the liquid to the liquid-vapor interface

required to supply the enthalpy of vaporization. Physically, you can think of it as:



Since convection is involved, we expect the fluid Prandtl number to be a relevant parameter.

In addition, similar to condensation, we end up sensibly heating the fluid and not all energy goes into phase change, hence the Jacob number is a relevant parameter.

$$Ja_f = \frac{c_{p,f} (T_0 - T_{sat})}{h_{fg}}$$

Experiments by Rohsenow (the father of boiling) show:

$$Nu = \frac{Ja_f^2}{C_{nb}^3 Pr_f^m} = \frac{h L_c}{k_f} \Rightarrow C_{nb} \text{ and } m \text{ are liquid and surface properties (Table 7.2, page 692 in Mills).}$$

For water, $m \approx 2.0$

$C_{nb} \approx 0.007$ to 0.013 (depends on surface material).

*All properties evaluated at T_{sat}

We see that: $q'' = h \Delta T \sim \Delta T^3$ for boiling. This strong dependence on ΔT is due to the rapid activation of nucleation sites at elevated ΔT (Since dp/dT is very large)

Example | Determine h and q'' for water boiling on a polished copper pot at 390 K and 1 atm.

$$\Delta T = T_o - T_{\text{SAT}} = 390\text{K} - 373.15\text{K} = 16.85\text{K}$$

We see from our graph on page 187 of notes that we are in the nucleate boiling regime ($5 < \Delta T < 30$).
Using our Rohsenow correlation:

$$\text{Nu} = \frac{\text{Ja}^2}{C_{\text{nb}}^3 \text{Pr}^m} \Rightarrow \text{All properties are evaluated at } T_{\text{SAT}}$$

$$T_{\text{SAT}} = 373.15\text{K}$$

$$\sigma = 58.9 \times 10^{-3}\text{N/m}$$

$$h_{\text{fg}} = 2.257 \times 10^6\text{J/kg}$$

$$k_f = 0.681\text{W/m}\cdot\text{K}$$

$$\rho_f = 958\text{kg/m}^3$$

$$c_{p,f} = 4212\text{J/kg}\cdot\text{K}$$

$$\text{Pr}_{,f} = 1.76$$

$$\text{Ja} = \frac{c_{p,f} (T_o - T_{\text{SAT}})}{h_{\text{fg}}} = \frac{(4212)(390 - 373.15)}{(2.257 \times 10^6)} = 3.145 \times 10^{-2}$$

From table 7.2 in Mills, $C_{\text{nb}} = 0.013$, $m = 2.0$

$$\text{Nu} = \frac{(3.145 \times 10^{-2})^2}{(0.013)^3 (1.76)^2} = 145$$

$$L_c = \left[\frac{\sigma}{(\rho_f - \rho_g)g} \right]^{1/2} = 2.5 \times 10^{-3}\text{m}$$

$$h = \frac{k_f}{L_c} \text{Nu} = \left(\frac{0.681}{2.5 \times 10^{-3}} \right) (145) = 39500\text{W/m}^2\cdot\text{K} \text{ or } \boxed{39.5\text{kW/m}^2\cdot\text{K}}$$

$$q'' = h \Delta T = 6.66 \times 10^5\text{W/m}^2 = \boxed{666\text{kW/m}^2 = q''}$$

Note, q'' and h are extremely high values. On the same order as condensation values. Much higher than h_{conv} , h_{rad} .

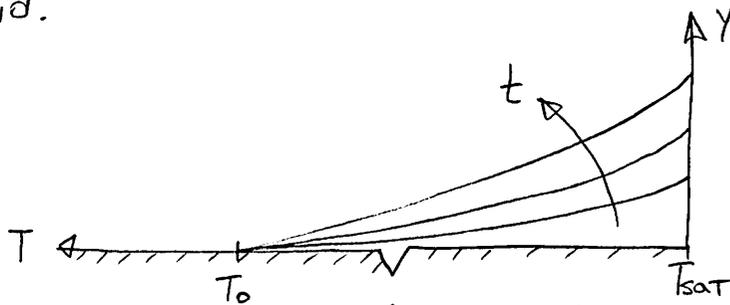
Looking at a more fundamental boiling model:

Mikic - Rohsenow Model (for nucleate boiling)

Bora Mikic (the professor who taught me this class) developed this model as part of a heat transfer final exam.

Warren Rohsenow didn't know how to solve it so he asked and when Mikic explained, Rohsenow saw the brilliance of it and published it together.

The model is based on transient heat conduction into the fluid.



\Rightarrow Mikic assumed that a majority of the heat is transferred to the thin fluid next to the surface.

We solved in the first part of the course that: (pg. 66 of notes)

$$q'' = \frac{h_f \Delta T}{\sqrt{\pi \alpha_f t}} \Rightarrow \text{heat flux due to transient heat transfer}$$

If we assume a frequency of bubble formation as f

$$f = \frac{1}{\tau}, \quad \alpha_f = \frac{h_f}{\rho_f c_{p,f}}$$

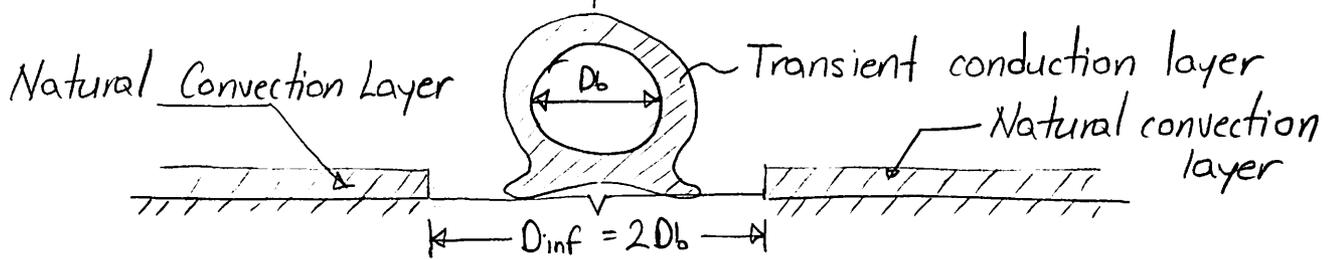
$$q'' = \frac{h_f \Delta T}{\sqrt{\pi \frac{h_f}{\rho_f c_{p,f}} t}} \Rightarrow \text{Note, this is instantaneous heat flux at } t.$$

$$q''_{AVE} = \frac{1}{\tau} \int_0^{\tau} q'' dt = f \int_0^{\frac{1}{f}} q'' dt \Rightarrow \text{Average heat flux for one nucleation site.}$$

$$q''_{AVE} = 2 \frac{h_f \Delta T}{\sqrt{\pi \frac{h_f}{\rho_f c_{p,f}}}} \cdot \sqrt{f}$$

So in the Mikic model, the bubbles departing act to bring in fresh cool (T_{sat}) water to the surface, and re-start the transient conduction process.

Assuming that the influence area of the bubble is πD_b^2



The bubble rising drags away πD_b^2 area of the superheated layer. So now we can solve for overall area heat flux:

$$q''_{\text{TOT}} = \pi D_b^2 n q''_{\text{AVE}} ; \quad n = \text{number of active sites per m}^2$$

$$q''_{\text{TOT}} = 2\sqrt{\pi} \sqrt{k_f \rho_f c_{p,f}} \sqrt{f} D_b^2 \cdot n \cdot \Delta T \quad \Rightarrow \text{Mikic-Rohsenow nucleate boiling model.}$$

The model is accurate when there is no overlap between transient conduction layers.

It's also more difficult than our previous Rohsenow correlation since f , and n need to be known a-priori.

Advantages of this model is that it is a lot more physical, and doesn't depend on tabulated parameters or fitting coefficients.

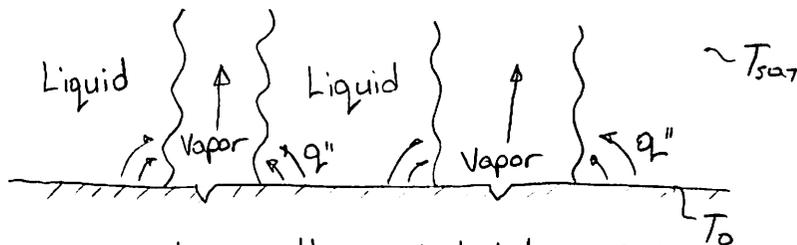
For more details, see Mikic & Rohsenow's paper on the dropbox.

END OF LECTURE 23

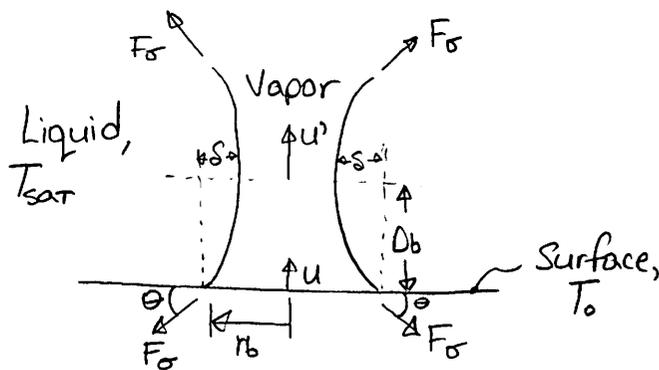
Critical Heat Flux (q''_{max})

We see on our boiling curve that the nucleate boiling regime reaches a maximum heat flux (at $\Delta T \approx 30$ for water) and then decreases. The reason for this peak is due to a vapor instability in rising bubble columns on the surface.

At high heat fluxes, bubbles coalesce as they rise (since f is very high) and form isolated columns:



We can analyze the stability of these columns. Looking at one column:



Assuming: No vapor generation upstream of the column. All vapor is generated at column base at u , near the contact line.

By conservation of mass:

$$u \pi r_b^2 = u' \pi (r_b - \delta)^2$$

$$u' = u \left(\frac{r_b}{r_b - \delta} \right)^2 = u \left(\frac{1}{1 - \delta/r_b} \right)^2, \text{ Assuming } \frac{\delta}{r_b} \ll 1$$

By Maclaurin series expansion: (Taylor series expansion about $x=0$)

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$u' \approx u \left(1 + 2 \frac{\delta}{r_b} \right),$$

Using the Bernoulli equation along a streamline inside the vapor column from u to u'

$$\Delta P = \frac{1}{2} \rho_g (u'^2 - u^2) \quad (1)$$

$$u'^2 = u^2 \left(1 + \frac{4s}{r_b} + 4 \frac{s^2}{r_b^2} \right)$$

Since $\frac{s}{r_b} \ll 1$, $\left(\frac{s}{r_b}\right)^2 \approx 0$, so: $u'^2 = u^2 \left(1 + \frac{4s}{r_b} \right) \quad (2)$

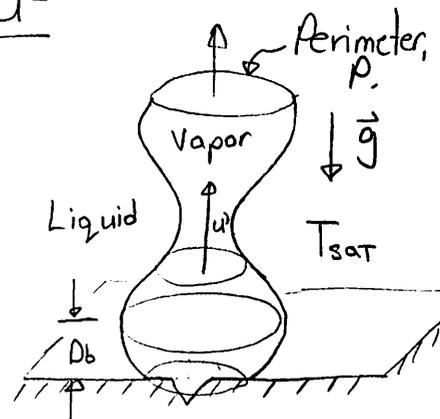
Back substituting (2) into (1), we obtain:

$$\Delta P = \frac{1}{2} \rho_g \left(u^2 + \frac{4su^2}{r_b} - u^2 \right) = \frac{2\rho_g s u^2}{r_b}$$

Now we can calculate a force balance:

$$F_{\Delta P} = 2\rho_g u^2 \frac{s}{r_b} (\pi r_b^2) = 2\rho_g u^2 s \pi r_b$$

$$F_{\sigma} = \rho \sigma \sin \theta \approx \rho \sigma \tan \theta = \sigma \frac{s}{2r_b} \pi r_b$$



Note, here $F_{\Delta P}$ & F_{σ} are forces on the CV of the vapor column, and we assume that $\theta \rightarrow 0$, so $\sin \theta \approx \tan \theta$. (For high heat fluxes, $\theta \rightarrow 0$ and our column is nearly flat and vertical)

By force ballance: $F_{\Delta P} \sim F_{\sigma}$

$$2\pi r_b \rho_g u^2 s \sim \pi \sigma s \quad (D_b = 2r_b)$$

$$u^2 \sim \frac{\sigma}{\rho_g D_b} \sim \frac{\sqrt{\sigma g (\rho_f - \rho_g)}}{\rho_g} \quad \left(\text{since } D_b \sim \left[\frac{\sigma}{(\rho_f - \rho_g) g} \right]^{1/2} \right), \text{ pg 187}$$

So $u \sim \frac{1}{\rho_g^{1/2}} [\sigma g (\rho_f - \rho_g)]^{1/4} \Rightarrow$ if velocity is $> u$, the column becomes unstable.

$$q_{\text{max}}'' \sim u \rho_g h_{fg} \sim h_{fg} \rho_g^{1/2} [\sigma g (\rho_f - \rho_g)]^{1/4}$$

So in experiments, we look for the following form

$$q''_{\max} = C_{\max} h_{fg} [\sigma g \rho_g^2 (\rho_f - \rho_g)]^{1/4} \Rightarrow \text{Critical heat flux based on Helmholtz instability.}$$

This correlation works very well. For large flat surfaces, $C_{\max} = 0.15$. In reality, it depends on the surface geometry and size. For more details, see pg. 694, 695 in Mills.

Note, this instability mechanism is also important in many other physical problems such as flags flapping. When the vapor columns become unstable, and the velocity induced pressure forces exceed the surface tension forces, the jet begins to oscillate (like a flag in the wind) and has the chance of merging with adjacent columns.

Example | Find q''_{\max} for boiling water on a flat surface

$$q''_{\max} = (0.15)(2.257 \times 10^6) [(0.0589)(9.81)(0.59)^2 (958 - 0.59)]^{1/4}$$

$$q''_{\max} = 1.26 \text{ MW/m}^2 = 126 \text{ W/cm}^2$$

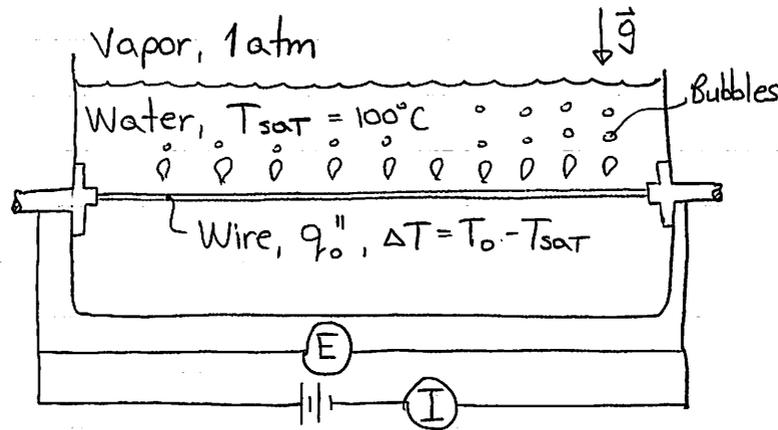
↳ This is usually labeled as the critical heat flux, or q''_{CHF} .

Note, much research has been going on in the past 10 years to show that q''_{\max} also depends on surface properties, such as wettability, surface roughness and oxidation.

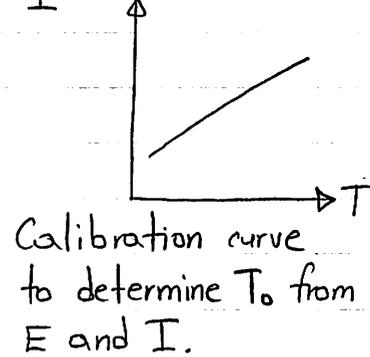
Film Boiling

To understand the regime of film boiling, we need to look at the history of the boiling curve.

In 1934, Nukiyama did an experiment of boiling on a nichrome wire, in a saturated pool of water:

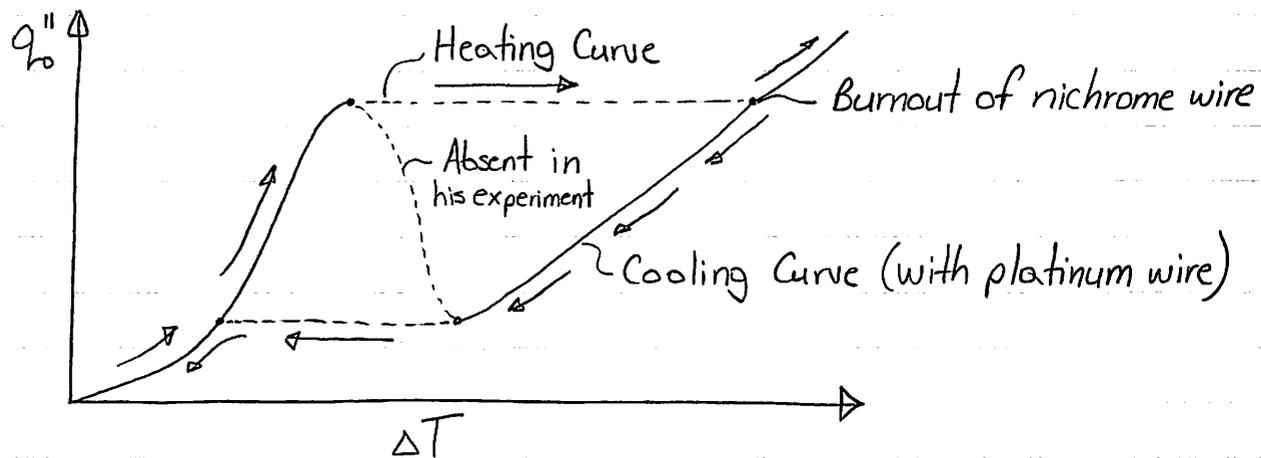


$$\frac{E}{I} = R(T)$$



From this setup, ΔT was the dependent variable, and q''_0 was the independent or controlled variable.

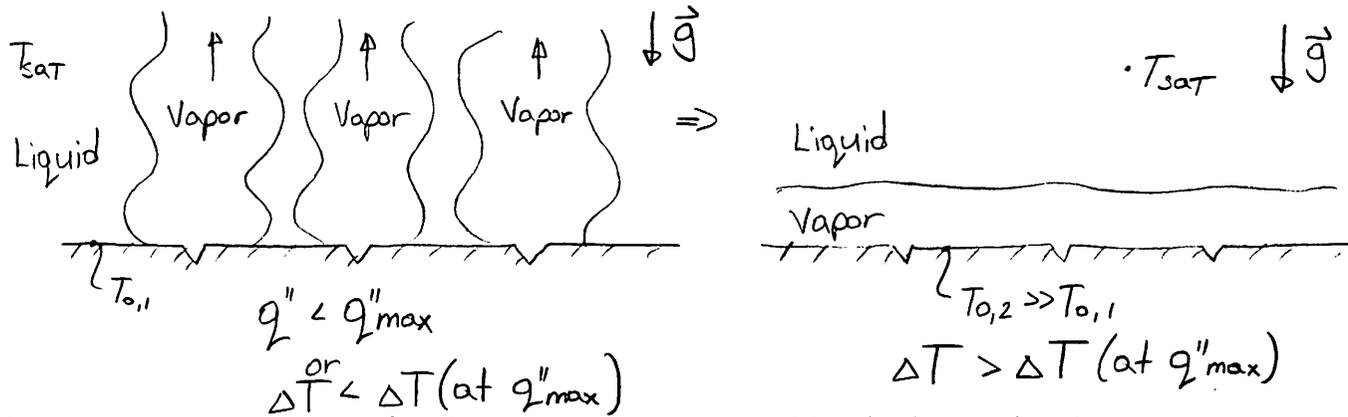
Nukiyama observed a fascinating phenomenon:



He noticed that at q''_{max} , the temperature of the wire T_0 jumped drastically to $> 1000^\circ\text{C}$ and his wire melted.

He re-did the experiment with a platinum wire ($T_{melt} > 2000^\circ\text{C}$) and once he cooled back down, he had a large hysteresis.

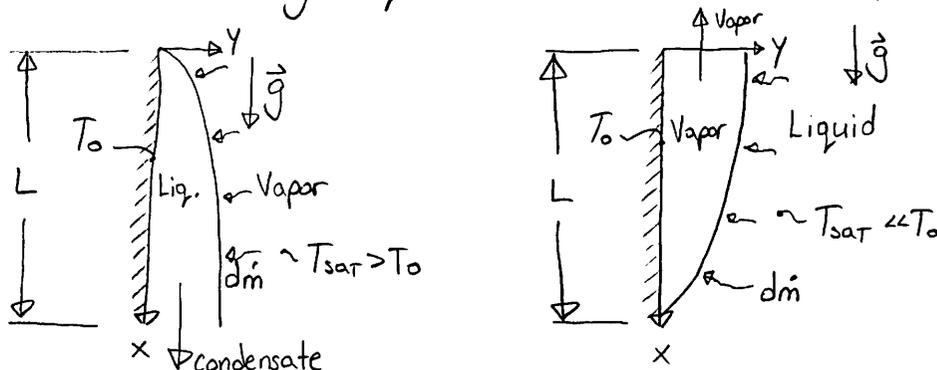
So what happened? The phenomena is related to the q''_{max} limit we discussed earlier. If we exceed q''_{max} , we have a Helmholtz instability which causes the merging of vapor columns and the formation of a thin vapor blanket.



The presence of the thin vapor blanket acts to severely impede heat transfer. Now the main mode is convection to vapor and radiation from the surface to the fluid. Since:

h_{conv} and $h_{rad} \ll h_{boiling}$, T_o jumps dramatically as Nukiyama observed.

The formation and growth of this thin vapor layer can be treated analogously to film condensation, but upside down.



We can show that:

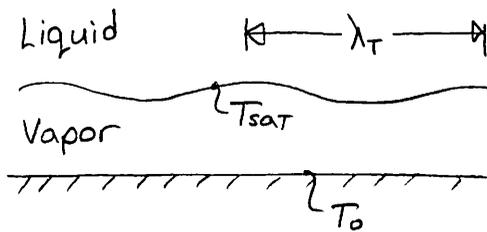
$$\bar{h} = C_{fb} \left[\frac{(\rho_f - \rho_g) g h_{fg}^3 k_g^3}{\nu_g L (T_o - T_{sat})} \right]^{1/4}$$

$$h_{fg}^3 = h_{fg} + 0.35 c_{p,v} (T_o - T_{sat})$$

\Rightarrow Film boiling heat transfer coefficient
 For a plane surface, $C_{fb} = 0.71$
 For a cylinder, $L=D$, $C_{fb} = 0.62$
 For a sphere, $L=D$, $C_{fb} = 0.67$

Minimum Heat Flux (q''_{min})

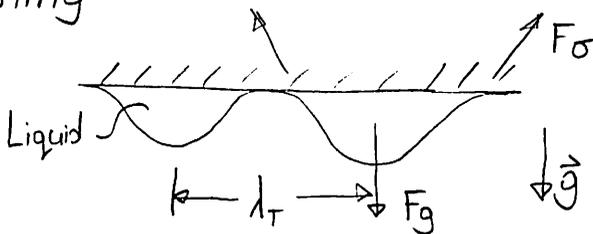
If we are in the film boiling regime, and q'' or ΔT is reduced, at a certain point the film will break down. This occurs when the vapor generation rate becomes too low to sustain Taylor instability waves at the liquid-vapor interface.



The interface is unstable to wavelengths on the order of:

$$\lambda_T \sim L_c \sim \left[\frac{\sigma}{(\rho_f - \rho_g)g} \right]^{1/2}$$

This criterion is intuitive since its nothing but a force balance between gravity and surface tension; think of droplets on a ceiling:



$$F_\sigma \sim \sigma \pi \lambda_T$$

$$F_g \sim (\rho_f - \rho_g)g \frac{4}{3} \pi \lambda_T^3$$

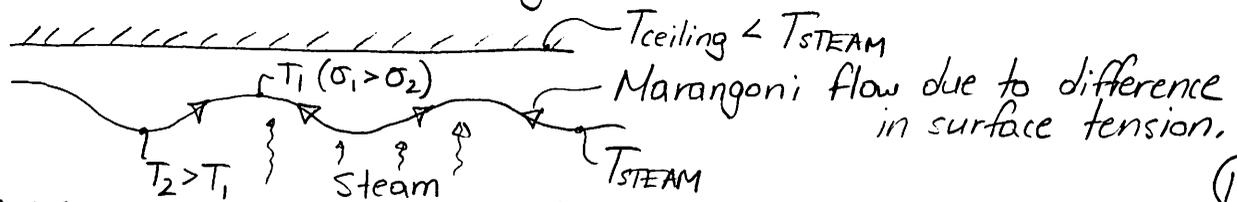
$$F_\sigma \sim F_g \text{ (at departure)} \Rightarrow \sigma \pi \lambda_T \sim (\rho_f - \rho_g)g \frac{4}{3} \pi \lambda_T^3$$

$$\lambda_T \sim \left[\frac{\sigma}{(\rho_f - \rho_g)g} \right]^{1/2}$$

Aside: $\frac{F_g}{F_\sigma} = \frac{(\rho_f - \rho_g)g \pi \lambda_T^3}{\sigma \pi \lambda_T} = \frac{(\rho_f - \rho_g)g \lambda_T^2}{\sigma} = Bo$

Bond Number = $\frac{\Delta \rho g L^2}{\sigma} = Bo$

Aside: Do you ever notice in the shower, when you look up at the ceiling, that the thin film of condensed water doesn't break up? There are droplets, but the film is intact. This is also due to a Rayleigh-Taylor instability, but Marangoni stresses hold the film together:



Using Taylor stability analysis, Zuber showed that:

$$q''_{min} = C_{min} \rho_g h_{fg} \left[\frac{\sigma g (\rho_f - \rho_g)}{(\rho_f + \rho_g)^2} \right]^{1/4}$$

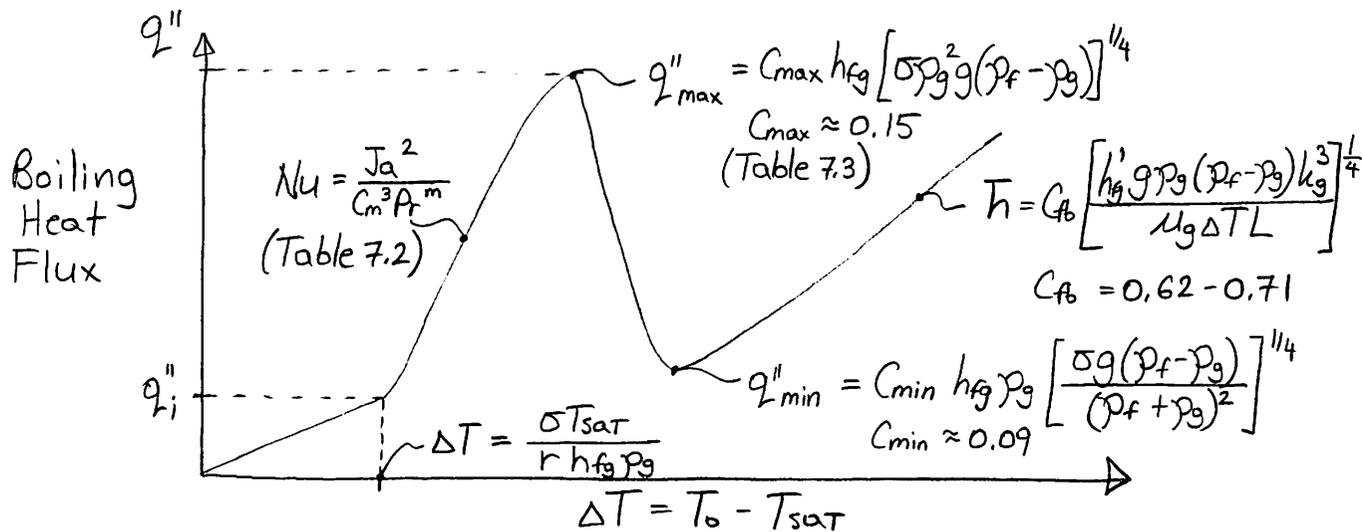
Where: $C_{min} = 0.09$ for large horizontal surfaces, and

$$C_{min} = 0.0464 \left[\frac{18}{L^{*2} (2L^{*2} + 1)} \right]^{1/4}; \quad \text{for horizontal cylinders}$$

$L^* = R/\lambda_T$; R = cylinder radius.

Note, ρ_g and ρ_f are evaluated at T_{sat} .

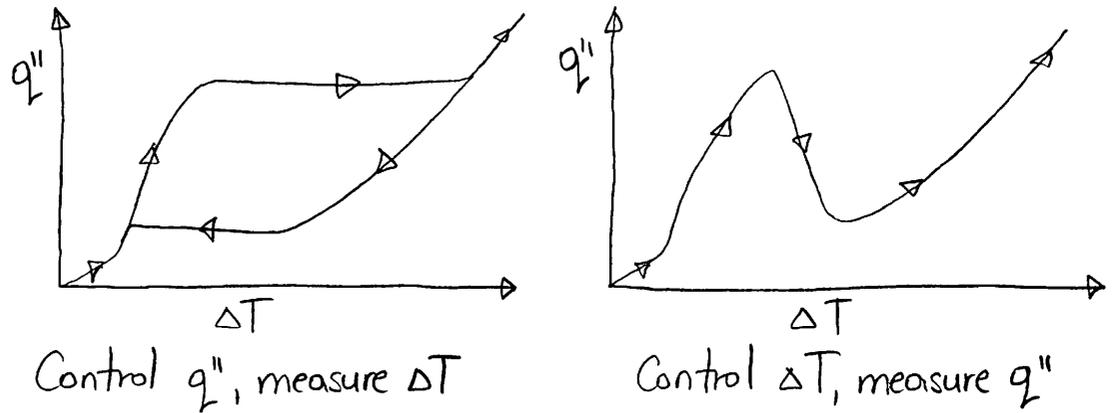
We now have a complete picture of the boiling phenomena of a stationary fluid. If we re-draw our pool boiling curve for a flat horizontal large surface, we would have:



Some Notes

- The transition regime is notoriously difficult to model analytically or numerically. It is characterized by unstable or partial film boiling. At any point the conditions can oscillate between film and nucleate boiling, analogous to the transition Reynolds number in pipe flows ($2300 < Re_d < 10000$), where flow oscillates between laminar & turb.

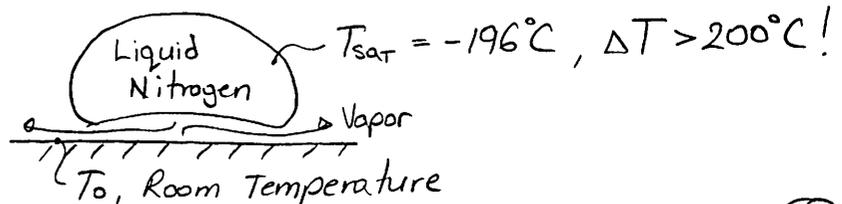
- 2) The transition region was not observed by Nukiyama since he controlled q'' and measured ΔT . If the experiment was set up to control ΔT and measure q'' , then the transition region would be observed and no hysteresis would exist.



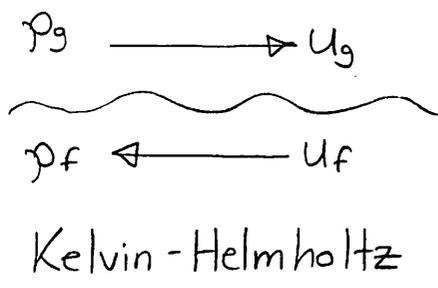
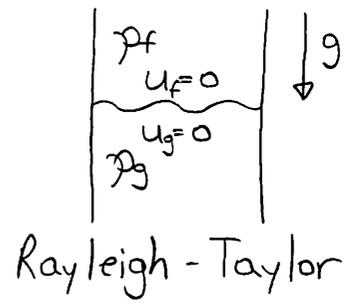
- 3) Note, in most boiling applications in industry (i.e. industrial power plants, nuclear power, etc...), q'' is the variable input and ΔT cannot be controlled. Hence, designers design the system so that q''_{max} or q''_{CHF} is never exceeded. This is why it's termed the critical heat flux, since if $q'' > q''_{CHF}$, the system will melt down. This is what happened at Chernobyl.

$$q''_{CHF} = q''_{max} > q'' \quad \text{or} \quad q'' \leq C_{max} h_{fg} [\sigma \rho_f^2 g (\rho_f - \rho_g)]^{1/4}$$

- 4) q''_{min} is sometimes called the Leidenfrost point or effect, after Johann Leidenfrost. A great demonstration of this is the apparent levitation of liquid nitrogen droplets when spilled on a lab surface. The superheat (ΔT) is so high that the droplet undergoes film boiling and evaporates very slowly:

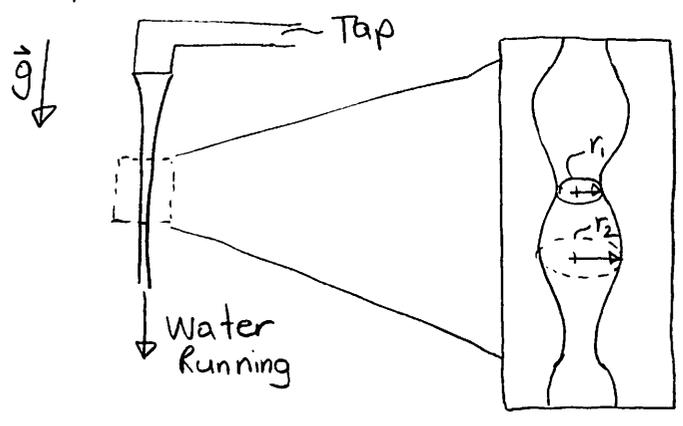


5) The difference in the instability mechanisms for q''_{min} and q''_{max} is that Helmholtz instabilities are shear driven due to a velocity of the gas phase. Taylor instability is surface tension driven and velocity is zero of both phases, with the denser phase on top of the less dense counterpart.



For more details, consult a graduate level fluid dynamics textbook or class.

6) For your interest only: a common place to observe the instabilities we've been discussing is your kitchen tap. At low enough flow rates, we observe a Rayleigh-Taylor type instability called the Rayleigh-Plateau instability.

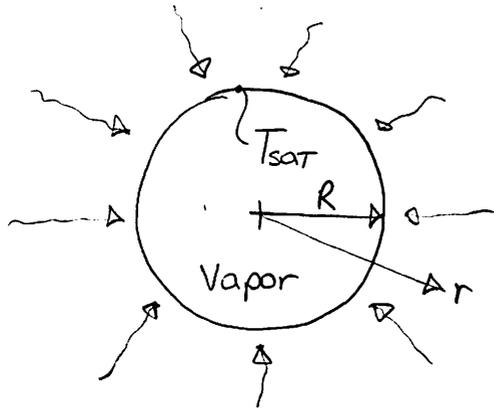


Since $r_1 < r_2$, the surface tension force ($\sim 2\pi r \sigma$) is larger and will tend to squeeze fluid to the r_2 zones. This causes droplet formation and breakup. Initiates when the fluid stream is very thin, $Fr \uparrow$.

In reality, this is a simplified view since there are other curvature effects that need consideration but nevertheless it is a great instability demonstration in fluid mechanics

Homogeneous Boiling (Bubble Growth in a Superheated Liquid)

So far, we have only focused on boiling of fluids on solid substrates (heterogeneous). What if no surface exists and boiling occurs inside the superheated fluid?



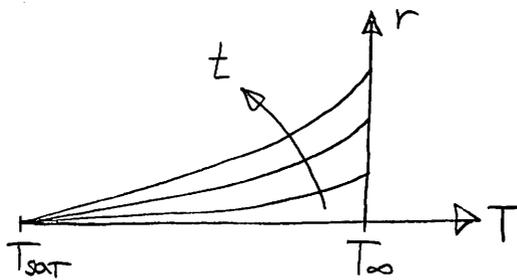
$$\sim T_{\infty} > T_{\text{sat}}$$

$$P_g \approx P_f$$

Assuming: $\frac{\delta_T}{R} \ll 1$ (flat plate model)

$P_g \approx P_f$ (large bubble)

We can treat this as a semi-infinite conduction problem



\Rightarrow The surrounding fluid cools and gives its energy to the bubble interface (T_{sat}) for evaporation.

$$q'' = \frac{k_f \Delta T}{\sqrt{\pi \alpha_f t}} = \frac{1}{\sqrt{\pi}} \frac{(k \rho C_p)_f^{1/2} \Delta T}{\sqrt{t}}$$

By an energy balance:

$$\underbrace{q''}_{\text{Energy in}} 4\pi R^2 = \underbrace{\frac{d}{dt} \left(\rho_g \frac{4}{3} \pi R^3 \right) h_{fg}}_{\dot{m} h_{fg}} = \rho_g \frac{4}{3} \pi R^2 h_{fg} \frac{dR}{dt}$$

Solving for $\frac{dR}{dt}$ explicitly: $\left[\frac{1}{\sqrt{\pi}} \frac{(k \rho C_p)_f^{1/2}}{\rho_g h_{fg}} \Delta T \right] \frac{1}{\sqrt{t}} = \frac{dR}{dt}$

Integrating:

$$R(t) = \frac{2}{\sqrt{\pi}} \frac{(k \rho C_p)_f^{1/2}}{\rho_g h_{fg}} \Delta T \sqrt{t}$$

Note we can make our solution simpler by multiplying by $\frac{C_{p,f}}{C_{p,f}}$

$$R = \frac{2}{\sqrt{\pi}} \frac{(k_f \rho_f C_p)_f^{1/2}}{\rho_f C_{p,f}} \left(\frac{C_{p,f} \Delta T}{h_{fg}} \right) \sqrt{t}$$

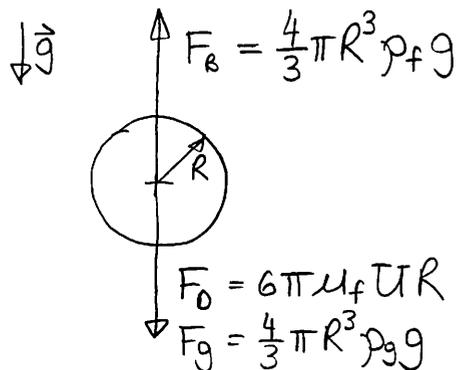
$$= \frac{2}{\sqrt{\pi}} \left(\frac{k_f}{C_{p,f} \rho_f} \right)^{1/2} \left(\frac{J_a \rho_f}{\rho_g} \right) \sqrt{t}$$

$$R(t) = \frac{2}{\sqrt{\pi}} \cdot Ja^* \sqrt{\alpha_f t} \quad , \quad \text{where} \quad Ja^* = \frac{C_p \Delta T}{h_{fg}} \cdot \frac{\rho_f}{\rho_g} \Rightarrow \text{Modified Jacob number.}$$

Note, the above solution is valid for diffusion controlled bubble growth, since we assume everything is in quasi-equilibrium. We assume no energy is required to move the liquid as the bubble grows. If the growth is fast, inertia must be accounted for with flow work. This latter case is called inertially controlled bubble growth and was solved by Rayleigh in the 1800's.

END OF LECTURE 24

Aside: To give you an example of flow work, lets do a quick calculation of the speed of a bubble rising in water, assuming Stokes flow ($Re < 0.1$)



$\downarrow \vec{g}$
 $F_B = \frac{4}{3} \pi R^3 \rho_f g$
 $F_0 = 6 \pi \mu_f U R$
 $F_g = \frac{4}{3} \pi R^3 \rho_g g$

Force balance at terminal speed:

$$F_B = F_g + F_0$$

$$\frac{24}{3} \pi R^2 \rho_f g = \frac{24}{3} \pi R^2 \rho_g g + \frac{3}{2} \pi \mu_f U R$$

$$\frac{2}{3} R^2 (\rho_f - \rho_g) g = 3 \mu_f U$$

$$U = \frac{2 R^2 (\rho_f - \rho_g) g}{9 \mu_f}$$

Lets try a $R=1\text{cm}$ bubble in water (air is bubble phase, $\rho_g \approx 1.2$)

$$U = \frac{2 (0.01)^2 (1000 - 1.2) (9.81)}{9 (8.94 \times 10^{-4})} \approx 243.8 \text{ m/s! Very large and not physically accurate.}$$

We haven't included flow work to move water, hence U is very large! (202)

Thermal Radiation

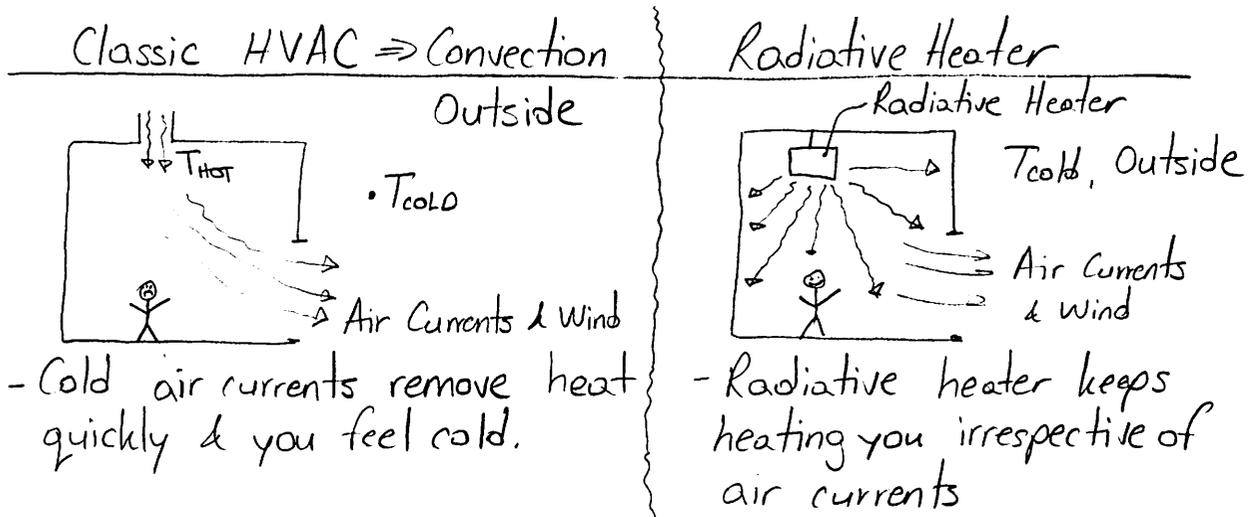
So far, we've dealt with the following:

Conduction } Heat transfer by a temperature gradient
 Convection } in a medium.

Radiation } - no medium required (can propagate in vacuum)
 } - is a form of energy emitted by all matter at a finite temperature.

Applications:

- 1) Global energy balance → earth receives all of its energy from the sun via radiation.
- 2) HVAC → Radiative heaters. Work well in areas where lots of "air changes" or opening of doors to the outside occurs, i.e. entrance ways to a theater, outside, warehouses.



Theory:

① Electromagnetic Theory (Maxwell) ⇒ waves

$$c = \lambda \nu = \frac{3.0 \times 10^8 \text{ m/s}}{n}$$

where $n = \text{index of refraction}$

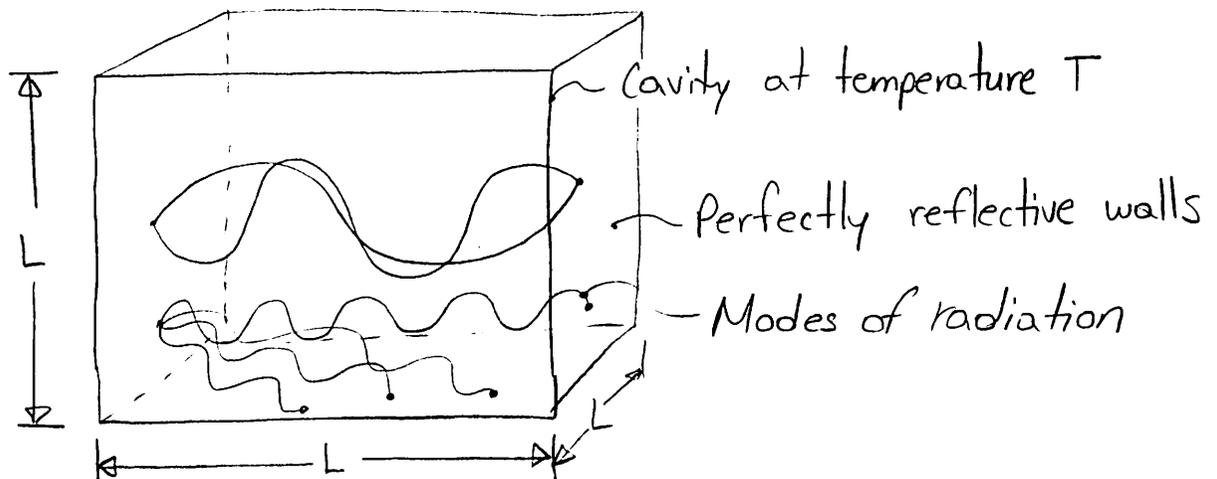
↳ Used for Rayleigh-Jeans Law.

$\nu = \frac{c}{\lambda}$
 $c = \text{speed of light}$

② Quantum Theory (Planck) \Rightarrow photons, packets of energy

$$e = \frac{hc}{\lambda} = h\nu, \text{ where } \begin{array}{l} e = \text{energy of a photon [J]} \\ c = \text{speed of light [m/s]} \\ \lambda = \text{wavelength of light [m]} \\ h = 6.6256 \times 10^{-34} \text{ J}\cdot\text{s} \end{array}$$

These two theories came to a conflict in early 1900's. The classical theory predicted an infinite "radiative energy" of a body at a finite temperature.



For higher frequency modes, more can fit in our finite box than lower frequency modes. Using analysis of standing waves, we can show that:

$$\frac{\# \text{ modes per unit wavelength}}{\text{Cavity volume}} = \frac{8\pi}{\lambda^4}$$

Using equipartition of energy, the energy per mode is $k_B T$.

$$E_{\text{mode}} = \frac{\frac{1}{2} k_B T \cdot (2)}{\text{Degree of freedom}} \quad \leftarrow 2 \text{ polarizations / mode}$$

where: $k_B = 1.38 \times 10^{-23}$

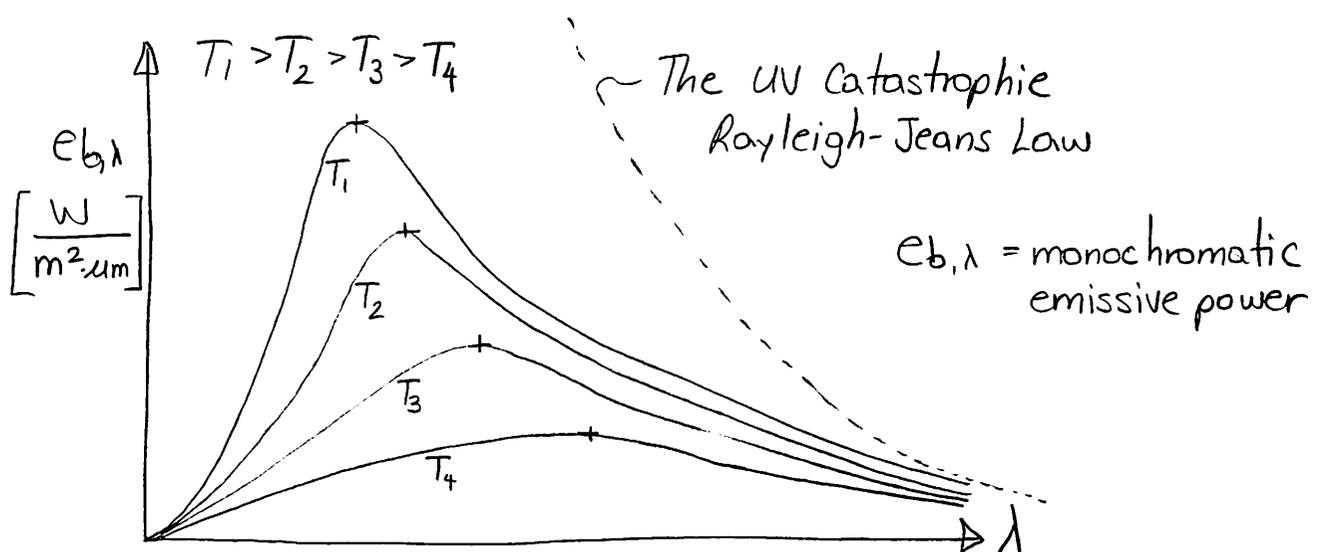
$$\frac{du}{d\lambda} = \frac{8\pi k_B T}{\lambda^4} \Rightarrow \text{Energy per unit volume per unit wavelength}$$

$$\text{So: } U(T) = \int_0^{\infty} dU = \int_0^{\infty} \frac{8\pi k_B T}{\lambda^4} d\lambda \rightarrow \infty \text{ as } \lambda \rightarrow 0$$

This was called the ultraviolet catastrophe. As the wavelengths reached the UV levels, energy shoots to infinity.

This was resolved by Planck. The main difference is:

- 1) Classical view assumed all modes were equally probable to be occupied. Adding a small amount of energy to any mode is possible.
- 2) Planck's assumption that energy is quantized (add the energy of a whole photon, or don't add it at all) treats the photons like particles. As we learned in kinetic theory, the probability of finding a particle at the average kinetic energy is higher than finding it at a much higher energy state. Hence, it is harder to find a photon far away from the average thermal energy & hence the probability of UV photons is low.



Planck derived his formula to be:

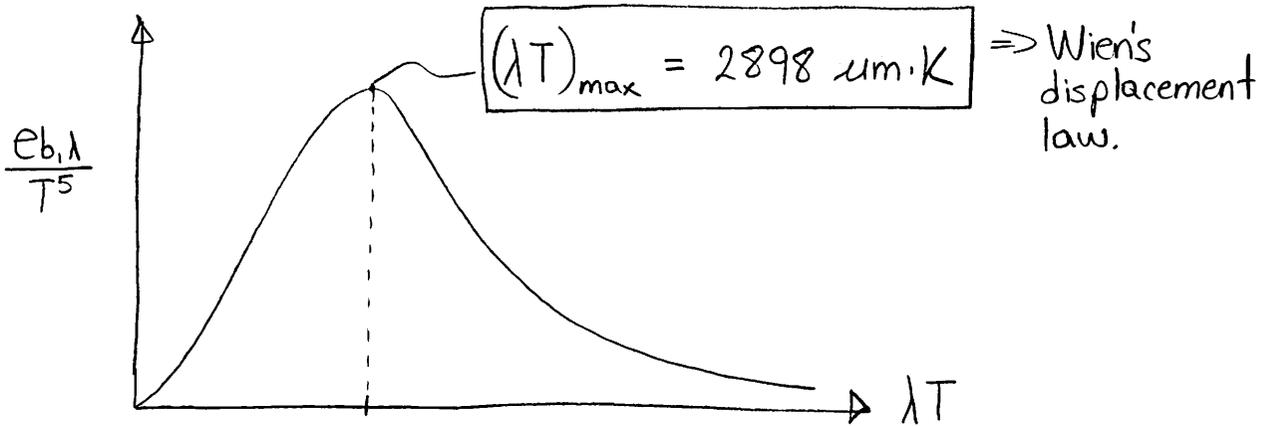
$$e_{b,\lambda} = \frac{2\pi hc^2 \lambda^{-5}}{e^{\frac{hc}{\lambda T}} - 1} = \frac{C_1 \lambda^{-5}}{e^{C_2/\lambda T} - 1}$$

Dividing through by T^5

$$\frac{e_{b,\lambda}}{T^5} = \frac{C_1 (\lambda T)^{-5}}{e^{C_2/\lambda T} - 1}$$

$$C_1 = 2\pi hc^2 = 3.742 \times 10^8 \text{ W}\mu\text{m}^4/\text{m}^2$$

$$C_2 = hc/k_B = 1.4389 \times 10^4 \mu\text{m}\cdot\text{K}$$



Visible range: $0.4 - 0.7 \mu\text{m}$ (blue to red)
 Thermal radiation: $0.1 - 100 \mu\text{m}$

So as energy increases ($T \uparrow$), wavelength is reduced and more of the emissive spectrum shifts to the visible range.

Example: hot iron bar \Rightarrow goes from no color \rightarrow red \rightarrow orange \rightarrow yellow \rightarrow white.

Note, if we want total emissive power:

$$e_b = \int_0^{\infty} e_{b,\lambda} d\lambda = T^5 C_1 \int_0^{\infty} \frac{(\lambda T)^{-5}}{e^{C_2/\lambda T} - 1} d\lambda$$

$$= T^4 \left[C_1 \int_0^{\infty} \frac{\eta^{-5}}{e^{C_2/\eta} - 1} d\eta \right], \quad \eta = \lambda T$$

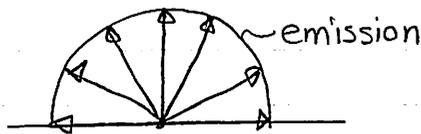
$$\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$$

So our total emissive power is:

$$e_b = \sigma T^4$$

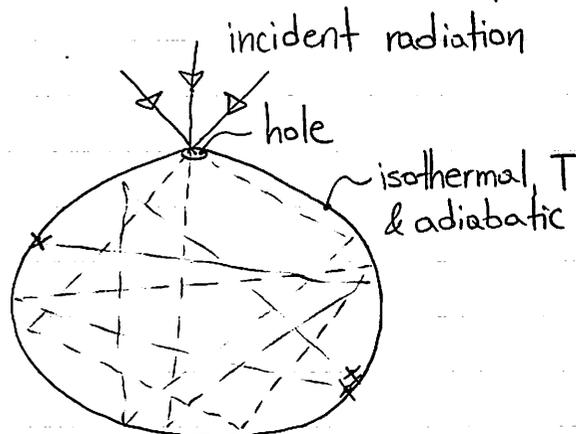
It is important to note, that the previous analysis is valid for "blackbodies." These are ideal radiators:

- 1) They absorb all incident (incoming) radiation
- 2) Emission of radiation is independent of direction (diffuse)



- 3) At a given temp. (T) and wavelength (λ), no surface can emit more energy.

Experimentally, a blackbody can be simulated by:



- all incident radiation is absorbed (multiple bounces)
- emitted radiation independent of direction
- the small hole is the blackbody

For non-blackbody surfaces, we can define a monochromatic emissivity ϵ_λ :

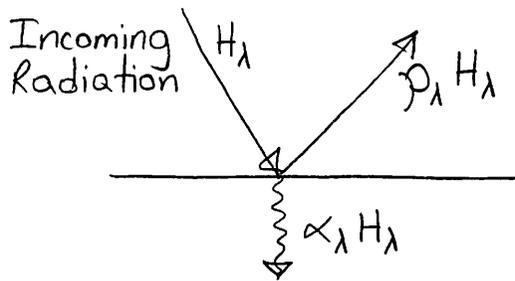
$$e_\lambda = \epsilon_\lambda e_{b,\lambda}$$

$$e = \int_0^\infty \epsilon_\lambda e_{b,\lambda} d\lambda = \epsilon e_b$$

$$\epsilon = \frac{\int_0^{\infty} \epsilon_{\lambda} e_{b,\lambda} d\lambda}{\int_0^{\infty} e_{b,\lambda} d\lambda}$$

\Rightarrow Spectrally averaged emissivity.

For a real surface, we also have reflection (ρ) and absorption (α)



From an energy balance:

$$\rho_{\lambda} + \alpha_{\lambda} = 1$$

In thermal equilibrium:

$$\epsilon_{\lambda} = \alpha_{\lambda} \Rightarrow \text{Kirchhoff's Law}$$

Note, this is valid for diffuse surfaces, otherwise $\epsilon_{\lambda,\theta} = \alpha_{\lambda,\theta}$

Similarly for absorption:

$$\alpha = \frac{\int_0^{\infty} \alpha_{\lambda} H_{\lambda} d\lambda}{\int_0^{\infty} H_{\lambda} d\lambda}$$

\Rightarrow Spectrally averaged absorption

Naturally from Kirchhoff's law we can say:

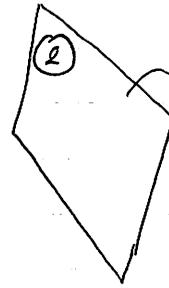
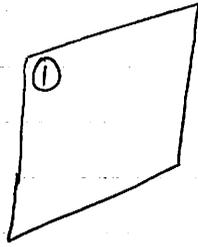
- 1) Good absorbers are good emitters
 - 2) Poor absorbers are poor emitters
- } experimentally established.

This is why high tech thermal blankets are made of reflective metallic coatings, since poor absorbers are poor emitters, and will keep you warm.

Also note, the above assumes transmission is zero. If not:

$$\alpha_{\lambda} + \rho_{\lambda} + \tau_{\lambda} = 1$$

View Factors and Blackbody Exchange



Blackbody surfaces

$$\left. \begin{aligned} q_{1 \rightarrow 2} &= F_{12} A_1 e_{b1} \\ q_{2 \rightarrow 1} &= F_{21} A_2 e_{b2} \end{aligned} \right\}$$

F_{ab} = view factor = fraction of radiation leaving surface "a" and reaching surface "b".

So our net exchange is:

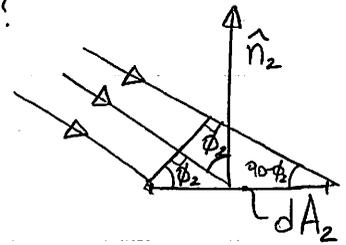
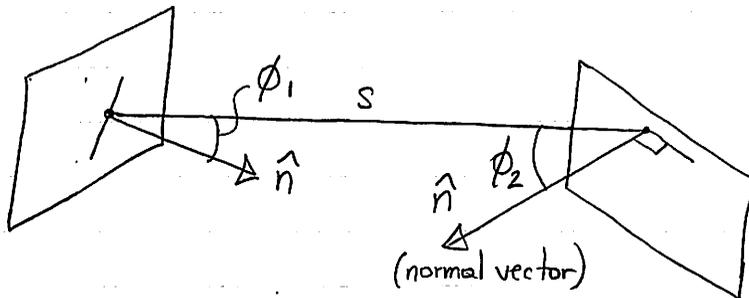
$$q_{12} = q_{1 \rightarrow 2} - q_{2 \rightarrow 1} = F_{12} A_1 e_{b1} - F_{21} A_2 e_{b2}$$

If $T_1 = T_2$, $e_{b1} = e_{b2}$

$$q_{12} = e_b (A_1 F_{12} - A_2 F_{21}) = 0 \quad (\text{no net energy transfer between bodies at same } T)$$

$$\therefore \boxed{A_1 F_{12} = A_2 F_{21}}$$

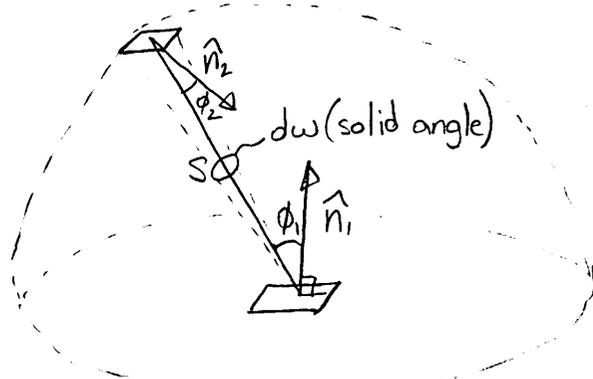
But what is our fraction F_{12} or F_{21} ?



$$A_1 F_{12} = B \int_{A_1} \int_{A_2} \frac{\cos \phi_1 \cos \phi_2}{s^2} dA_1 dA_2 \quad ; \quad B = \text{constant}$$

Aside:

Think of the previous expression in terms of small differential areas:



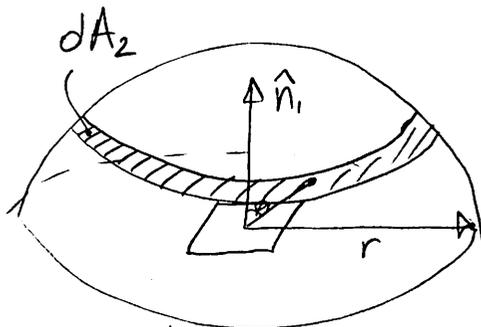
$$d\omega = \frac{\cos \phi_2 dA_2}{s^2} \text{ (Solid Angle)}$$

$$E_{\text{out}} = I_{\text{out}} \cos \phi_1 dA_1 d\omega$$

$$I_{\text{out}} = \frac{\sigma T^4}{\pi} \text{ (Intensity)}$$

You can solve the rest. Try it!

Going a step further



$$A_1 = dA_1$$

$$s = r$$

$$\phi_2 = 0$$

$$dA_2 = 2\pi r^2 \sin \phi_1 d\phi_1$$

Substituting back into our relation: $\underbrace{A_1}_{dA_1} F_{12} = B \int \int \frac{\cos \phi_1 \cos \phi_2}{s^2} dA_1 dA_2$

$$F_{12} = B \cdot 2\pi \int_0^{\pi/2} \cos \phi_1 \sin \phi_1 d\phi_1$$

$$F_{12} = 1$$

$$1 = B \cdot 2\pi \int_0^{\pi/2} \sin \phi_1 d(\sin \phi_1) = B \cdot 2\pi \left. \frac{(\sin \phi_1)^2}{2} \right|_0^{\pi/2} = \pi B = 1$$

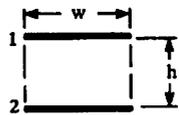
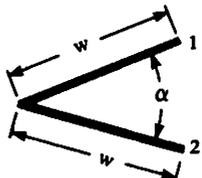
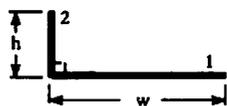
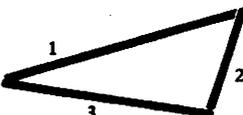
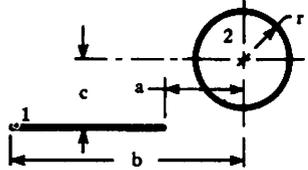
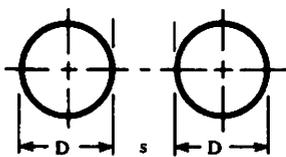
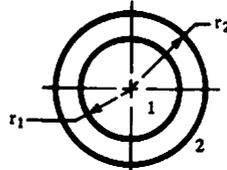
$$B = \frac{1}{\pi}$$

So our view factor becomes:

$$A_1 F_{12} = A_2 F_{21} = \frac{1}{\pi} \int \int \frac{\cos \phi_1 \cos \phi_2}{s^2} dA_1 dA_2$$

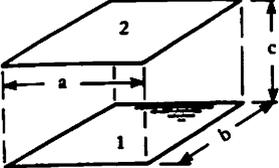
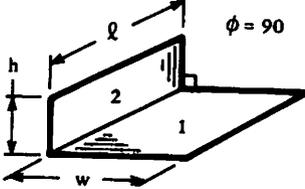
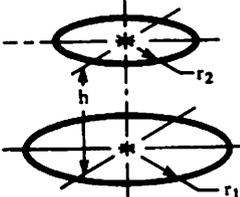
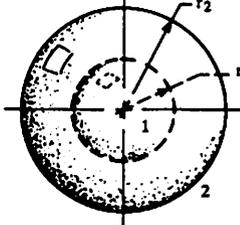
⇒ Look up in Mills Table 6.1, pg. 541-543.

Table 10.2 View factors for a variety of two-dimensional configurations (infinite in extent normal to the paper)

Configuration	Equation
1. 	$F_{1-2} = F_{2-1} = \sqrt{1 + \left(\frac{h}{w}\right)^2} - \left(\frac{h}{w}\right)$
2. 	$F_{1-2} = F_{2-1} = 1 - \sin(\alpha/2)$
3. 	$F_{1-2} = \frac{1}{2} \left[1 + \frac{h}{w} - \sqrt{1 + \left(\frac{h}{w}\right)^2} \right]$
4. 	$F_{1-2} = (A_1 + A_2 - A_3)/2A_1$
5. 	$F_{1-2} = \frac{r}{b-a} \left[\tan^{-1} \frac{b}{c} - \tan^{-1} \frac{a}{c} \right]$
6. 	Let $X = 1 + s/D$. Then: $F_{1-2} = F_{2-1} = \frac{1}{\pi} \left[\sqrt{X^2 - 1} + \sin^{-1} \frac{1}{X} - X \right]$
7. 	$F_{1-2} = 1$, $F_{2-1} = \frac{r_1}{r_2}$, and $F_{2-2} = 1 - F_{2-1} = 1 - \frac{r_1}{r_2}$

Adapted From Lienhard & Lienhard "A Heat Transfer Textbook" (2012)

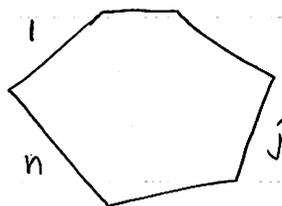
Table 10.3 View factors for some three-dimensional configurations

Configuration	Equation
<p>1. </p>	<p>Let $X = a/c$ and $Y = b/c$. Then:</p> $F_{1-2} = \frac{2}{\pi XY} \left\{ \ln \left[\frac{(1 + X^2)(1 + Y^2)}{1 + X^2 + Y^2} \right]^{1/2} - X \tan^{-1} X - Y \tan^{-1} Y + X\sqrt{1 + Y^2} \tan^{-1} \frac{X}{\sqrt{1 + Y^2}} + Y\sqrt{1 + X^2} \tan^{-1} \frac{Y}{\sqrt{1 + X^2}} \right\}$
<p>2. </p>	<p>Let $H = h/l$ and $W = w/l$. Then:</p> $F_{1-2} = \frac{1}{\pi W} \left\{ W \tan^{-1} \frac{1}{W} - \sqrt{H^2 + W^2} \tan^{-1} (H^2 + W^2)^{-1/2} + H \tan^{-1} \frac{1}{H} + \frac{1}{4} \ln \left[\frac{(1 + W^2)(1 + H^2)}{1 + W^2 + H^2} \right] \times \left[\frac{W^2(1 + W^2 + H^2)}{(1 + W^2)(W^2 + H^2)} \right]^{W^2} \left[\frac{H^2(1 + H^2 + W^2)}{(1 + H^2)(H^2 + W^2)} \right]^{H^2} \right\}$
<p>3. </p>	<p>Let $R_1 = r_1/h$, $R_2 = r_2/h$, and $X = 1 + (1 + R_2^2) / R_1^2$. Then:</p> $F_{1-2} = \frac{1}{2} \left[X - \sqrt{X^2 - 4(R_2/R_1)^2} \right]$
<p>4. </p>	<p>Concentric spheres:</p> $F_{1-2} = 1, \quad F_{2-1} = (r_1/r_2)^2, \quad F_{2-2} = 1 - (r_1/r_2)^2$

Now we can formulate our analysis as:

$$q_{12} = \frac{A_1 F_{12} (e_{b1} - e_{b2})}{A_1 F_{12} e_{b1} - A_2 F_{21} e_{b2}} \Leftrightarrow \begin{array}{c} \frac{1}{A_1 F_{12}} \\ \text{---} \text{---} \text{---} \\ e_{b1} \xrightarrow{q_{12}} e_{b2} \end{array}$$

If we have an enclosure: (Radiation exchange inside a body)

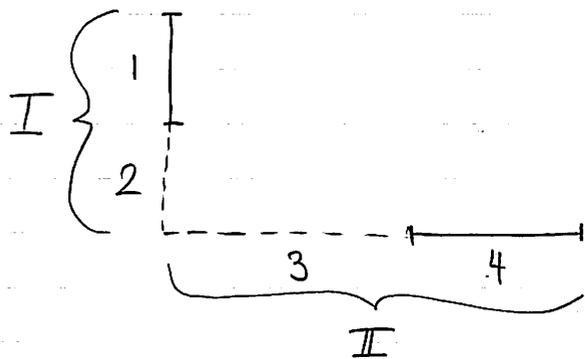


$$\sum_{j=2}^n F_{ij} = 1$$

Note, only true for flat or convex surfaces. If concave

$\sum_{j=2}^n F_{ij} \neq 1$ since $F_{i1} \neq 0$ (Surface sees itself).

Example Consider the following: (Solve for F_{14})



$$A_1 F_{14} = A_I F_{I4} - A_2 F_{24}$$

$$\cancel{A_I} F_{I4} = \cancel{A_I} F_{II} - \cancel{A_I} F_{I3} \Rightarrow F_{I4} = \underbrace{F_{II} - F_{I3}}$$

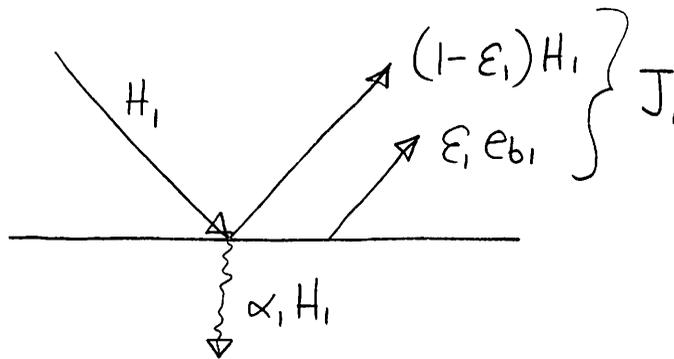
$$\cancel{A_2} F_{24} = \cancel{A_2} F_{2II} - \cancel{A_2} F_{23} \Rightarrow F_{24} = \underbrace{F_{2II} - F_{23}}$$

Our charts have these factors: $F_{II}, F_{I3}, F_{2II}, F_{23}$ Can easily look these up.

Gray Body Radiation

If $\epsilon \neq 1$, and $\epsilon \neq f(\lambda, \theta)$, we call this a gray body. Here, we can assume that $\epsilon = \alpha$.

diffuse & independent of wavelength



Flux of energy that irradiates the surface.

$H_1 = \text{irradiance } \left[\frac{W}{m^2} \right]$

$J_1 = \text{radiosity } \left[\frac{W}{m^2} \right]$

Total flux of radiat energy away from the surface.

$$J_1 = \underbrace{(1-\epsilon_1)H_1}_{\text{Reflected energy}} + \underbrace{\epsilon_1 e_{b1}}_{\text{Emitted energy}}$$

Aside: Note, gray body analysis is very useful in real life since most surfaces don't behave as blackbodies and have some finite emissivity $\epsilon < 1$ & $\alpha < 1$

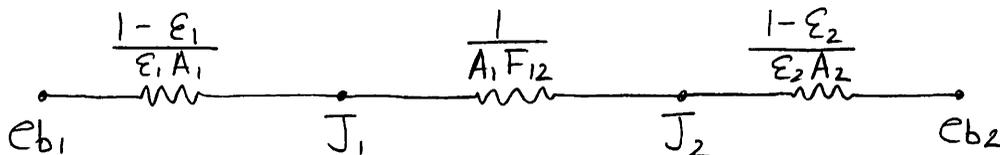
$$H_1 = \frac{J_1 - \epsilon_1 e_{b1}}{1 - \epsilon_1}$$

$$\frac{q_{1,NET}}{A_1} = (J_1 - H_1) = J_1 - \frac{J_1 - \epsilon_1 e_{b1}}{1 - \epsilon_1} = \frac{J_1(1 - \epsilon_1) - J_1 + \epsilon_1 e_{b1}}{1 - \epsilon_1} = \frac{\epsilon_1}{1 - \epsilon_1} (e_{b1} - J_1)$$

$\frac{q_{1,NET}}{A_1} = \frac{\epsilon_1}{1 - \epsilon_1} (e_{b1} - J_1)$

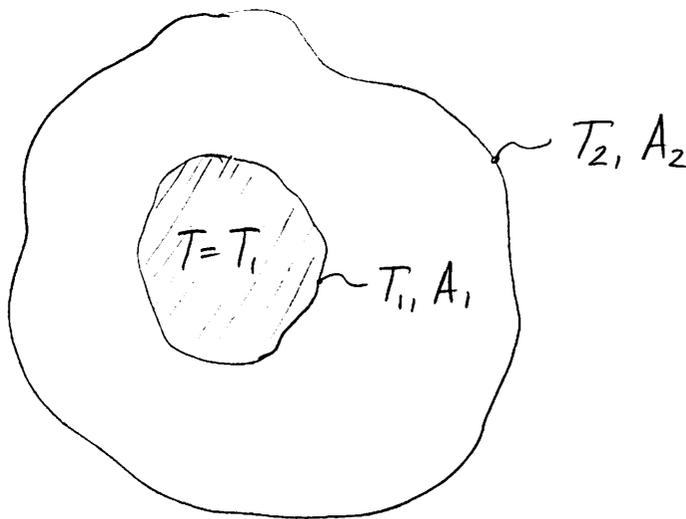
 \Rightarrow Net heat flux into surface 1.

We can reformulate this in terms of a thermal resistance



Note, if you set $\epsilon_1 = \epsilon_2 = 1$, you get the blackbody solution.

Gray Body Enclosures



By an energy balance:

$$A_1 H_1 = A_2 \underbrace{J_2 F_{21}}_{A_1 J_2 F_{12}} + A_1 J_1 F_{11}$$

$$H_1 = J_2 F_{12} + J_1 F_{11}$$

↳ Net energy coming to ①

$$\frac{q_{1 \rightarrow 2, NET}}{A_1} = J_1 - J_2 F_{12} - J_1 F_{11}$$

$$= \underbrace{(J_1 - J_2)}_{\text{Since } (1-F_{11})=F_{12}} F_{12}$$

So we have the same solution as before, but assuming that body 2 is much larger than body 1, we get $F_{12} = 1$.

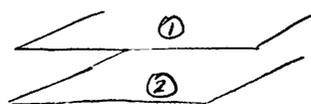
$$q_{1 \rightarrow 2} = \frac{A_1 (e_{b1} - e_{b2})}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \frac{A_1}{A_2} \left(\frac{1}{\epsilon_2} - 1\right)}$$

① $A_2 \gg A_1$; $F_{12} = 1$

$$q_{1-2} = A_1 \epsilon_1 (e_{b1} - e_{b2}) \Rightarrow \text{Smaller body behaves like a black body.}$$

Physically, this result comes from the fact that the emitted and reflected radiation from ① bounces around multiple times and is absorbed by ② before it can hit ① again. Hence $\epsilon_2 = 1$.
 $\alpha_2 = 1$.

② Parallel Plates:

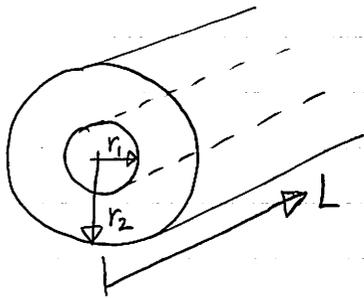


$$F_{12} = 1 = F_{21}$$

$$A_1 = A_2$$

$$q_{1-2} = \frac{A (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

③ Concentric Cylinders

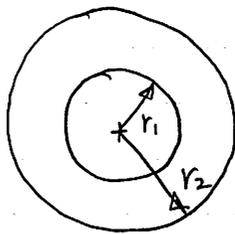


$$F_{12} = 1$$

$$\frac{A_1}{A_2} = \frac{r_1}{r_2}$$

$$q_{r1-2} = \frac{2\pi r_1 L (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{r_1}{r_2} \left(\frac{1}{\epsilon_2} - 1 \right)}$$

④ Concentric Spheres

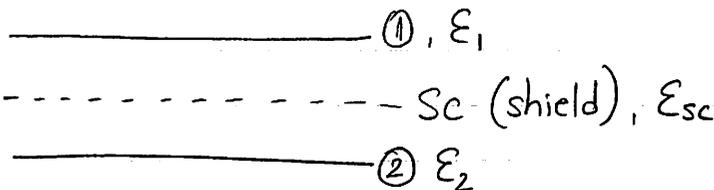


$$\frac{A_1}{A_2} = \frac{r_1^2}{r_2^2} ; F_{12} = 1$$

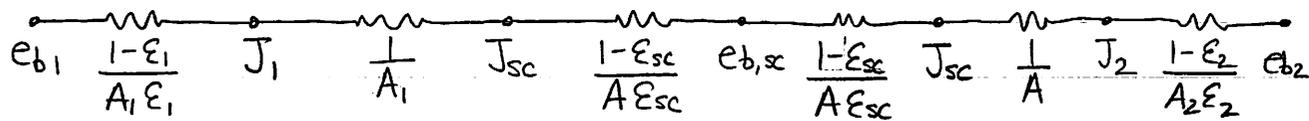
$$q_{r1-2} = \frac{4\pi r_1^2 (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{r_1^2}{r_2^2} \left(\frac{1}{\epsilon_2} - 1 \right)}$$

Radiation Shields

They are often used to reduce heat losses



Drawing out our resistance diagram



$$q_{12} = \frac{A (e_{b1} - e_{b2})}{\left(\frac{1}{\epsilon_1} - 1 \right) + 1 + \left(\frac{1}{\epsilon_2} - 1 \right) + 2 \left(\frac{1}{\epsilon_{sc}} - 1 \right) + 1}$$

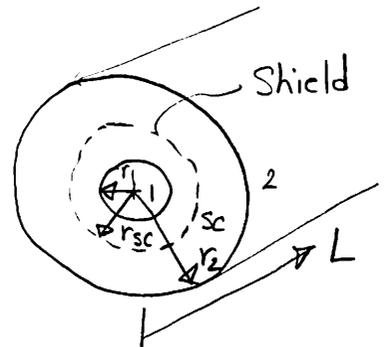
$$q_{r12} = \frac{A (e_{b1} - e_{b2})}{\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1\right) + \left(\frac{2}{\epsilon_{sc}} - 1\right)} \Rightarrow \text{For 1 shield}$$

For n shields in series:

$$q_{r12} = \frac{A (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{2n}{\epsilon_{sc}} - (n+1)}$$

For cylindrical shields (n of them)

$$q_{r12} = \frac{2\pi r_1 L (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{r_1}{r_2} \left(\frac{1}{\epsilon_2} - 1\right) + 2 \left(\frac{1}{\epsilon_{sc}} - 1\right) \sum_{j=1}^n \frac{r_1}{r_{sc,j}}}$$

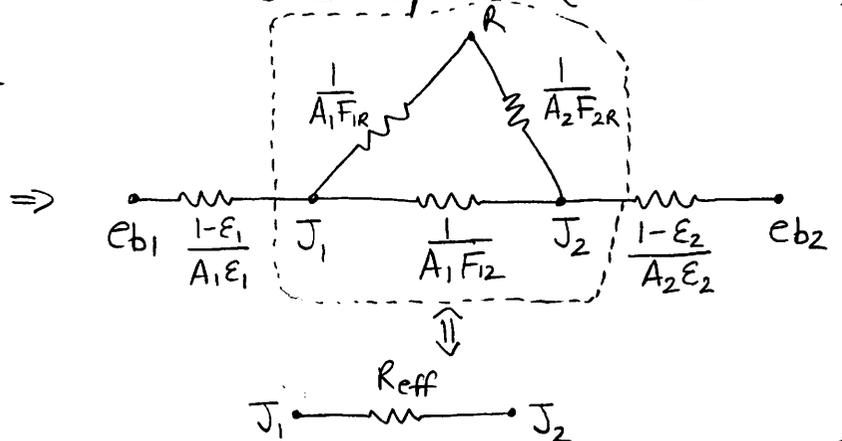
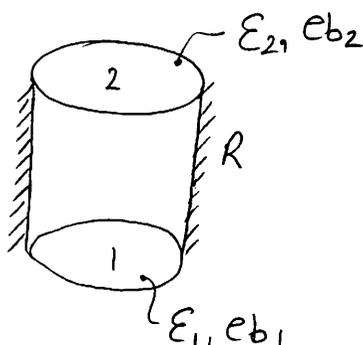


For spherical shields (n of them)

$$q_{r12} = \frac{4\pi r_1^2 (e_{b1} - e_{b2})}{\frac{1}{\epsilon_1} + \frac{r_1^2}{r_2^2} \left(\frac{1}{\epsilon_2} - 1\right) + 2 \left(\frac{1}{\epsilon_{sc}} - 1\right) \sum_{j=1}^n \frac{r_1^2}{r_{sc,j}^2}}$$

Note, for spherical & cylindrical shields, you want r_{sc} to be close to r_1 since you want the shield to see as little of itself as possible.

Looking back at our adiabatic example: ($e_b = J$ at R)

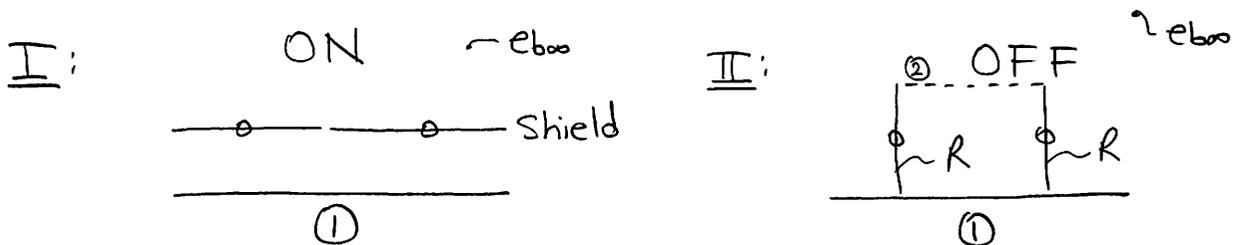


$$\frac{1}{R_{eff}} = \frac{1}{\frac{1}{A_1 F_{1R}} + \frac{1}{A_2 F_{2R}}} + A_1 F_{12}$$

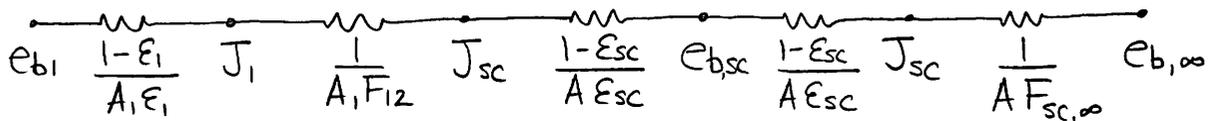
$$q_{12} = \frac{e_{b1} - e_{b2}}{\frac{1}{A_1} \left(\frac{1}{\epsilon_1} - 1 \right) + \frac{1}{A_2} \left(\frac{1}{\epsilon_2} - 1 \right) + R_{eff}}$$

⇒ Note, our solution does not depend on ϵ_R since $e_{bR} = J_R$ or $q'' = 0$ at R .

Example Consider a rotatable radiation shield shown below. Calculate its effectiveness in the on & off positions:

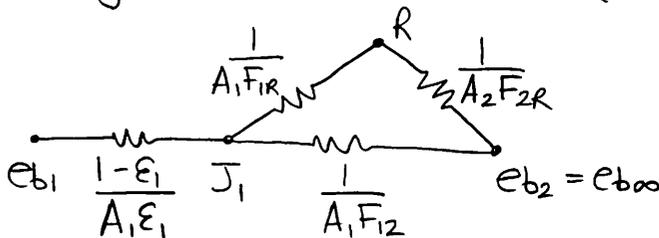


Solving the on case first:



$$q_{1\infty} = \frac{A_1 (e_{b1} - e_{b\infty})}{\frac{1}{\epsilon_1} + \frac{2}{\epsilon_{sc}} - 1}, \quad F_{sc,\infty} = 1$$

Looking at the OFF case (surfaces are reflective since $q'' = 0$ at the shields)



$$\left. \begin{aligned} F_{12} &= \sqrt{2} - 1 \\ F_{1R} &= 2 - \sqrt{2} \end{aligned} \right\} \text{From tables}$$

$$q_{12} = \frac{A_1 (e_{b1} - e_{b\infty})}{\frac{1}{\epsilon_1} - 1 + \sqrt{2}}$$

Aside: Note, the shields in II are adiabatic due to symmetry $q'' = 0$

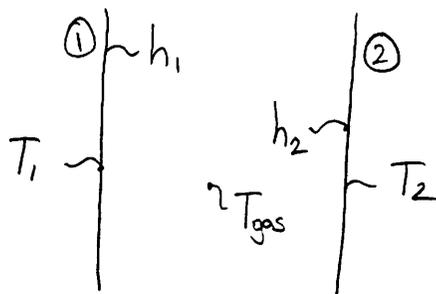
$R_I \gg R_{II}$, Assuming $\epsilon_1 = 0.5$; $R =$ thermal radiative resistance.
 $\epsilon_{sc} = 0.1$

$$\frac{R_I}{R_{II}} = \frac{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_{sc}} - 1}{\frac{1}{\epsilon_1} - 1 + \sqrt{2}} = 13.2$$

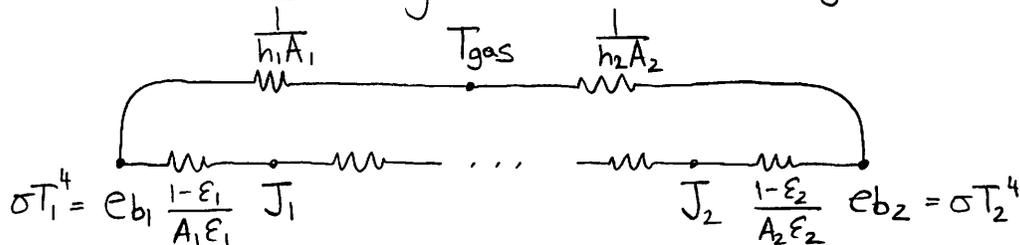
So when the shield is in the ON position, it is 13.2 times better at limiting heat transfer!

Multi-Mode Heat Transfer (Radiation + Convection)

Assuming we have radiation and convection for example, our analysis is similar:

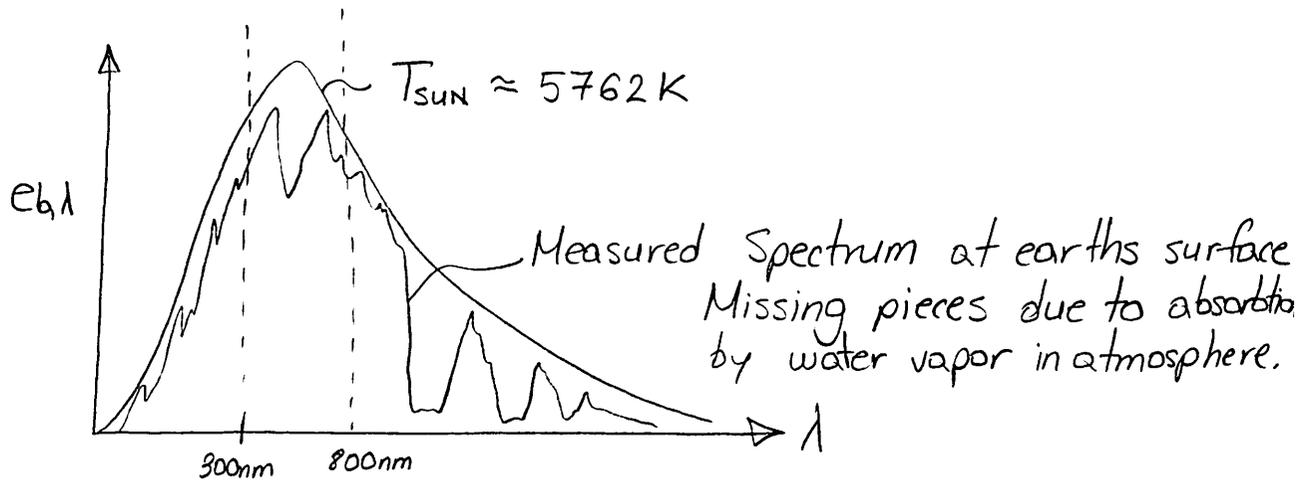


Our resistance diagram becomes: (T_{gas} is unknown)



We would have to solve the complete diagram, and it may require iteration, if T_1 or T_2 are not provided and instead q'' is provided.

Solar Radiation



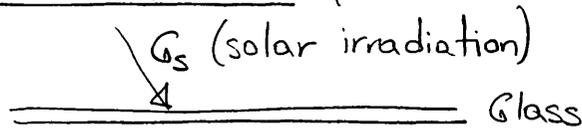
$q_{\text{sun}} = 1353 \text{ W/m}^2$ (Outside the atmosphere)

$q_{\text{earth}} = 636 \text{ W/m}^2$ (On the earth's surface)

Note, the total arriving energy from the sun to the earth is $\approx 1.7 \times 10^{14} \text{ kW}$. The US peak demand is $\approx 1 \times 10^9 \text{ kW}$.

Our eyes are most sensitive to visible light (400-700nm) where the peak intensity from the sun is. This is simply due to evolution.

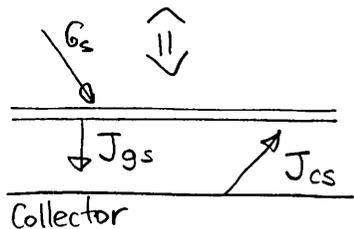
Solar Collectors: (Detailed Analysis)



glass: $\tau_{gs}, \rho_{gs}, \alpha_{gs}$

collector: ρ_{cs}, α_{cs}

Collector



$$\begin{aligned} J_{gs} &= G_s \tau_{gs} + \rho_{gs} J_{cs} \\ J_{cs} &= \rho_{cs} J_{gs} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{Reflected energy } J_{cs} \\ \text{from glass back to} \\ \text{collector.} \end{array}$$

$$J_{gs} = G_s \tau_{gs} + \rho_{gs} \rho_{cs} J_{gs}$$

$$J_{gs} = \frac{G_s \tau_{gs}}{1 - \rho_{gs} \rho_{cs}}$$

$$Q_{\text{collector},s} = \alpha_{cs} J_{gs} = G_s \frac{\tau_{gs} \alpha_{cs}}{1 - \rho_{gs} \rho_{cs}} \Rightarrow \text{Heat Collected in the solar spectrum.}$$

So now if we examine the IR range:

$$Q_{\text{coll-glass}} = \frac{E_{bc} - E_{bg}}{\frac{1}{\epsilon_c} + \frac{1}{\epsilon_g} - 1} \quad (\text{Two parallel plates})$$

If we look at the glass heat input: (From the solar spectrum)

$$\underbrace{\alpha_{gs}}_{\text{solar absorptivity}} (G_s + J_{cs}) = \alpha_{gs} G_s \left[1 + \frac{\rho_{cs} \tau_{gs}}{1 - \rho_{gs} \rho_{cs}} \right]$$

We also have heat input from the collector to the glass:

$$\alpha_{gs} G_s \left[1 + \frac{\rho_{cs} \tau_{gs}}{1 - \rho_{gs} \rho_{cs}} \right] + \frac{E_{bc} - E_{bg}}{\frac{1}{\epsilon_c} + \frac{1}{\epsilon_g} - 1} = \epsilon_g (E_{bg} - E_{b\infty}) + h(T_g - T_\infty)$$

↳ Energy balance at the glass.

Solving the above equations can give us our collector efficiency. Note here we had to split up the two spectral ranges since $\epsilon = \text{constant} = f(\lambda)$

Current collectors use micro/nanotechnology to achieve high absorptivity in the solar spectrum ($300\text{nm} < \lambda < 2500\text{nm}$) and low thermal emissivity in the IR spectrum ($\lambda > 2500\text{nm}$).

Mass Transfer

Lets begin with some definitions:

$$\rho_i = \text{mass concentration of species } i \text{ [kg/m}^3\text{]}$$

$$c_i = \text{molar concentration of } i \text{ [kmol/m}^3\text{]}$$

$$\rho = \sum \rho_i, \quad m_i = \frac{\rho_i}{\rho}$$

$m_i = \text{mass fraction of } i$
 $\rho = \text{total mass of mixture}$

$$c = \sum c_i, \quad x_i = \frac{c_i}{c}$$

$x_i = \text{mole fraction of } i$

$$c_i = \frac{\rho_i}{M_i}, \quad M = \sum x_i M_i$$

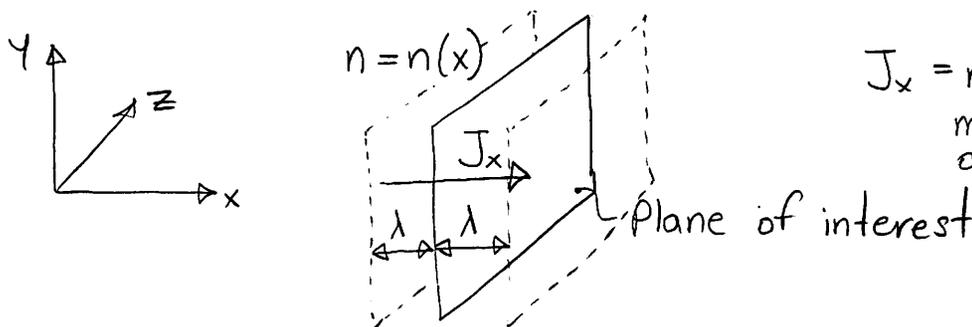
$P_i = \text{partial pressure of } i$

$P = \text{total pressure}$

$$c = \frac{\rho}{M}, \quad \frac{1}{M} = \sum \frac{m_i}{M_i}$$

$$\frac{P_i}{P} = x_i, \quad P = \sum P_i$$

Now lets consider the self-diffusion of a gas.



$J_x = \text{mean number of molecules crossing unit area of plane per unit time.}$

Similar analysis to when we derived viscosity on page 146 of notes.

$$J_x = \frac{1}{6} \bar{v} n_1(x-\lambda) - \frac{1}{6} \bar{v} n_1(x+\lambda)$$

$$= \frac{1}{6} \bar{v} (n_1(x-\lambda) - n_1(x+\lambda)) = \frac{1}{6} \bar{v} \left(-2 \frac{\partial n_1}{\partial x} \lambda \right)$$

$$\boxed{J_x = -D \frac{\partial n_1}{\partial x}} \quad \text{and} \quad \boxed{D = \frac{1}{3} \bar{v} \lambda} \Rightarrow \text{Coefficient of self diffusion. [m}^2\text{/s]}$$

↳ Fick's Law

Note Fick's Law doesn't only apply to self-diffusion. As long as the molecules are similar (ie. N_2 & O_2), it still works.

Other ways of writing it are:

$$j_{m,x} = -D_{i,m} \frac{\partial \rho_i}{\partial x} = -D_{i,m} \rho \frac{\partial m_i}{\partial x} \Rightarrow D_{i,m} = \text{diffusion coefficient of } i \text{ in } m.$$

$$\vec{j}_{m,i} = -D_{i,m} \rho \nabla m_i$$

$$J_{m,x} = -D_{i,m} \frac{\partial C_i}{\partial x} = -D_{i,m} \cdot C \cdot \frac{\partial x_i}{\partial x}$$

$$\vec{J}_{m,i} = -D_{i,m} \cdot C \cdot \nabla x_i$$

For a Lagrangian system:

$$\frac{DC_i}{Dt} = D_{i,m} \nabla^2 C_i$$

Mass Transfer

$$\frac{DT}{Dt} = \alpha \nabla^2 T$$

Heat Transfer

Our governing equation is:

$$\frac{DC_i}{Dt} = \frac{\partial C_i}{\partial t} + \vec{v} \cdot \nabla C_i = D_{i,m} \nabla^2 C_i$$

At steady state and $\vec{v} = 0$

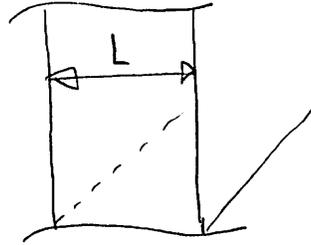
$$\nabla^2 C_i = 0, \quad \nabla^2 x_i = 0, \quad \nabla^2 \rho_i = 0, \quad \nabla^2 m_i = 0$$

If we have a 1-D problem and steady, with $\vec{v} = 0$

$$N_i = \frac{\Delta C_i}{R_m} \longleftrightarrow \text{Mass transfer resistance}$$

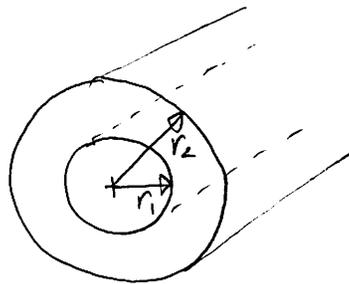
So our mass transfer resistances are:

Slab:



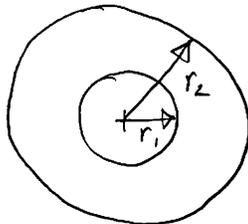
$$R_m = \frac{L}{D_{1,2} A}$$

Cylinder:



$$R_m = \frac{\ln(r_2/r_1)}{2\pi L D_{1,2}}$$

Sphere:

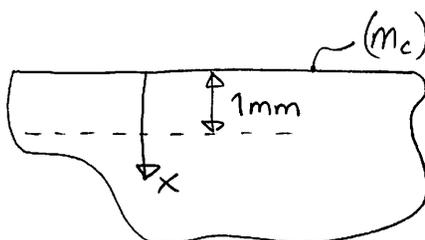


$$R_m = \frac{1}{2\pi D_{1,2}} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

Example Case hardening of mild steel.

Mild steel 0.2% or carbon (mass fraction). $m_c = 0.002$
 $t = 0, x = 0 \quad (m_c)_{x=0} = 0.015$
 $t = ?, x = 1\text{mm} \quad (m_c)_{x=1\text{mm}} = 0.008$ } $D_{c,s} = 5.6 \times 10^{-10} \text{ m}^2/\text{s}$

At what time, t , will this happen $(m_c)_{x=1\text{mm}} = 0.008$?
 To solve, we first need to model the problem. We can use the semi-infinite body approach.



$$\frac{(m_c)_{x=1\text{mm}} - (m_c)_\infty}{(m_c)_{x=0\text{mm}} - (m_c)_\infty} = \text{erfc} \left(\frac{x}{2\sqrt{D_{c,s}t}} \right)$$

$$\frac{6}{13} = 0.4615 = \text{erfc}(\beta) \Rightarrow \text{Look in Mills pg. 924}$$

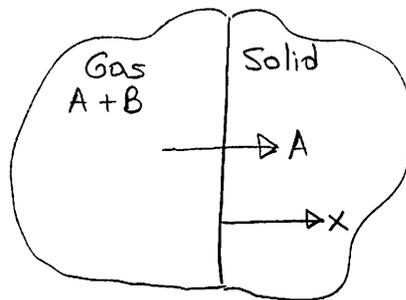
$$\beta = 0.52$$

So our solution becomes:

$$0.52 = \frac{10^{-3} \text{ m}}{2\sqrt{Dt}} \Rightarrow \boxed{t = 1650 \text{ s} = 28 \text{ minutes}}$$

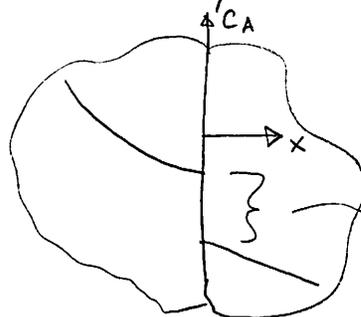
Where heat and mass transfer begin to differ is the fact that mass transfer allows for equilibrium to be reached with a concentration difference.

For example:

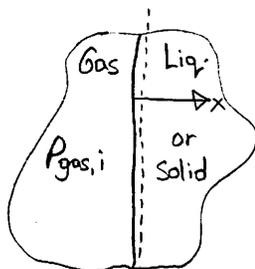


$$-CD_{AB} \frac{\partial x_A}{\partial x} \Big|_{x=0} = J_{A,s} \Big|_{x=0}$$

So our concentration may look like:



To deal with this, we can use Henry's Law to solve for mole fraction just inside:



$$\boxed{P_{\text{gas},i} = X_{i,u} \cdot H} ; H = \text{Henry's constant}$$

~ Just inside the Liquid or Solid.

H (1 atm (10^5 Pa)) \Rightarrow in Water

	290K	300K	320K	330K	340K
O_2	38000	45000	52000	57000	65000
CO_2	1280	1710	2170	3220	4500
Air	62000	74000	92000	99000	1224

If we treat the gas and solid as a solution, we can say:

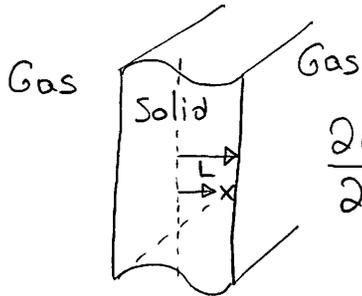
$$C_{i,u} = S P_{\text{gas},i}$$

$P_{\text{gas},i}$ = partial pressure of gas i next to the interface

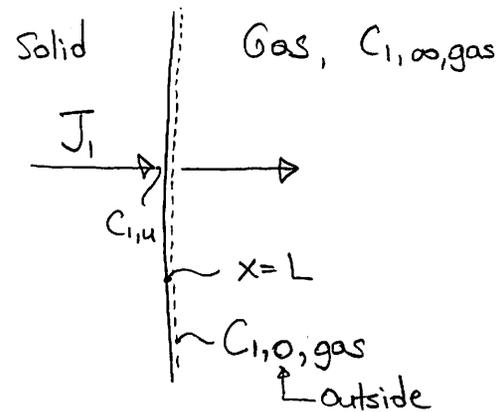
$C_{i,u}$ = kilomoles of gas i per cubic meter of solid $\left[\frac{\text{kmol}}{\text{m}^3}\right]$

S = solubility

So now if we look at a slab problem again:



$$\frac{\partial C_i}{\partial t} = D_{1,2} \nabla^2 C_i \Leftrightarrow$$



Our boundary condition becomes:

$$- D_{1,2} \left(\frac{\partial C_i}{\partial x} \right) \Big|_{x=L} = J_1 = h_m (C_{i,0,gas} - C_{i,\infty,gas}) \quad \textcircled{1}$$

Inside the solid

Outside, in the gas
 h_m = mass transfer coeff. $\left[\frac{\text{m}}{\text{s}}\right]$

Now we can solve our problem.

Multiplying $\textcircled{1}$ by S/S ($x_{i,u} = x_{i,gas} \cdot S$)

$$- D_{1,2} \left(\frac{\partial C_i}{\partial x} \right) = \frac{h_m}{S} [(C_{i,0} - C_{i,\infty})_{\text{gas}} \cdot S] \quad \textcircled{2}$$

Non-dimensionalizing equation $\textcircled{2}$

$$\bar{x} = \frac{x}{L}, \quad \phi = \frac{C_i - C_{i,u,\infty}}{C_{i,u,0} - C_{i,u,\infty}}$$

$$d\bar{x} = \frac{dx}{L}, \quad d\phi = \frac{dC_i}{C_{i,u,0} - C_{i,u,\infty}}$$

$C_i = C(x=L, t)$ inside solid

$C_{i,u,\infty} = C(t \rightarrow \infty)$ inside

$C_{i,u,0} = C(t \rightarrow 0)$ inside

$C_{i,0,gas} = C$ at interface outside in gas

$C_{i,\infty,gas} = C(x \rightarrow \infty)$ outside

$$-D_{1,2} \frac{1}{L} \frac{\partial \phi}{\partial \bar{x}} (C_{1,u,0} - C_{1,u,\infty}) = \frac{h_m}{S} [(C_{1,0} - C_{1,\infty})_{\text{gas}} \cdot S]$$

But note that $C_{1,0,\text{gas}} \cdot S = C_1$ and $C_{1,\infty,\text{gas}} \cdot S = C_{1,u,\infty}$

$$-D_{1,2} \frac{1}{L} \frac{\partial \phi}{\partial \bar{x}} (C_{1,u,0} - C_{1,u,\infty}) = \frac{h_m}{S} (C_1 - C_{1,u,\infty})$$

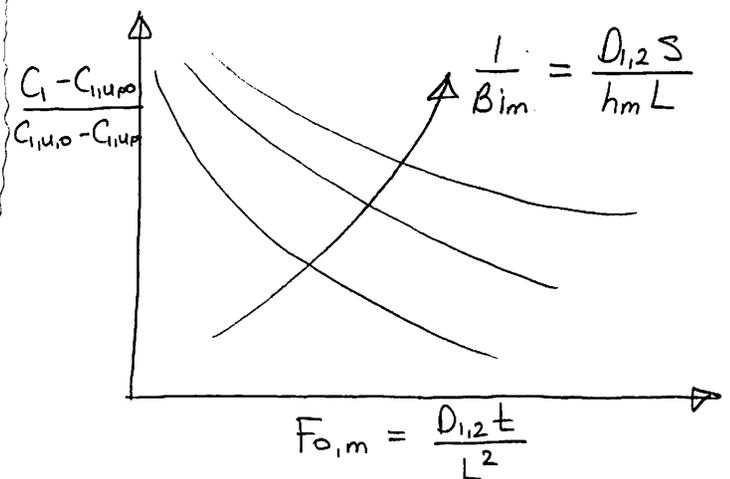
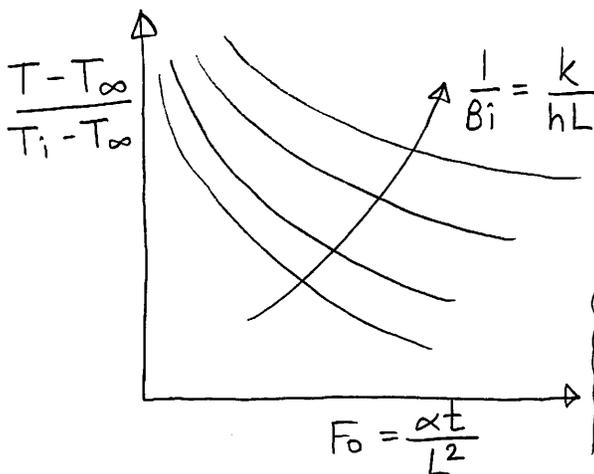
$$-D_{1,2} \frac{1}{L} \frac{\partial \phi}{\partial \bar{x}} = \frac{h_m}{S} \underbrace{\left(\frac{C_1 - C_{1,u,\infty}}{C_{1,u,0} - C_{1,u,\infty}} \right)}_{\phi}$$

$$-D_{1,2} \frac{1}{L} \frac{\partial \phi}{\partial \bar{x}} = \frac{h_m \phi}{S}$$

$$\boxed{-\frac{\partial \phi}{\partial \bar{x}} = \left(\frac{h_m L}{D_{1,2} S} \right) \phi} \Rightarrow \text{Similar to heat transfer solution. Just added factor } S.$$

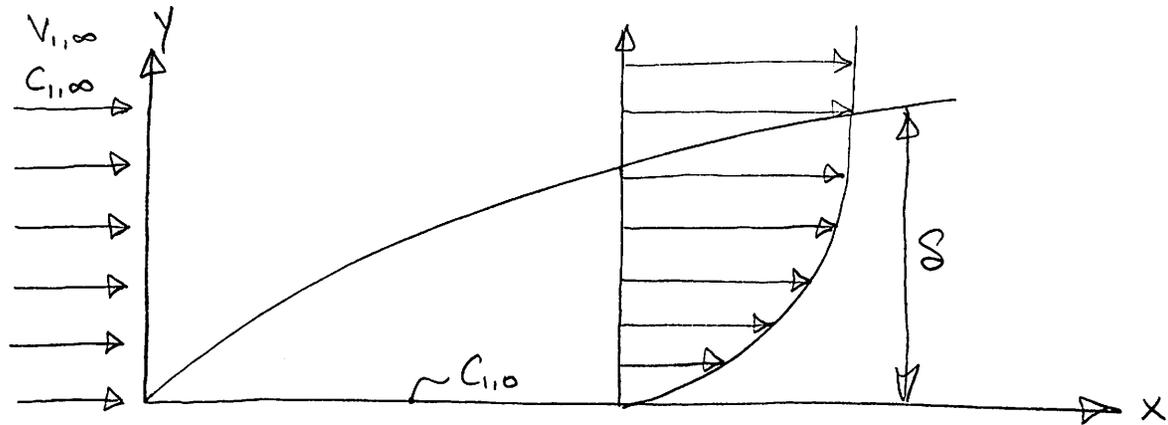
$Bi_m = \text{mass transfer Biot number}$
 $= \frac{\text{internal diffusion resistance}}{\text{external convection resistance}}$

If we look back at our heat transfer analogy

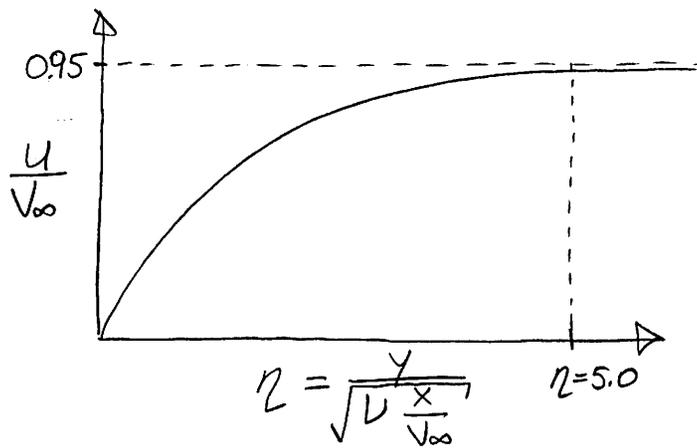


Convective Mass Transfer (External)

If we examine air blowing over a flat plate



We've already solved this problem:

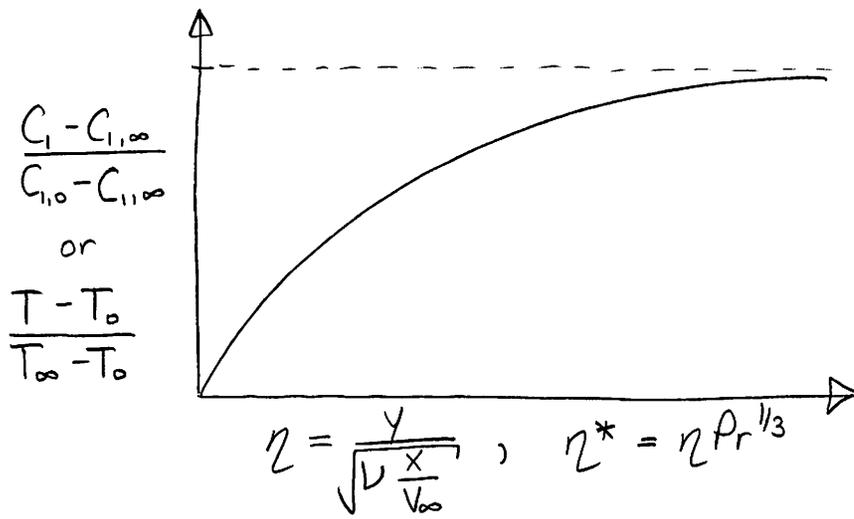


⇒ See page 84 of notes.

Now if we look at the mass transfer analogy (N-S eqn.)

$$u \frac{\partial C_1}{\partial x} + v \frac{\partial C_1}{\partial y} = D_{1,g} \frac{\partial^2 C_1}{\partial y^2}$$

Note, this is the exact same equation as in heat transfer, so we can make a really simple analogy and our mass transfer results will be the same



⇒ See page 89 of notes

For the heat transfer case: $Pr = \frac{\nu}{\alpha}$

For the mass transfer case: $Sc = \frac{\nu}{D_{1,2}}$

Schmidt number
 ν = kinematic viscosity
 $D_{1,2}$ = diffusivity (mass)

Our solutions become:

$$Nu = \frac{hx}{k} = 0.332 Re_x^{1/2} Pr^{1/3}$$

$$\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.664 Re_L^{1/2} Pr^{1/3}$$

For mass transfer:

$$Sh = \frac{h_m x}{D} = 0.332 Re_x^{1/2} Sc^{1/3} \quad \text{Sherwood number}$$

We can now develop a Colburn analogy for mass transfer to express the mass transfer coefficient in terms of friction factor, f : (pg. 90 of notes)

$$C_{f,x} = \frac{\tau}{\frac{1}{2} \rho V_\infty^2} = \frac{2a_2}{\sqrt{Re_x}}, \quad a_2 = 0.332$$

$$Nu_x = a_2 Re_x^{1/2} Pr^{1/3} \quad (\text{Flat plate})$$

$$\left(\frac{Nu_x}{Re_x Pr} \right) = \frac{1}{2} \frac{2a_2}{\sqrt{Re_x}} \frac{1}{Pr^{2/3}} = \frac{1}{2} C_{f,x} \frac{1}{Pr^{2/3}}$$

$$St = \frac{h}{\rho C_p V_\infty} \quad (\text{Stanton number}) \equiv \frac{\text{heat transferred to a fluid}}{\text{thermal capacity of the fluid}}$$

So our heat transfer analogy was:

$$St \cdot Pr^{2/3} = \frac{C_{f,x}}{2} \Rightarrow \text{See page 90 of notes for more.}$$

For mass transfer, we have:

$$\left(\frac{Sh_x}{Re_x Sc} \right) = \frac{1}{2} \frac{2a_2}{\sqrt{Re_x}} \frac{1}{Sc^{2/3}} = \frac{1}{2} C_{f,x} \frac{1}{Sc^{2/3}} = St_m$$

By analogy (you can prove it)

$$\boxed{St_m \cdot Sc^{2/3} = \frac{C_{f,x}}{2}} \quad \text{Mass transfer Colburn analogy}$$

$$\boxed{St_m = \frac{h_m}{V_\infty}} \quad \text{Mass transfer Stanton number}$$

Notice that we can combine the two analogies

$$St \cdot Pr^{2/3} = St_m \cdot Sc^{2/3}$$

$$\frac{h}{\rho C_p V_\infty} Pr^{2/3} = \frac{h_m}{V_\infty} Sc^{2/3}$$

$$\boxed{\frac{h}{h_m} = (\rho C_p) \left(\frac{\alpha}{D_{1,2}} \right)^{2/3} = \rho \cdot C_p \cdot Le^{2/3}}, \quad \text{where } \boxed{Le = \frac{\alpha}{D_{1,2}}}$$

\equiv Lewis number

Note this means we can measure mass transfer (h_m) and use it to back calculate heat transfer (h) \Rightarrow very powerful

Example Dry air at atmospheric pressure blows across a thermometer which is enclosed in a damp cover. This is a classical wet bulb measurement. The temperature reads 18.3°C . What is the dry air temperature, T_∞ ?

The heat needed to evaporate the water comes from the surrounding air (at steady state).

$$hA(T_\infty - T_w) = \dot{m}_w h_{fg} \quad (1) \Rightarrow \text{Heat transfer}$$

$$\dot{m}_w = h_m A (p_w - p_\infty) \quad (2) \Rightarrow \text{Mass transfer}$$

Combining (1) and (2)

$$hA(T_\infty - T_w) = h_m A (p_w - p_\infty) h_{fg} \quad (3)$$

Mass concentrations


Using our Colburn analogies

$$p_a C_p Le^{2/3} (T_\infty - T_w) = (p_w - p_\infty) h_{fg} \quad (3)$$

We know C_w corresponds to the saturation conditions at T_w
From steam tables:

$$p_{\text{sat}} = 2107 \text{ Pa at } T_{\text{sat}} = 18.3^\circ\text{C}$$

$$p_w = \frac{p_{\text{sat}}}{R_w T_w} = \frac{(2107)(18)}{(8315)(291.3)} = 0.01566 \text{ kg/m}^3$$

For the other properties:

$$p_\infty = 0 \text{ (dry air)}$$

$$p_a = \text{air density} = \frac{\rho}{RT} = \frac{101325}{(287)(291.3)} = 1.212 \text{ kg/m}^3$$

$$C_p = 1.004 \text{ kJ/kg}$$

$$\frac{\alpha}{D_{1,2}} = Le = \frac{Sc}{Pr} = 0.845$$

$$h_{fg} = 2.456 \text{ MJ/kg}$$

Back substituting into equation ③

$$T_\infty - T_w = \frac{(0.01566 - 0)(2.456 \times 10^6)}{(1.212)(1004)(0.845)^{2/3}} = 35.36^\circ\text{C}$$

$$\therefore \boxed{T_\infty = 53.69^\circ\text{C}} \Rightarrow \text{Dry bulb temperature}$$

Mass Transfer Summary

Heat Transfer	Mass Transfer
h [W/m ² ·K]	h_m [m/s]
Re [dimensionless]	Re
$Nu = \frac{hL}{k_f} \equiv \text{Nusselt}$	$Sh = \frac{h_m L}{D_{1,2}} \equiv \text{Sherwood}$
$Nu = f(Re, Pr)$	$Sh = f(Re, Sc)$
$Pr = \frac{\nu}{\alpha} \equiv \text{Prandtl}$	$Sc = \frac{\nu}{D_{1,2}} \equiv \text{Schmidt}$
$Ra = \frac{g\beta\Delta TL^3}{\nu\alpha} \equiv \text{Rayleigh}$	$Ra_m = \frac{g\Delta\rho L^3}{\mu D_{1,2}} \equiv \text{Rayleigh mass transf. (Natural conv.)}$
	$Le = \frac{\alpha}{D_{1,2}}$