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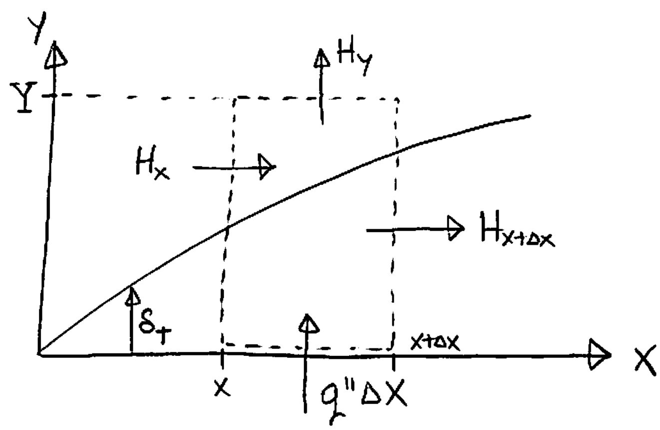
Energy Integral (Heat Transfer)

Similar to what we just did, we can develop an energy balance on our CV:

*Assuming viscous dissipation heating is negligible.



H_x = enthalpy flow in the x plane

Writing out our terms:

$$H_x = \left(\int_0^Y \rho c_p u T\,dy\right)\bigg|_x$$

$$H_{x+\Delta x} = \left(\int_0^Y \rho c_p u T\,dy\right)\bigg|_{x+\Delta x} = \overbrace{\left(\int_0^Y \rho c_p u T\,dy\right)\bigg|_x + \frac{\partial}{\partial x}\left(\int_0^Y \rho c_p u T\,dy\right)\Delta x + \dots}^{\text{Taylor Series Expans.}}$$

$$H_Y = (\rho v c_p T)\big|_Y \cdot \Delta X$$

$$q''\Delta x = -k\frac{\partial T}{\partial y}\bigg|_0 \cdot \Delta x$$

Substituting all of our terms into our energy equation

$$-H_{x+\Delta x} + H_x - H_Y + q''\Delta X = 0 \quad \text{or} \quad H_{x+\Delta x} - H_x + H_Y = q''\Delta X$$

$$\left[\cancel{\left(\int_0^Y \rho c_p u T\,dy\right)\bigg|_x} + \frac{\partial}{\partial x}\left(\int_0^Y \rho c_p u T\,dy\right)\bigg|_x \Delta x + \text{H.O.T}\right] - \cancel{\left(\int_0^Y \rho c_p u T\,dy\right)\bigg|_x}$$

$$+ (\rho v c_p T)\big|_Y \cdot \cancel{\Delta X} = -k\frac{\partial T}{\partial y}\bigg|_0 \cdot \cancel{\Delta X}$$

⇒ Divide through by ΔX and let ΔX → 0.

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From continuity (solved during momentum integral calculation)

$$v\big|_Y = -\frac{\partial}{\partial x}\int_0^Y u\,dy \quad \Rightarrow \text{Back substitute into our } M_y \text{ term:}$$

I assumed here ρ & c_p = constants

$$M_y = (\rho v c_p T)\big|_Y \cdot \Delta x = -\left(\rho c_p \frac{\partial}{\partial x}\int_0^Y u\,dy \cdot T\right)\Big|_Y \cdot \Delta x$$

$$M_y = -\left[\rho c_p \frac{\partial T}{\partial x}\Big|_Y \int_0^Y u\,dy + \rho c_p T\big|_Y \int_0^Y \frac{\partial u}{\partial x}\,dy\right]\Delta x$$

Back substituting (dropping the Δx since we already canceled it).

$$\frac{\partial}{\partial x}\left(\int_0^Y \rho c_p u T\,dy\right)\Big|_x - \rho c_p T\big|_Y \frac{\partial}{\partial x}\int_0^Y u\,dy - \rho c_p \frac{\partial T}{\partial x}\Big|_Y \int_0^Y u\,dy = -k\frac{\partial T}{\partial y}\Big|_0$$

Assuming ρc_p = constant, & knowing $\alpha = \frac{k}{\rho c_p}$

$$\boxed{\frac{\partial}{\partial x}\int_0^Y u(T_\infty - T)\,dy = \alpha\frac{\partial T}{\partial y}\Big|_0 - \frac{\partial T}{\partial x}\int_0^Y u\,dy}$$

→ Integral Boundary Layer Equation for Energy

Usually, we assume two things to simplify:

1) Integrate to $Y = \delta_T$ since for $y > \delta_T \Rightarrow$ integrals become 0.

2) The T_∞ = constant. For the above equation, it doesn't need to be however.

$$\boxed{\frac{\partial}{\partial x}\int_0^Y u(T_\infty - T)\,dy = \alpha\frac{\partial T}{\partial y}\Big|_0} \quad \Rightarrow T_\infty = \text{constant} \quad (1)$$

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So let's try the same approach we used for the momentum integral:

$$\theta(\eta_T) = \frac{T - T_0}{T_\infty - T_0} \quad ; \quad \eta_T = \frac{y}{\delta_T} \quad \Rightarrow \quad \begin{aligned} d\theta &= (T_\infty - T_0)\, dT \\ dy &= \delta_T\, d\eta_T \end{aligned}$$

We see right away that:

$$\frac{T - T_\infty}{T_0 - T_\infty} = 1 - \frac{T - T_0}{T_\infty - T_0} = 1 - \theta$$

Back substituting into 1 $\Rightarrow$ Also $\phi = \frac{u}{U_\infty}$

$$\frac{\partial}{\partial x}\int_0^1 U_\infty \phi (1-\theta)\cancel{(T_0 - T_\infty)}\,\delta_T\, d\eta_T = \frac{\alpha}{\delta_T}\frac{\partial \theta}{d\eta_T}\bigg|_0 \cdot \cancel{(T_\infty - T_0)}$$

$$\frac{\partial}{\partial x}\int_0^1 \delta_T \phi (1-\theta)\,\partial \eta_T = \frac{\alpha}{U_\infty \delta_T}\frac{\partial \theta}{\partial \eta_T}\bigg|_0 \qquad 2$$

Our B.C.'s are:

$\theta(\eta_T = 0) = 0$ (Wall temperature, T_0)
$\theta(\eta_T = 1) = 1$ (Free stream temperature, T_∞)
$\theta'(\eta_T = 1) = 0$ (No temp. gradient at b.l. edge)
$\theta''(\eta_T = 0) = 0$ (Linear Temp. profile at wall)

Using the identical clever math trick as before:

$$\theta(\eta^*) = \frac{\delta}{\delta_T}\phi \quad ; \quad \eta^* = \eta\, Pr^{1/3}$$

We can check the rationality of our assumption

$$\left[\phi = \frac{3}{2}\eta - \frac{1}{2}\eta^3 = \frac{3}{2}\cdot\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\cdot\frac{\delta}{\delta_T}$$

$$\theta = \frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y^3}{\delta^2\delta_T}\right)$$ $\Rightarrow$ Note here we can simplify

We've implicity assumed that $u \not\approx U_\infty$, i.e. $Pr \approx 1$ or $Pr \gg 1$
Because of this assumption, we can safely say that $\delta > \delta_T$
So comparing our two terms near the wall (where it's important)

$$\frac{\frac{3}{\cancel{2}}\left(\frac{\cancel{y}}{\cancel{\delta_T}}\right)}{\frac{1}{\cancel{2}}\left(\frac{y^{\cancel{3}}}{\delta^2\cancel{\delta_T}}\right)} = \frac{3\delta^2}{y^2} \gg 1 \Rightarrow \text{Since for energy solution } (\theta)\ \ 0 < y < \delta_T,\ y_{max} = \delta_T$$

$$= \frac{3\delta^2}{\delta_T^2} \gg 1 \quad \text{since } Pr \approx 1 \text{ or } Pr \gg 1$$

So this allows us to safely say that the second term in our θ solution is negligible compared to the first.

Note, this is also possible because our B.C.'s in non-dimensional momentum (ϕ) and energy (θ) are identical.

Now we have the following:

$$\theta(\eta^*) = \frac{3}{2}\frac{y}{\delta_T} \Rightarrow \text{Back substitute into } ② \quad \left(\phi = \frac{\delta_T}{\delta}\theta\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\delta_T^2}{\delta}\right)\int_0^1 \theta(1-\theta)\,d\eta_T = \frac{\alpha}{\nu\delta_T}\left(\frac{\nu}{U_\infty}\theta'|_0\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\delta_T^2}{\delta}\right) = \frac{1}{Pr\,\delta_T}\cdot\frac{\nu}{U_\infty}\underbrace{\left[\frac{\theta'|_0}{\int_0^1 \theta(1-\theta)\,d\eta_T}\right]}_{\beta}$$

$\Rightarrow$ We've defined this before!

Rearanging:

$$\delta_T\frac{\partial}{\partial x}\left(\frac{\delta_T^2}{\delta}\right) = \frac{1}{Pr}\underbrace{\left(\frac{\nu}{U_\infty}\beta\right)}_{=\delta\frac{\partial\delta}{\partial x}}$$

(Derived this before)

$$\delta_T \frac{\partial}{\partial x}\left(\frac{\delta_T^2}{\delta}\right) = \frac{\delta}{Pr}\frac{\partial \delta}{\partial x}\cdot\left(\frac{\delta^{1/2}}{\delta^{1/2}}\right)$$ ⇒ Integrate both sides

$$\int_{x_0}^{x} \frac{\delta_T}{\delta^{1/2}} \frac{\partial}{\cancel{\partial x}}\left(\frac{\delta_T^2}{\delta}\right) = \int_{x_0}^{x}\frac{1}{Pr}\,\delta^{1/2}\,\frac{\partial\delta}{\cancel{\partial x}}$$ ⇒ Note, my bounds of integration are x_0 to x. I will explain later!

$$\Rightarrow \underbrace{\int_{x_0}^{x}\frac{\delta_T}{\delta^{3/2}}\, d(\delta_T^2)} = \int_{x_0}^{x} \frac{1}{Pr}\,\delta^{1/2}\,d\delta$$

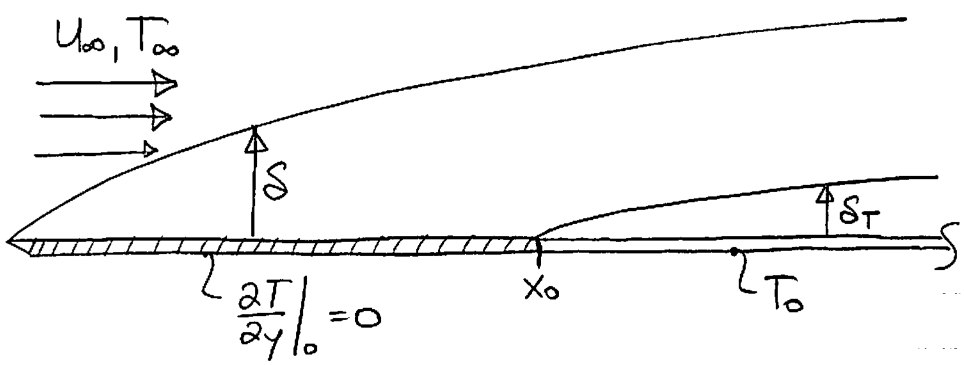
$$\int_{x_0}^{x}\frac{\delta_T}{\delta^{3/2}}\,2\delta_T\, d\delta_T = \int_{x_0}^{x}\frac{2\delta_T^2}{\delta^{3/2}}\,d\delta_T = \frac{2}{3}\frac{\delta_T^3}{\delta^{3/2}}\Big|_{x_0}^{x}$$

Our second integral is more trivial: $\int_{x_0}^{x} \delta^{1/2}\,d\delta = \frac{2}{3}\delta^{3/2}\Big|_{x_0}^{x}$

$$\cancel{\frac{2}{3}}\,\frac{\delta_T^3}{\delta^{3/2}}\Big|_{x_0}^{x} = \cancel{\frac{2}{3}}\,\delta^{3/2}\Big|_{x_0}^{x}$$

$$\left(\frac{\delta_T^2}{\delta}\right)^{3/2} = \frac{1}{Pr}\,\delta^{3/2}\left[1-\left(\frac{\delta_0}{\delta}\right)^{3/2}\right]$$

So the reason we integrated from x_0 instead of 0 is because it allows us to get a much more general and powerful solution. The physical picture is the following:



So for this situation, at x_0, $\delta_T = 0$ and $\delta_0 = C\sqrt{x}$.

Note: $\frac{\delta}{x} \sim \frac{C}{Re_x^{1/2}} \Rightarrow \delta \sim \sqrt{x}$

$$\left(\frac{\delta_T}{\delta}\right)^3 = \frac{1}{Pr}\left(1-\left(\frac{X_o}{X}\right)^{3/4}\right)$$ ⇒ I subbed in $\delta_o = C\sqrt{x_o}$ & $\delta = C\sqrt{x}$

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}\cdot\left[1-\left(\frac{X_o}{X}\right)^{3/4}\right]^{-1/3}}$$ ⇒ Note, we didn't assume a θ profile and got this! Only assumed $\delta \sim \sqrt{x}$ which is obvious!

Note, if we check our result with $X_o = 0$, we obtain the same familiar result as before:

$$\boxed{\frac{\delta}{\delta_T} = Pr^{1/3}}$$ ⇒ $X_o = 0$, $Pr \geq 1$.

So how do we calculate heat transfer?

$$q''\Big|_{y=0} = -k\frac{\partial T}{\partial y}\Big|_0 = -\frac{k}{\delta_T}\cdot\frac{\partial\theta}{\partial\eta_T}\cdot\underbrace{(T_\infty - T_o)}_{-\Delta T}$$ ⇒ Remember from before: $\partial\theta = (T_\infty - T_o)\partial T$, $\partial y = \delta_T\partial\eta_T$

For $x > x_o$

$$q''\Big|_{y=0} = \frac{k\Delta T}{\delta_T}\theta'\Big|_0 = \frac{k\Delta T}{x}\frac{\theta'|_0}{\left(\frac{\delta_T}{\delta}\right)\left(\frac{\delta}{x}\right)}$$

We know that:

$$\frac{\delta_T}{\delta} = Pr^{-1/3}\left[1-\left(\frac{X_o}{X}\right)^{3/4}\right]^{1/3} \quad \text{and} \quad \frac{\delta}{x} = \frac{\sqrt{2\beta}}{Re_x^{1/2}}$$ ⇒ Back substitute

$$q''\Big|_{y=0} = \left(\frac{k\Delta T}{x}\right)\frac{Pr^{1/3}Re_x^{1/2}\theta'|_0}{\left[1-\left(\frac{X_o}{X}\right)^{3/4}\right]^{1/3}\sqrt{2\beta}}$$ ⇒ Expand this by substituting β

$$\boxed{q''\Big|_{y=0} = \left(\frac{k\Delta T}{x}\right)\left[\frac{\theta'|_0}{2}\int_0^1 \theta(1-\theta)\,d\eta_T\right]^{1/2}\cdot\frac{Re_x^{1/2}Pr^{1/3}}{\left[1-\left(\frac{X_o}{X}\right)^{3/4}\right]^{1/3}}}$$

To solve, we need to assume a temperature profile $\theta(\eta_T)$.

However there is something nice we can use about this solution and our previous similarity solutions:

$$\left[\frac{\theta'|_0}{2}\int_0^1 \theta(1-\theta)\,d\eta_T\right]^{1/2} \neq f(x_0)$$

We know that in our limit of $x_0 = 0$, this term must be 0.332 (similarity solution). Hence:

$$\boxed{\left[\frac{\theta'|_0}{2}\int_0^1 \theta(1-\theta)\,d\eta_T\right]^{1/2} = 0.332}$$

$\Rightarrow$ Try it for a few profiles.

Let's see how good we are. We already assumed before:

$$\theta = \frac{3}{2}\eta_T - \frac{1}{2}\eta_T^3$$

$$\theta'|_0 = \frac{3}{2}$$

$$\int_0^1 \theta(1-\theta)\,d\eta_T = \int_0^1\left(\frac{3}{2}\eta_T - \frac{1}{2}\eta_T^3\right)\left(1 - \frac{3}{2}\eta_T + \frac{1}{2}\eta_T^3\right)d\eta_T$$

$$= \int_0^1\left(\frac{3}{2}\eta_T - \frac{9}{4}\eta_T^2 - \frac{1}{3}\eta_T^3 + \frac{3}{2}\eta_T^4 - \frac{1}{4}\eta_T^6\right)d\eta_T$$

$$= \left[\cancel{\frac{3}{4}} - \cancel{\frac{9}{12}} + \frac{3}{10} - \frac{1}{8} - \frac{1}{28}\right] = 0.13928$$

$$\left[\frac{\theta'|_0}{2}\int_0^1 \theta(1-\theta)\,d\eta_T\right]^{1/2} = \left[\frac{3}{4}\left(\frac{3}{10} - \frac{1}{8} - \frac{1}{28}\right)\right]^{1/2} = 0.3232$$

So with $\theta = \frac{3}{2}\eta_T - \frac{1}{2}\eta_T^3 \Rightarrow$ Our constant is 0.3232!

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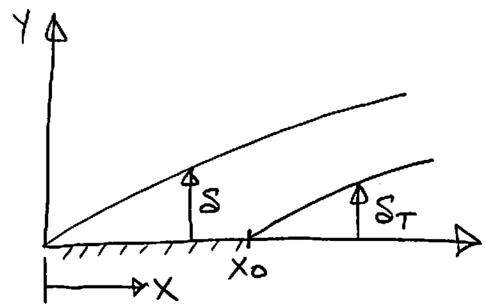
So finally, our solution becomes:

$$\boxed{q''\big|_{y=0} = 0.332\,\frac{k\Delta T}{x}\cdot\frac{Re_x^{1/2}\,Pr^{1/3}}{\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}} \Rightarrow$$



$$h = \frac{q''\big|_{y=0}}{\Delta T} = \frac{0.332k}{x}\ldots$$

$$\boxed{Nu_x = 0.332\,\frac{Re_x^{1/2}\,Pr^{1/2}}{\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}}}$$

$\Rightarrow$ Laminar

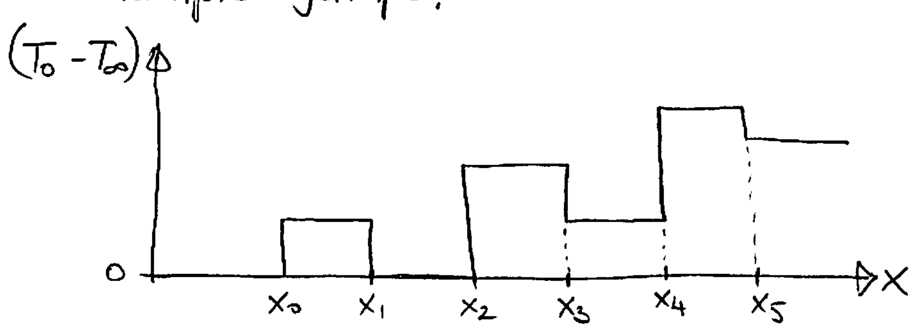
$\Rightarrow$ From $Nu_x = \frac{hx}{k} = \frac{q''|_0}{\Delta T}\cdot\frac{x}{k}$

$\Rightarrow U_\infty, \Delta T = \text{const.}$

Our result kind of makes sense, however be cariful. It is valid for $x > x_0$. For $x < x_0$, weve defined the plate to be adiabatic, so our solution gives us nor-physical answers.

Arbitrary Varying Temperature Difference

What if instead of just one simple temperature jump like above, we had an arbitrarily varying wall temperature with multiple jumps?



Looking at our energy equation again reveals the solution:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial y^2} \Rightarrow \text{Linear Differential Equation}$$

We can use <u>Duhamel's Theorem</u> to say that superposition of any number of stepwise variations is a valid solution.

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What Duhamel's Theorem implies is the following:

$$T = \sum_1^n T_i \quad \Rightarrow \quad T_i = \text{a particular solution to the energy equation for specific boundary cond.}$$

↳ Try substituting this in, you will see that it works out.

At the surface, we know:

$$q''\Big|_0 = -k \left.\frac{\partial T}{\partial y}\right|_0 = -k \sum_1^n \left.\frac{\partial T_i}{\partial y}\right|_0$$

Here, we can invoke that each particular solution corresponds to a temperature jump ΔT_i. For each particular solution, we can say:

$$-k \left.\frac{\partial T_i}{\partial y}\right|_0 = h_i \Delta T_i$$

Hence, for the full solution, we can say:

$$q'' = \sum_1^n h_i \Delta T_i$$

Translating this to our previous solution, we say:

$$\boxed{q'' = \sum_1^n h(x, x_{o,i}) \Delta T_i} \quad (1)$$

h = heat transfer coefficient which describes heat flow at location x when only one step ΔT_i in the wall temperature occurs at location $x_{o,i}$.

Note, this method of solution (superposition) can also be applied to other linear ODEs & PDEs, (i.e, wave equation).

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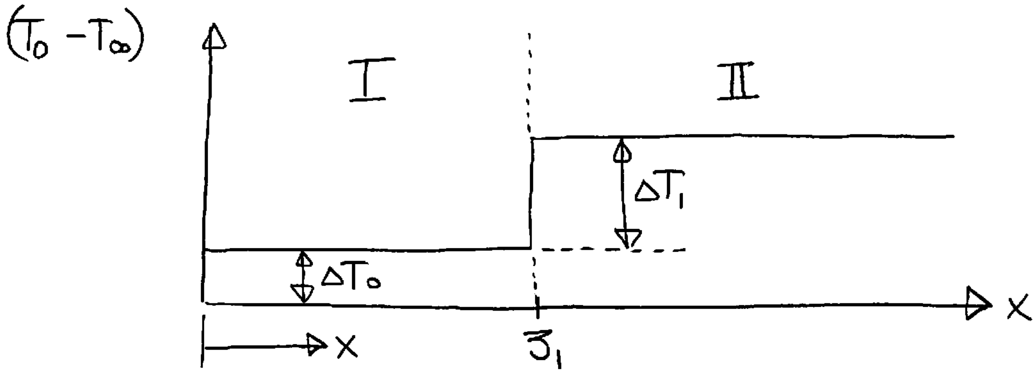
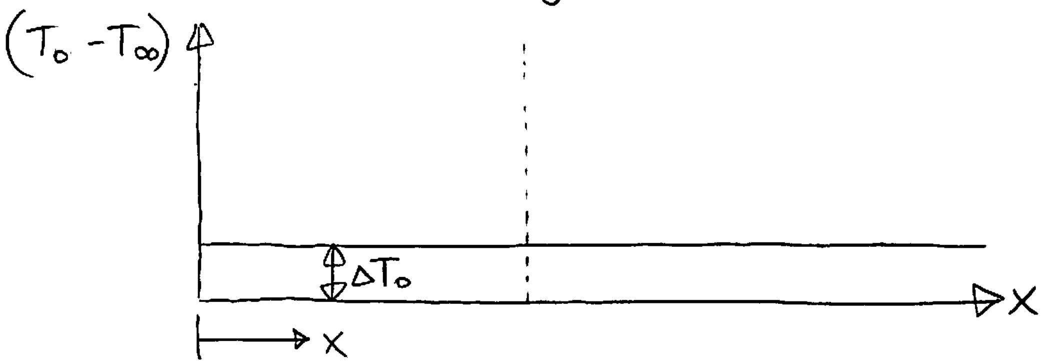


$*\ \zeta = x_0$ ↑ change of notation from here.

⇕ Equivalent to



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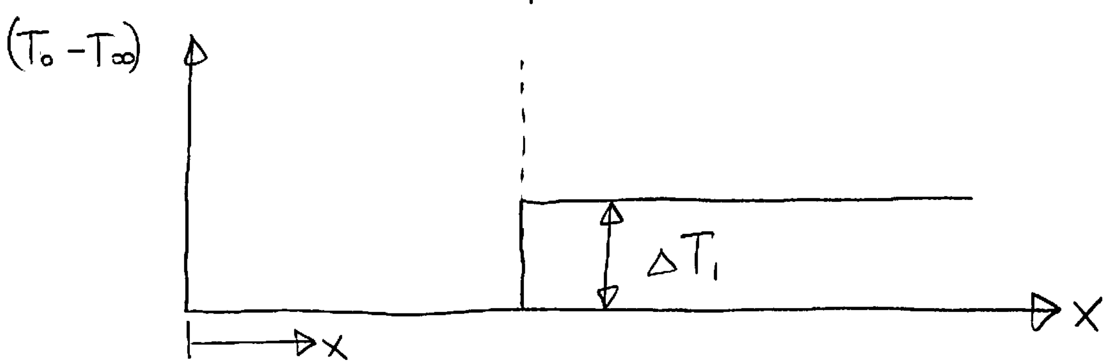


In general, we can say:

$$(T - T_\infty) = (T_0 - T_\infty) f(x, y, \zeta)$$

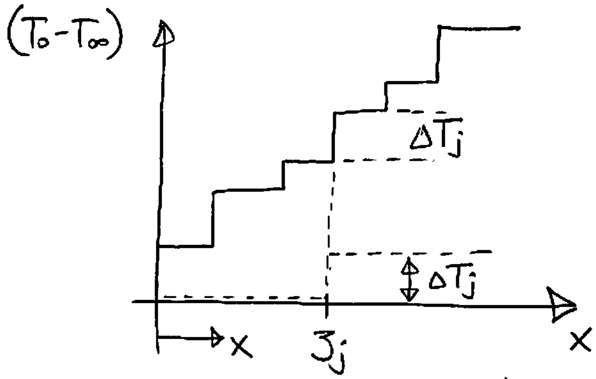
$$\boxed{(T - T_\infty) = \sum_{j=0}^{n} \Delta T_j \cdot f(x, y, \zeta_j)} \quad ; \quad \zeta_n < x < \zeta_{n+1}$$

Similarly: (from equation 1 on the previous page)

$$\boxed{q''\big|_0 = 0.332 \left(\frac{k}{x}\right) Re_x^{1/2} Pr^{1/3} \sum_{j=0}^{n} \frac{\Delta T_j}{\left[1 - \left(\frac{\zeta_j}{x}\right)^{3/4}\right]^{1/3}}} \quad ; \quad \zeta_n < x < \zeta_{n+1} \qquad (2)$$

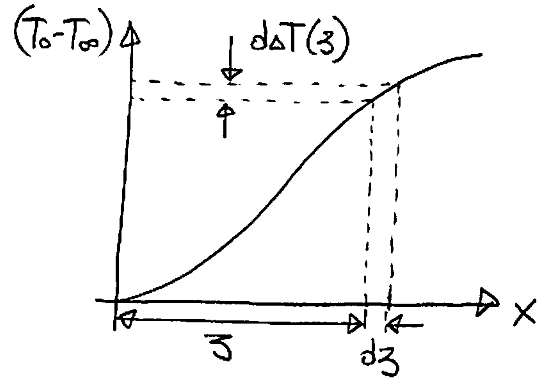
↳ U_∞ = constant; Laminar; ΔT_j = constant

This is valid for a stepwise variation in wall temperature. What if the variation is continuous?



Stepwise (Eq. ②)

vs.



Continuous

For the continuous case, we can change ① to:

$$q''\Big|_0 = \int_0^x h(x,\xi)\, d\Delta T(\xi)$$

To represent this integral in terms of our independent variable, ξ, we use:

$$d\Delta T(\xi) = \frac{d\Delta T(\xi)}{d\xi} \cdot d\xi$$

$$q''\Big|_0 = \int_0^x h(x,\xi)\, \frac{d\Delta T(\xi)}{d\xi}\, d\xi$$

Or in our notation: $\dfrac{d\Delta T(\xi)}{d\xi} = \dfrac{d(T_0 - T_\infty)}{d\xi}$

$$\boxed{q''\Big|_0 = 0.332\left(\frac{k}{x}\right) Re_x^{1/2}\, Pr^{1/3} \int_0^x \frac{\dfrac{d(T_0 - T_\infty)}{d\xi}}{\left[1 - \left(\dfrac{\xi}{x}\right)^{3/4}\right]^{1/3}} \cdot d\xi}$$

↳ Laminar; U_∞ = constant ; Continuous temperature difference variation.