\[
\sqrt{\frac{\partial T}{\partial y}} = \frac{U_0}{2x} \sqrt{\frac{x}{U_0}} \left( n f' - f \right) \cdot \frac{\partial T}{\partial y} \cdot \frac{\partial n}{\partial y} \sqrt{\frac{U_0}{c_p}}
\]

\[
\sqrt{\frac{\partial T}{\partial y}} = \frac{U_0}{2x} \left( n f'T' - fT' \right)
\]

Now for the right hand side:

\[
\alpha \frac{\partial^2 T}{\partial y^2} = \alpha \frac{\partial}{\partial y} \left( \frac{\partial T}{\partial y} \right) = \alpha \frac{\partial}{\partial y} \left( \frac{\partial T}{\partial n} \cdot \frac{\partial n}{\partial y} \right) = \alpha \frac{\partial}{\partial y} \left( \frac{\partial}{\partial n} \left( \frac{\partial T}{\partial n} \cdot \frac{\partial n}{\partial y} \right) \right)
\]

\[
\alpha \frac{\partial^2 T}{\partial y^2} = \alpha \left( \frac{\partial T}{\partial y} \right)^2 \cdot T'' = \frac{U_0 \alpha}{UX} \cdot \frac{T''}{T''}
\]

And the viscous term:

\[
\frac{U}{c_p} \left( \frac{\partial (2U)}{\partial y} \right)^2 = \frac{U}{c_p} \cdot \left( \frac{\partial (2U)}{\partial n} \cdot \frac{\partial n}{\partial y} \right)^2 = \frac{U}{c_p} \left( U_0 \alpha f'' \sqrt{\frac{U_0}{UX}} \right)^2
\]

\[
\frac{U}{c_p} \left( \frac{\partial (2U)}{\partial y} \right)^2 = \frac{U_0^3}{c_p UX} \left( f'' \right)^2
\]

Putting it all together:

\[
- \frac{U_0 n}{2x} f'T' + \frac{U_0 n}{2x} \frac{f'T'}{2x} - \frac{U_0 n}{2x} fT' = \frac{U_0 \alpha}{UX} \cdot \frac{T''}{T''} + \frac{U_0^2}{c_p UX} \left( f'' \right)^2
\]

\[
\frac{1}{Pr} T'' + \frac{U_0^2}{c_p} \left( f'' \right)^2 + \frac{1}{2} Pr fT' = 0
\]

\[
T'' + \frac{U_0^2 Pr}{c_p} \left( f'' \right)^2 + \frac{1}{2} Pr fT' = 0 \tag{1}
\]

So now we need to solve equation (1). Note, we can see that the equation is linear and that we have already solved for \( f \) before (Blasius).

Let \( \Theta(x) = \frac{T - T_{\infty}}{U_0^2/2c_p} \tag{2} \)
Substituting (2) into (1)

\[ \Theta'' + \frac{1}{2} \rho R f \Theta' + 2 \rho R (f'')^2 = 0 \]

\[ \Rightarrow \text{Second order, linear.} \quad (3) \]

Using Duhamel's theorem, we can solve this by superposition of a particular and homogeneous solution.

\[ \Theta(n) = \Theta_p(n) + \Theta_c(n) \]

For the particular solution, let's assume adiabatic wall conditions: \( \Rightarrow \) We can do this due to Duhamel's theorem (Linear)

\[ \Theta_p(n) \Rightarrow \frac{\partial \Theta_p}{\partial n} \bigg|_0 = 0 \quad , \quad \Theta_p(n \to \infty) = 0 \]

To solve we can use the integrating factor method or solve numerically using a shooting scheme:

\[ \Theta_p(n=0) = \Theta_{Aw}(n=0) = \int_0^\infty \frac{\int_0^n \exp \left( \int_0^x \frac{1}{2} \rho R f \, dx \right) 2 \rho R (f'')^2 \, dx}{\exp \left( \int_0^x \frac{1}{2} \rho R f \, dx \right)} \, dx \]

We can numerically integrate the above solution for various \( \text{Pr} \) and using \( f(n) \) solution from Blasius (Tabulated result)

\[ \Theta_p(0) = \begin{cases} \text{Pr}^{1/2} & ; & 0.5 \leq \text{Pr} \leq 47 \\ 1.9 \text{Pr}^{1/3} & ; & \text{Pr} \geq 47 \end{cases} \quad \Rightarrow \text{Numerical solution} \]

\[ \Rightarrow \text{Note, only solution at wall. Not full solution.} \]

So for the adiabatic wall:

\[ \Theta_p(0) = \Theta_{Aw}(0) = \frac{T_{Aw} - T_\infty}{\frac{1}{2} U_{\infty}^2 / \text{Pr}} = r \quad \Rightarrow \text{recovery factor} \]

\[ T_{Aw} = T_\infty + r \frac{U_{\infty}^2}{2 \text{Pr}} \]

\[ r = \begin{cases} \text{Pr}^{1/2} & ; & 0.5 \leq \text{Pr} \leq 47 \\ 1.9 \text{Pr}^{1/3} & ; & \text{Pr} \geq 47 \end{cases} \]
So the recovery factor represents the fraction of kinetic energy "recovered" by an adiabatic wall.

Note, let's check the limit of the solution: for Pr = 1

\[ r = 1 \Rightarrow T_{AW} = T_0 + \frac{U_{in}^2}{2c} \Rightarrow \text{Same as isentropic solution (stagnation point).} \]

Now we need the homogeneous solution

\[ \Theta_+ (n) \Rightarrow \Theta_+ (n = 0) = \text{constant (constant wall temp.)} \]

\[ \Theta_+ (n \to \infty) = 0 ; \quad \Theta_+ '' + \frac{1}{2} \Pr \Theta_+ ' = 0 \quad (4) \]

\[ \Theta_+ = \frac{T - T_0}{T_{in} - T_0} \Rightarrow \text{Eq. 55} \quad \text{Homogeneous Equation} \]

So:

\[ \Theta(n) = \Theta_0 + C_1 \Theta_+ + C_2 \Rightarrow \text{Back substitute into (3) to check.} \]

Aside: Remember the following, for a nonhomogeneous linear ODE

\[ y'' + p(t) y' + q(t) y = g(t) \]

\[ y = y_c + y_p \]

Where \( y_p \) = any particular (specific) solution that satisfies the nonhomogeneous equation

\[ y_H = C_1 y_1 + C_2 y_2 \] is the general solution to the homogeneous equation: (complementary soln)

\( y_1 \) & \( y_2 \) are linearly independent solutions.

\[ y'' + p(t) y' + q(t) y = 0 \Rightarrow \text{Homogeneous eqn.} \]

Back to our problem, we can now solve for \( C_1 \) & \( C_2 \), using our overall b.c.'s.
Just to be sure though, we can back substitute our assumed solution into \((\ref{eqn:assumption})\) and see if it makes sense:

\[
\Theta'' + \frac{1}{2} Pr f' \Theta' + 2 Pr \left(f''\right)^2 = 0
\]

\[
\frac{\partial}{\partial n} \left( \frac{\partial}{\partial n} \left( \Theta_p + C_i \Theta_h + C_2 \right) \right) + \frac{1}{2} Pr f \frac{\partial}{\partial n} \left( \Theta_p + C_i \Theta_h + C_2 \right) + 2 Pr f' \left( f'' \right)^2 = 0
\]

\[
\Theta_p'' + \frac{1}{2} Pr f \Theta_p' + 2 Pr \left(f''\right)^2 + C_i \Theta_h'' + \frac{1}{2} Pr f C_i \Theta_h' = 0
\]

\[
= 0 \ (\text{Complementary solution}) = 0 \ (\text{Homogeneous solution})
\]

Now back to our B.C.'s

\[
\Theta(\eta \rightarrow \infty) = 0 \ or \ T(\eta \rightarrow \infty) = T_{\infty}
\]

\[
\Theta_h(\eta \rightarrow \infty) = \frac{T_{\infty} - T_0}{T_\infty - T_0} = 1
\]

\[
\Theta_p(\eta \rightarrow \infty) = 0 \ (\text{From numerical solution})
\]

\[
0 = 0 + C_i (1) + C_2 \ \Rightarrow \ C_i = -C_2
\]

Our second b.c. is: \(\Theta(\eta=0)\)

\[
\Theta_h(\eta=0) = \frac{T_0 - T_\infty}{T_\infty - T_0} = 0
\]

\[
\Theta_p(\eta=0) = \Theta_{AW}(\eta=0) \ (\text{Just a name change}) \ \text{or} \ \Theta_p(0)
\]

\[
\frac{T_0 - T_\infty}{U_{\infty}/2C_p} = \Theta_{AW}(0) + 0 + C_2 \ \Rightarrow \ C_2 = \frac{T_0 - T_\infty}{U_{\infty}/2C_p} - \Theta_{AW}(0)
\]

Back substituting everything into our complete solution

\[
\Theta = \frac{T - T_{\infty}}{U_{\infty}/2C_p} = \Theta_p + \left( \frac{T_0 - T_{\infty}}{U_{\infty}/2C_p} - \frac{T_{0,AW} - T_{\infty}}{U_{\infty}/2C_p} \right) (1 - \Theta_h)
\]
Rearranging, we obtain:

\[ T - T_\infty = \Theta_0 \frac{U_\infty^2}{2c_p} + (T_0 - T_{0,Aw})(1 - \Theta_\infty) \]

Now we can evaluate the heat flux (note, \( \Theta_0 \) still is solved numerically). \( \Theta_\infty \) we solved before (pg. 55 of notes).

\[ q''|_0 = -k \frac{\partial T}{\partial y}|_0 \]

\[ \frac{\partial T}{\partial y}|_0 = \frac{2\Theta}{2c_p} U_\infty^2 - (T_0 - T_{Aw}) \frac{\partial \Theta}{\partial y}|_0 \] ; note \( \frac{\partial \Theta}{\partial y}|_0 = 0 \) (Adiabatic)

\[ \frac{\partial T}{\partial y}|_0 = 0 - (T_s - T_{Aw}) \frac{\partial \Theta}{\partial y}|_0 \]

\[ q''|_0 = \frac{\partial T}{\partial y]|_s = k (T_s - T_{Aw}) \frac{\partial \Theta}{\partial y}|_0 \frac{\partial y}{U_\infty} \]

\[ q''|_0 = k (T_s - T_{Aw}) \frac{\partial \Theta}{\partial y}|_0 \frac{\partial y}{U_\infty} \]

But we've already solved for \( \Theta|_0 \) on pg. 56

\[ \Theta(n^*) = f'(n) \]

\[ \frac{\partial \Theta}{\partial n} = \frac{\partial \Theta}{\partial n^*} \frac{\partial n^*}{\partial n} \Rightarrow n^* = n \rho \frac{1}{3} \Rightarrow \frac{\partial n^*}{\partial n} = \rho \frac{1}{3} \]

\[ \frac{\partial \Theta}{\partial n}|_0 = \frac{\partial \Theta}{\partial n^*}|_0 \rho \frac{1}{3} = f''(0) \rho \frac{1}{3} = 0.332 \rho \frac{1}{3} \]

\[ \frac{\partial \Theta}{\partial n}|_0 = 0.332 \rho \frac{1}{3} \Rightarrow \] Back substitute into \( q''|_0 \)

\[ q''|_0 = k (T_0 - T_{Aw}) \frac{0.332 \rho \frac{1}{3}}{\sqrt{DX/U_\infty}} \] \( \Rightarrow \) Same result as before but \( T_\infty = T_{Aw} \)

So we can say for flows with viscous dissipation
So for viscous heating b.k.'s:

\[ q'' \left|_{o} = h (T_{o} - T_{aw}) \right. \]
\[ h = k \frac{0.332 Pr^{1/3}}{\sqrt{Ux/U_{\infty}}} \Rightarrow \text{Same as before} \]
\[ Nu_{x} = \frac{hx}{k} = 0.332 Pr^{1/3} Re_{x}^{1/2} \Rightarrow \text{Same as before} \]

The only change is instead of using \( T_{o} \), we should use \( T_{aw} \). Very cool!

Let's test the limits of this:

\[ T_{aw} = T_{o} + r \frac{U_{\infty}^2}{2C_{p}} \quad r = \begin{cases} \Pr^{1/2} ; & 0.5 \leq \Pr < 47 \\ 1.9 \Pr^{1/3} ; & \Pr \geq 47 \end{cases} \]

For \( \Pr = 1 \), \( r = 1 \) so

\[ T_{aw} = T_{o} + \frac{U_{\infty}^2}{2C_{p}} \Rightarrow \text{Same as previous solution (stagnation pt.)} \]

For:

\[ q'' \left|_{o} = h (T_{o} - T_{o} - \frac{U_{\infty}^2}{2C_{p}}) \Rightarrow \frac{U_{\infty}^2}{2C_{p}(T_{o} - T_{o})} \ll 1 \]

Then \( T_{aw} = T_{o} \) and our old results are valid.

Here we see why we defined the Eckert number like we did:

\[ Ec = \frac{U_{\infty}^2}{C_{p}(T_{o} - T_{o})} \Rightarrow \text{It's nothing but a measure of the negligibility of the recovery factor term.} \]

Note to be more rigorous, we should say:

\[ Ec = \frac{r U_{\infty}^2}{2C_{p}(T_{o} - T_{o})} \ll 1 \text{ for } T_{aw} = T_{o} \text{ since } r = f(\Pr) \]

Sometimes our condition is written as \( \Pr \cdot Ec \ll 1 \)
So how about our second condition (temperature dependent properties). Well, Eckert found a miraculous solution.

As long as we take our properties at: (as long as $C_v\approx\text{const}$)

$$T_R = T_o + 0.5(T_o - T_m) + 0.22(T_{aw} - T_m)$$

for $Ma<20$, $Pr<15$.

All properties are evaluated at $T_R$, including $Pr,\rho,\mu, k$.

Note, this also takes into account compressibility effects.

This also works for calculating shear stress & local skin friction coefficient. Use $T_R$ as the reference temperature.

Example #1: For turbulent flows, it's been shown that:

$$r \approx Pr^{1/3} \quad \text{for all} \quad Pr, \quad \text{and} \quad 0.4 \leq Ma < 8, \quad \text{and gas flow}$$

Find $T_{aw}$ for the space shuttle re-entry. ($Ma = 5$)

\[
V = 5(343\text{m/s}) = 1715\text{ m/s}
\]

\[
C_p,\text{Air} \approx 1000\text{ J/kg.K}
\]

\[
Pr,\text{Air} \approx 0.62 \quad (\text{at} \quad T = 400\text{°C})
\]

$$r = Pr^{1/3} = 0.88 \quad \Rightarrow \quad T_{aw} = -40\text{°C} + (0.88)\frac{(1715\text{m/s})^2}{2(1000\text{ J/kg.K})}$$

$$T_{aw} \approx 1254\text{°C}$$

Note, now we should iterate to get properties at $T_R$. $C_p$ won't change that much, but $Pr$ will.

Also, at very high gas velocities ($Ma>20$), the wall gets so hot that ionization of the gas occurs, which needs to be taken into account. This ionization is why we lose radio communication with re-entry vehicles. The charged ions act as a blocking mechanism for EM waves.
Example #2 | Flow past a flat plate (air)

\[ U_\infty = 130 \text{ m/s} \quad T_w = 550 \text{K} \]
\[ T_\infty = 500 \text{K} \quad L = 0.1 \text{m} \]

First let's check to see if viscous heating is important:
\[ Pr \cdot Ec = Pr \frac{U_\infty^2}{\rho_c \cdot \Delta T} = 0.23 \Rightarrow \text{Need to consider}\]
\[ Re_L = 310,000 \leq 5.0 \times 10^5 \quad \text{(Laminar)} \]
\[ Ma = 0.28 \approx \text{incompressible} \]

Now we can solve since all the conditions fit our developed correlations.

\[ T_{aw} = T_\infty + \frac{r U_\infty^2}{2 \rho} \quad r = Pr^{1/2} = (0.68)^{1/2} = 0.82 \]
\[ T_{aw} = 500 \text{K} + \frac{0.82 \cdot (130 \text{m/s})^2}{2 \cdot (1000 \text{J/kg.K})} = 506.9 \text{K} \]
\[ T_R = T_\infty + 0.5 (T_o - T_\infty) + 0.22 (T_{aw} - T_\infty) \]
\[ = 500 \text{K} + 25 \text{K} + 0.22 (6.9 \text{K}) \]
\[ T_R = 526.5 \text{K} \Rightarrow \rho_{\text{Air}} = 0.00125 \text{ kg/m}^3, \quad C_{p,\text{Air}} = 1050 \text{J/kg.K} \]
\[ \mu_{\text{Air}} = 2.67 \times 10^{-5} \text{kg/m.s}, \quad Pr \approx 0.68, \quad \lambda_{\text{Air}} = 0.03 \text{W/m.K} \]

\[ q''_{10} = h L^2 (T_o - T_{aw}) \]
\[ h = \frac{Nu_L \cdot \lambda_{\text{Air}}}{L} = \frac{\lambda_{\text{Air}}}{L} \cdot (0.664 Re_L^{1/2} Pr^{1/3}) = 136.3 \text{ W/m}^2 \cdot \text{K} \]
\[ [q''_{10}] = 58.84 \text{ W} \quad \text{[q''_{10,false} = hA(T_o - T_\infty) = 68 W]} \]