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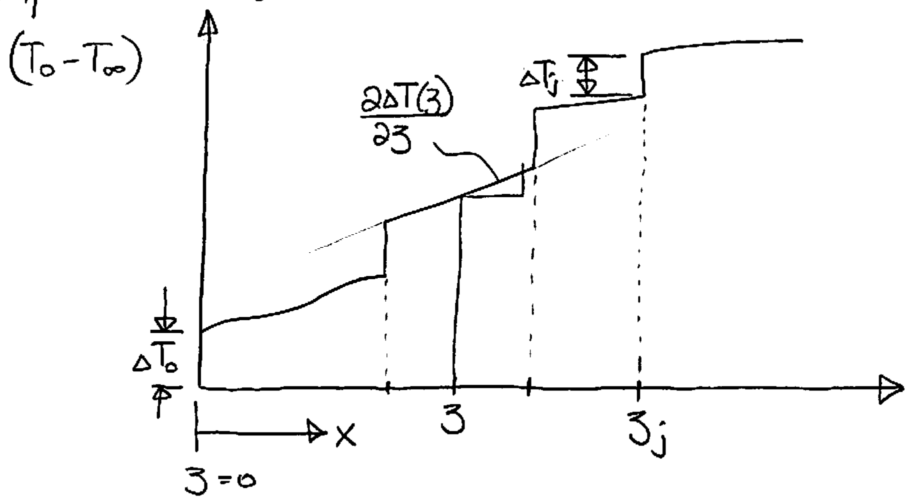
For a combination of temperature jumps and continuous temp. increase:



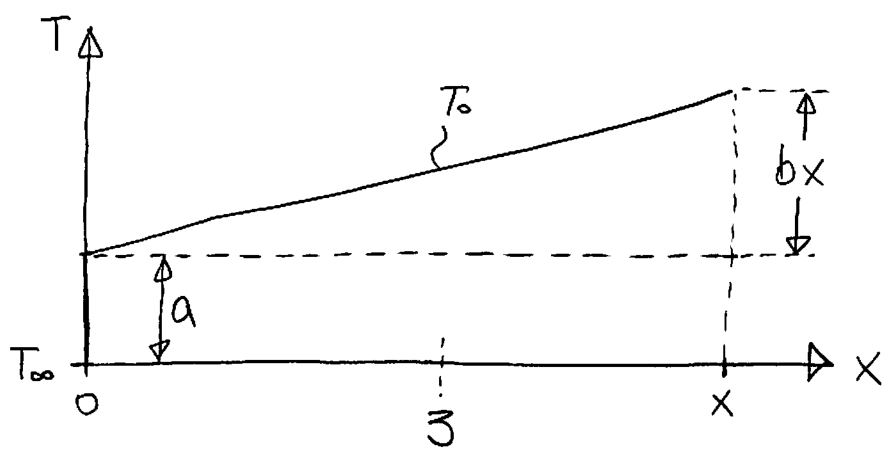
We use the following to solve:

$$\boxed{q''|_o = \int_0^x h(x,\xi)\frac{\partial \Delta T(\xi)}{\partial \xi}\, d\xi + \sum_0^n h(x,\xi_j)\Delta T_j}$$

↳ U_∞ = constant; Laminar; Continuous & Discrete ΔT variation

Reminder: $$\boxed{h(x,\xi) = \frac{0.332\, k}{x} Pr^{1/3} Re_x^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3}}$$

<u>Example</u> Plate with a step and linear surface-temp. variation:



$$T_o = T_\infty + a + bx$$
$$(T_o - T_\infty) = a + bx$$

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$$\frac{\partial \Delta T}{\partial x} = b = \frac{\partial \Delta T(\xi)}{\partial \xi}$$

Since we have a step increase (at leading edge) and a linear increase, we must use a hybrid approach:

$$q''\big|_o = \int_0^x h(x,\xi)\frac{\partial \Delta T(\xi)}{\partial \xi}d\xi + \sum_0^n h(x,\xi_j)\Delta T_j$$

For our case:

$$h(x,\xi) = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3} \quad ; \quad \frac{\partial \Delta T(\xi)}{\partial \xi} = b$$

$$h(x,\xi_j) = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2}\left[1-\left(\frac{\xi_o}{x}\right)^{3/4}\right]^{-1/3} \Rightarrow \xi_o = 0$$

so our solution becomes: $\Delta T_j = a$

$$h(x,\xi_o) = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2}$$

Putting everything togeter, we obtain:

$$q''\big|_o = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2}\left\{\int_0^x\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3} \cdot b\, d\xi + a\right\}$$

To solve these problems, we can use a powerful trick:

$$\boxed{\int_0^1 Z^{m-1}(1-Z)^{n-1}dZ = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}}$$

$\Rightarrow \Gamma$ = gamma fund.

$\Rightarrow$ Your new best friend

For our case, we first need to get our integral in this form:
We will use a change of variables:

$$s = \frac{\xi}{x} \Rightarrow \int_0^x\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3}d\xi = \int_0^1 x\left[1-s^{3/4}\right]^{-1/3}ds$$

$$ds \cdot x = d\xi \Rightarrow$$

Now we can do one more change of variables:

$$\left.\begin{aligned} \lambda &= 1 - s^{3/4} \Rightarrow \left(1-s^{3/4}\right)^{1/3} = \lambda^{1/3} \\ s &= (1-\lambda)^{4/3} \Rightarrow ds = -\tfrac{4}{3}(1-\lambda)^{1/3}\, d\lambda \end{aligned}\right\} \text{Substitute back in}$$

$$\int_0^1 \frac{x\,ds}{\left(1-s^{3/4}\right)^{1/3}} = \int_1^0 \frac{x\left(-\frac{4}{3}(1-\lambda)^{1/3}\right)d\lambda}{\lambda^{1/3}} = A$$

But since $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$A = \int_0^1 \left(\frac{4}{3}x\right)\frac{(1-\lambda)^{1/3}}{\lambda^{1/3}}\,d\lambda \Rightarrow \text{Looks a lot like what we need}$$

$$= \left(\frac{4}{3}x\right)\int_0^1 \lambda^{-1/3}(1-\lambda)^{1/3}\,d\lambda = \left(\frac{4}{3}x\right)\int_0^1 \lambda^{\frac{2}{3}-1}(1-\lambda)^{4/3-1}\,d\lambda$$

So $m = \frac{2}{3}$, $n = \frac{4}{3}$ $\Rightarrow$ Using a Gamma function table, or Wolfram alpha (online):

$$\Gamma\left(\frac{2}{3}\right) = 1.4, \quad \Gamma\left(\frac{4}{3}\right) = 0.89, \quad \Gamma\left(\frac{2}{3}+\frac{4}{3}\right) = 1.0$$

$$\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)} = 1.246$$

So our integral term becomes:

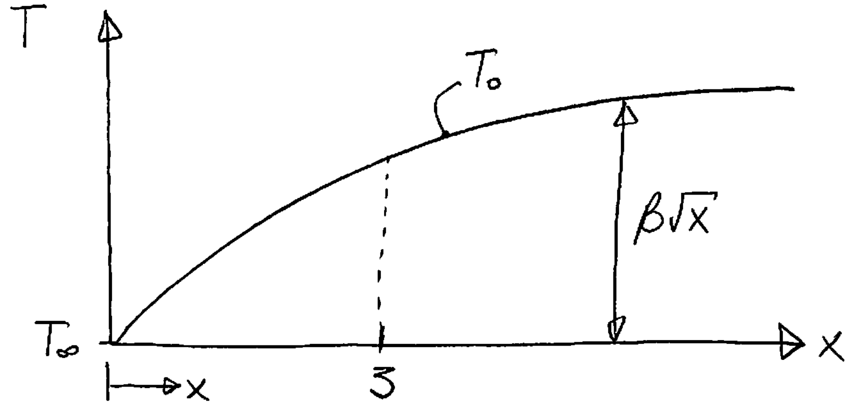
$$\int_0^x \left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3} d\xi = \left(\frac{4}{3}x\right)(1.246) = 1.661x$$

Back substituting:

$$\boxed{q''\big|_0 = 0.332\left(\frac{k}{x}\right) Pr^{1/3}\, Re_x^{1/2}\left(1.661\,bx + a\right)}$$

$\Rightarrow$ Result makes sense, If $b = 0$, we obtain our old solution.

Example #2 Continuous temp. change at the wall from leading edge with $(T_o - T_\infty) = \beta\sqrt{x}$



$$T_o = T_\infty + \beta\sqrt{x}$$

$$\frac{d\Delta T(\xi)}{dx} = \frac{1}{2}\beta\frac{1}{\sqrt{x}} \Rightarrow \frac{\partial\Delta T(\xi)}{\partial\xi} = \frac{\beta}{2\sqrt{\xi}} \quad \text{(since } x=\xi \text{ in our domain)}$$

$$h(x,\xi) = \frac{0.332k}{x} Pr^{1/3} Re_x^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{-1/3}$$

$$q''|_o = \int_0^x h(x,\xi)\frac{\partial\Delta T(\xi)}{\partial\xi}\,d\xi \Rightarrow \text{Back substitute the above expressions}$$

$$q''|_o = \frac{0.332k}{2x} Pr^{1/3} Re_x^{1/2}\int_0^x \frac{\beta\,d\xi}{\xi^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{1/3}}$$

We can use the exact same change of variable here:

$$s = \frac{\xi}{x} \Rightarrow \text{since } x=\xi \text{ in our domain, our integral bounds go from } [0,x] \text{ to } [0,1]$$

$$A = \int_0^x \frac{\beta\,d\xi}{\xi^{1/2}\left[1-\left(\frac{\xi}{x}\right)^{3/4}\right]^{1/3}} = \int_0^1 \frac{\beta(x\,ds)}{\xi^{1/2}(1-s^{3/4})^{1/3}} \Rightarrow \text{Note } \frac{x}{\xi^{1/2}} = \underbrace{\left(\frac{x}{\xi}\right)^{1/2}}_{s^{-1}}\cdot x^{1/2}$$

Doing another change of variable now: (just like before)

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$$\left.\begin{aligned} \lambda &= \left(1-s^{3/4}\right) \\ s &= (1-\lambda)^{4/3} \Rightarrow ds = -\frac{4}{3}(1-\lambda)^{1/3}\,d\lambda \\ s^{1/2} &= (1-\lambda)^{2/3} \end{aligned}\right\}$$ Back substitute into our integral

$$A = \int_0^1 \frac{\beta x^{1/2}\,ds}{s^{1/2}\left(1-s^{3/4}\right)^{1/3}} = \int_1^0 \frac{\left(\beta x^{1/2}\right)\left(-\frac{4}{3}(1-\lambda)^{1/3}\right)d\lambda}{(1-\lambda)^{2/3}\,\lambda^{1/3}}$$

$$= \frac{4}{3}\beta x^{1/2}\int_0^1 \frac{(1-\lambda)^{1/3}}{(1-\lambda)^{2/3}\lambda^{1/3}}\,d\lambda = \frac{4}{3}\beta x^{1/2}\int_0^1 (1-\lambda)^{-1/3}\lambda^{-1/3}\,d\lambda$$

This now looks a lot like our Gamma integral!

$$A = \frac{4}{3}\beta x^{1/2}\int_0^1 \lambda^{\left(\frac{2}{3}-1\right)}(1-\lambda)^{\left(\frac{2}{3}-1\right)}\,d\lambda \Rightarrow m = \frac{2}{3},\ n = \frac{2}{3}$$

$$\Gamma\left(\tfrac{2}{3}\right) = 1.355, \quad \Gamma\left(\tfrac{4}{3}\right) = 0.89$$

$$A = \frac{4}{3}\beta x^{1/2}\cdot\left(\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)}\right) = \frac{4}{3}\beta x^{1/2}(2.06)$$

$$q''|_o = 0.332\left(\frac{k}{x}\right)Pr^{1/3}Re_x^{1/2}\,\beta x^{1/2}\left(\frac{1}{2}\cdot\frac{4}{3}\cdot 2.06\right)$$

But note that $\beta x^{1/2} = \Delta T(3)$ or $\Delta T(x)$, so

$$\boxed{q''|_o = 0.455\left(\frac{k}{x}\right)Pr^{1/3}Re_x^{1/2}\cdot\Delta T}$$

$$\boxed{h_x = \frac{q''|_o}{\Delta T(x)} = 0.455\left(\frac{k}{x}\right)Pr^{1/3}Re_x^{1/2}}$$

$$\boxed{Nu_x = \frac{h_x x}{k} = 0.455\,Pr^{1/3}Re_x^{1/2}}$$ $\Rightarrow$ Laminar, U_∞ = constant, $q''|_o$ = const.

Note however for our case, $\Delta T = \beta\sqrt{x} \Rightarrow$ Let's sub it in and simplify.

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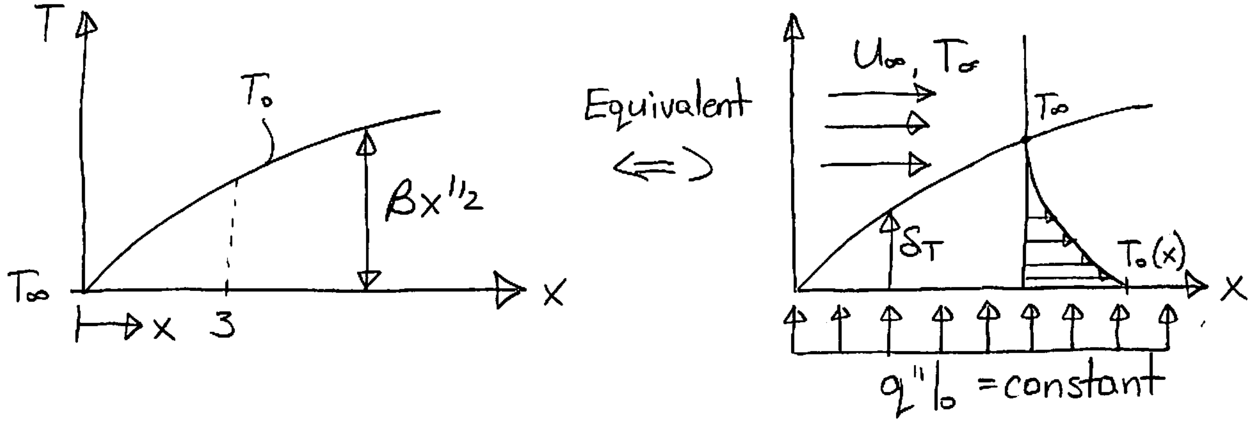
$$q''\big|_0 = 0.455\left(\frac{k}{x}\right) Pr^{1/3}\, Re_x^{1/2}\, \Delta T$$

$$= 0.455\left(\frac{k}{\cancel{x}}\right) Pr^{1/3}\left(\frac{U_\infty \cancel{x}}{\nu}\right)^{1/2} \cdot \beta\, \cancel{x^{1/2}}$$

$$\boxed{q''\big|_0 = 0.455\, k\, Pr^{1/3}\, \beta \left(\frac{U_\infty}{\nu}\right)^{1/2}} \neq f(x)\,!$$

So we have just solved the constant heat flux boundary condition case by accident! This implies:



So now you know how the wall temperature varies for a perscribed uniform heat flux b.c. ⇒ it's $T_0 \sim \sqrt{x}$

So what would $\bar{h}$ be?

$$\bar{h} = \frac{q''|_0}{\overline{\Delta T}} \Rightarrow \text{Look on page 60 of notes} \Rightarrow \bar{h} = \frac{q''|_0}{\frac{1}{L}\int_0^L \Delta T\, dx}$$

$$\frac{1}{L}\int_0^L \beta\sqrt{x}\, dx = \frac{1}{L}\frac{2}{3}\beta\left[x^{3/2}\right]\Big|_0^L = \frac{1}{L}\frac{2}{3}\beta L^{3/2} = \frac{2}{3}\beta L^{1/2}$$

$$\bar{h} = \frac{q''|_0}{\frac{2}{3}\beta L^{1/2}} = \frac{0.455\, k\, Pr^{1/3}\cancel{\beta}\left(\frac{U_\infty}{\nu}\right)^{1/2}}{\frac{2}{3}\cancel{\beta} L^{1/2}} = 0.6825\left(\frac{k}{L}\right) Pr^{1/3}\, Re_L^{1/2}$$

$$\boxed{\overline{Nu_L} = \frac{\bar{h}L}{k} = 0.6825\, Pr^{1/3}\, Re_L^{1/2}}$$

↳ Laminar, U_∞ = constant, $q''|_0$ = constant

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Viscous Energy Dissipation

So far, we have assumed that $Ec \ll 1 \left(= \frac{U_\infty^2}{C_p (T_o - T_\infty)} \right)$ ⇒ pg 61
or that heating of the fluid through viscous dissipation is negligible compared to heating from the wall.

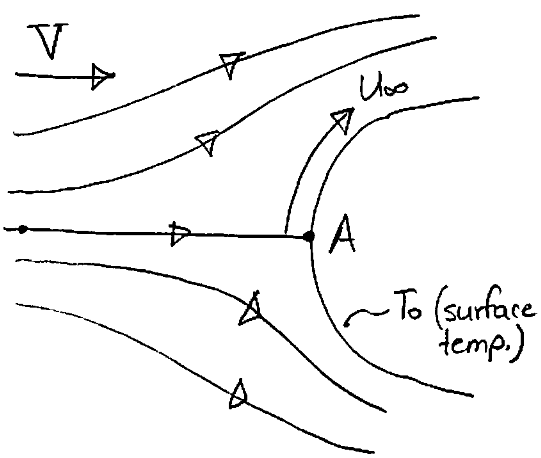
For convection at high velocities, we need to consider 2 things:

1) Conversion of mechanical energy into thermal energy, resulting in temperature variation in the fluid.

2) Variation of fluid properties due to temperature variation.

Let's look at consideration #1 first.

For a <u>reversible process</u> such as stagnation points in potential flow, we already know how to handle it:



From before:

$$\frac{1}{2}\rho V^2 + P = \text{constant}$$

$$P_A = P_\infty + \frac{1}{2}\rho V^2 \quad \text{(via Bernouli along a stream line)}$$

But for a gas with isentropic behavior $PV = nRT$, so we can say:

$$\frac{1}{2}\cancel{\rho} V^2 + \cancel{\rho} C_p T = \text{constant} \quad \text{(same streamline)}$$

$$\boxed{T_A = T_\infty + \frac{V^2}{2C}}$$ ⇒ T_A is the gas temp. at A.

So to solve high velocity stagnation point problems (reversible) it's easy:

$$\boxed{q''|_A = h(T_o - T_A)}$$

Note, we didn't look at this before since we usually dealt with low speed flows.

Example| Space shuttle re-entry. Shuttle re-enters at up to Ma = 25. By the time it reaches lower altitude, it decreases down to Ma ≈ 5. What is T_A?

$$Ma = 5 \Rightarrow V = 5\,(\text{Speed of sound}) = 5\,(343\ m/s)$$

$$V = 1715\ m/s$$ ⇒ Note, here we are assuming compressibility is not important.

$$T_A = T_\infty + \frac{V^2}{2c_p} \quad ; \quad c_{p,Air} \cong 1000\ J/kg\cdot K$$

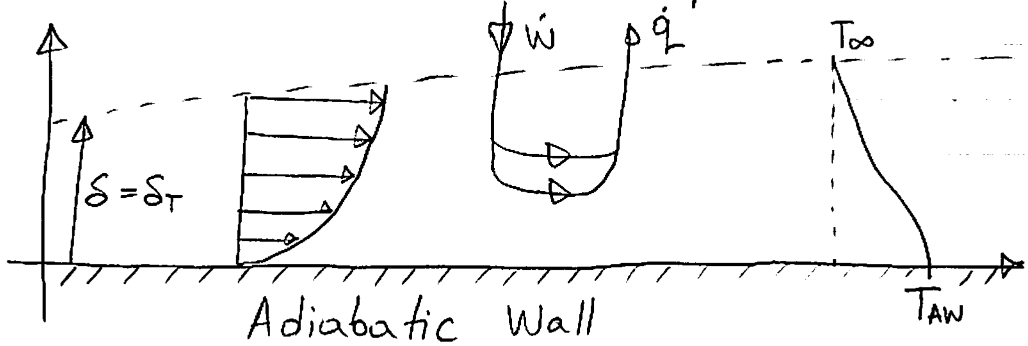
$$T_A = -40^\circ C + \frac{(1715\ m/s)^2}{2(1000)}$$

$$\therefore \boxed{T_A \approx 1430^\circ C}$$ ⇒ Pretty hot!

Note, this is true for any fast moving aircraft or projectile. Must be designed with thermal considerations in mind.

This solution is actually ok for stagnation points, but it's difficult to extend to flat plates.

For high velocity b.l. problems, velocity gradients within the b.l. cause mechanical to thermal energy conversion via viscous shear. Let's consider a simple case:



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Note, the shear-to-thermal and kinetic-to-thermal energy conversion processes are fundamentally different. The shear mechanism is an <u>irreversible process.</u> Usually, most real cases involve both.

For shear on a flat plate, all the energy generated must escape by molecular or eddy conduction. At steady state, the wall will reach an <u>adiabatic wall temperature</u>, T_{AW}.

This implies that Pr should be important in determining T_{AW}. We expect:

$$Pr \uparrow, \quad T_{AW} \uparrow \quad \text{due to high } \nu, \text{ low } \alpha.$$

$$Pr \downarrow, \quad T_{AW} \downarrow \quad \text{due to low } \nu, \text{ high } \alpha.$$

Let's do the math: Remember way back on pg. 14

$$\underbrace{\rho \frac{De}{Dt}}_{=0 \text{ Steady}} + e \underbrace{\left(\frac{D\rho}{Dt} + \rho \nabla \cdot V \right)}_{=0 \text{ (Incompressible)}} = -\nabla \cdot q'' + \underbrace{q'''}_{=0 \text{ (no heat gen.)}} - \underbrace{P \nabla \cdot V}_{=0 \text{ (Incompressible)}} + \mu \Phi$$

Assuming 2D, Incompressible, we simplified our energy eqn. to:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \Phi$$

$$\Phi = 2\left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \Rightarrow 2D$$

Let's compare our terms with scaling:

$$\left.\begin{aligned} u &\sim U_\infty \\ x &\sim L \\ v &\sim U_\infty \left(\frac{\delta}{L} \right) \\ y &\sim \delta \end{aligned}\right\} \quad \begin{aligned} &\frac{\partial u}{\partial x} \sim \frac{U_\infty}{L} \; ; \quad \frac{\partial u}{\partial y} \sim \frac{U_\infty}{\delta} \\ &\frac{\partial v}{\partial y} \sim U_\infty \frac{\cancel{\delta}}{\cancel{\delta} L} \; ; \quad \frac{\partial v}{\partial x} \sim U_\infty \frac{\delta}{L^2} \end{aligned}$$

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$$\left.\begin{array}{l} \dfrac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \sim \dfrac{\frac{\cancel{U_\infty}}{L}}{\frac{\cancel{U_\infty}}{\delta}} \sim \dfrac{\delta}{L} \ll 1 \Rightarrow \dfrac{\partial u}{\partial x} \ll \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} \sim \dfrac{\frac{U_\infty}{L}}{\frac{U_\infty}{\delta}} \sim \dfrac{\delta}{L} \ll 1 \Rightarrow \dfrac{\partial v}{\partial y} \ll \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y}} \sim \dfrac{\cancel{U_\infty}\frac{\delta}{L^2}}{\frac{\cancel{U_\infty}}{\delta}} \sim \left(\dfrac{\delta}{L}\right)^2 \ll 1 \Rightarrow \dfrac{\partial v}{\partial x} \ll \dfrac{\partial u}{\partial y} \end{array}\right\} \text{So } \frac{\partial u}{\partial y} \text{ dominates.}$$

$$\therefore \boxed{\Phi = \left(\frac{\partial u}{\partial y}\right)^2} \Rightarrow \text{2D flow,}$$

So our energy equation becomes:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \underbrace{\frac{\nu}{c_p}}_{\frac{\alpha Pr}{c_p}}\left(\frac{\partial u}{\partial y}\right)^2$$

Now we'll go back to our familiar tricks (similarity)

$$f = f(\eta) \quad , \quad T = T(\eta)$$

$$u = \frac{\partial \psi}{\partial y} \quad , \quad v = -\frac{\partial \psi}{\partial x} \quad , \quad \psi = \sqrt{\nu x U_\infty}\, f$$

$\Rightarrow$ see pg. 64 of notes for derrivation.

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}}$$

We defined $f' = \dfrac{u}{U_\infty} \Rightarrow u = U_\infty f'$

Let's do each term one by one:

$$u\frac{\partial T}{\partial x} = (U_\infty f')\frac{\partial T}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = (U_\infty f')T'\overbrace{\left(-\frac{\eta}{2x}\right)}^{\frac{\partial \eta}{\partial x}\text{ (pg. 54 of notes)}} = -\frac{U_\infty \eta}{2x} f'T'$$