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$$v\frac{\partial T}{\partial y} = \frac{U_\infty}{2x}\sqrt{\frac{x\nu}{U_\infty}}\left(\eta f' - f\right)\cdot\frac{\partial T}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} \qquad \sqrt{\frac{U_\infty}{\nu x}}$$

$$v\frac{\partial T}{\partial y} = \frac{U_\infty}{2x}\left(\eta f' T' - f T'\right)$$

Now for the right hand side:

$$\alpha\frac{\partial^2 T}{\partial y^2} = \alpha\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = \alpha\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right) = \alpha\frac{\partial \eta}{\partial y}\left(\frac{\partial}{\partial \eta}\left(\frac{\partial T}{\partial \eta}\right)\cdot\frac{\partial \eta}{\partial y}\right)$$

$$\alpha\frac{\partial^2 T}{\partial y^2} = \alpha\left(\frac{\partial \eta}{\partial y}\right)^2\cdot T'' = \frac{U_\infty \alpha}{\nu x}T''$$

And the viscous term:

$$\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 = \frac{\nu}{c_p}\cdot\left(\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)^2 = \frac{\nu}{c_p}\left(U_\infty f''\cdot\sqrt{\frac{U_\infty}{\nu x}}\right)^2$$

$$\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 = \frac{\cancel{\nu} U_\infty^3}{\cancel{\nu} c_p x}\left(f''\right)^2$$

Putting it all together: $\quad \frac{\alpha}{\nu} = \frac{1}{Pr}$

$$-\cancel{\frac{U_\infty \eta}{2x}f'T'} + \cancel{\frac{U_\infty \eta}{2x}f'T'} - \frac{\cancel{U_\infty}}{2\cancel{x}}fT' = \frac{\cancel{U_\infty}\alpha}{\nu \cancel{x}}T'' + \frac{U_\infty^{\cancel{3}\,2}}{c_p \cancel{x}}\left(f''\right)^2$$

$$\frac{1}{Pr}T'' + \frac{U_\infty^2}{c_p}\left(f''\right)^2 + \frac{1}{2}fT' = 0$$

$$\boxed{T'' + \frac{U_\infty^2 Pr}{c_p}\left(f''\right)^2 + \frac{1}{2}Pr\, f T' = 0} \quad (1)$$

So now we need to solve equation (1). Note, we can see that the equation is linear and that we have already solved for f before (Blasius).

Let $\quad \theta(\eta) = \dfrac{T - T_\infty}{U_\infty^2/2c_p} \quad (2)$

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Substituting ② into ①

$$\boxed{\theta'' + \frac{1}{2}Pr\, f\, \theta' + 2Pr\,(f'')^2 = 0} \quad ③$$

⇒ Second order, linear.

Using Duhamel's theorem, we can solve this by superposition of a particular and homogeneous solution.

$$\theta(\eta) = \theta_p(\eta) + \theta_c(\eta)$$

For the <u>particular solution</u>; let's assume adiabatic wall conditions: ⇒ We can do this due to Duhamel's theorem (Linear T)

$$\theta_p(\eta) \Rightarrow \left.\frac{\partial \theta_p}{\partial \eta}\right|_0 = 0 \quad , \quad \theta_p(\eta \to \infty) = 0$$

To solve we can use the integrating factor method or solve numerically using a shooting scheme:

$$\theta_p(\eta = 0) = \theta_{AW}(\eta = 0) = \int_0^\infty \frac{\int_0^\eta \exp\left(\int_0^\eta \frac{1}{2} Pr\, f\, d\eta\right) 2Pr\,(f'')^{1/2}\, d\eta}{\exp\left(\int_0^\eta \frac{1}{2} Pr\, f\, d\eta\right)}\, d\eta$$

We can numerically integrate the above solution for various Pr and using $f(\eta)$ solution from Blasius (Tabulated result)

$$\boxed{\theta_p(0) = \begin{cases} Pr^{1/2} \ ; & 0.5 \le Pr \le 47 \\ 1.9\, Pr^{1/3} \ ; & Pr \ge 47 \end{cases}}$$

⇒ Numerical solution

⇒ Note, only solution at wall. Not full solution.

So for the adiabatic wall:

$$\theta_p(0) = \theta_{AW}(0) = \frac{T_{AW} - T_\infty}{\frac{1}{2} U_\infty^2 / c_p} = r \equiv \text{recovery factor}$$

$$\boxed{T_{AW} = T_\infty + r\frac{U_\infty^2}{2c_p}} \ , \quad \boxed{r = \begin{cases} Pr^{1/2} \ ; & 0.5 \le Pr < 47 \\ 1.9\, Pr^{1/3} & Pr \ge 47 \end{cases}}$$

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So the recovery factor represents the fraction of kinetic energy "recovered" by an adiabatic wall.

Note, let's check the limit of the solution: for $Pr = 1$

$$r = 1 \Rightarrow T_{AW} = T_\infty + \frac{U_\infty^2}{2C} \Rightarrow \text{Same as isentropic solution (stagnation point).}$$

Now we need the <u>homogeneous solution</u>

$$\theta_H(\eta) \Rightarrow \theta_H(\eta = 0) = \text{constant (constant wall temp.)}$$

$$\theta_H(\eta \to \infty) = 0 \quad ; \quad \boxed{\theta_H'' + \frac{1}{2} Pr f \theta_H' = 0} \ ④$$

$$\theta_H = \frac{T - T_0}{T_\infty - T_0} \Rightarrow \text{pg. } ⑤⑤ \qquad \hookrightarrow \text{Homogeneous Equation}$$

So:

$$\theta(\eta) = \theta_p + C_1 \theta_H + C_2 \Rightarrow \text{Back substitute into ③ to check.}$$

Aside: Remember the following, for a nonhomogeneous linear ODE

$$y'' + p(t) y' + q(t) y = g(t)$$

$$y = y_c + y_p$$

Where $y_p \equiv$ any particular (specific) solution that satisfies the nonhomogeneous equation

$y_H \equiv C_1 y_1 + C_2 y_2$ is the general solution to the homogeneous equation: (complementary sol'n)

y_1 & y_2 are linearly independent solutions.

$$y'' + p(t) y' + q(t) y = 0 \Rightarrow \text{Homogeneous eqn.}$$

Back to our problem, we can now solve for C_1 & C_2, using our overall b.c.'s

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Just to be sure though, we can back substitute our assumed solution into ③ and see if it makes sense:

$$\Theta'' + \frac{1}{2} Pr f \Theta' + 2 Pr (f'')^2 = 0$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial}{\partial \eta}\left(\Theta_p + C_1 \Theta_H + C_2\right)\right) + \frac{1}{2} Pr f \frac{\partial}{\partial \eta}\left(\Theta_p + C_1 \Theta_H + C_2\right) + 2 Pr f''^2 = 0$$

$$\underbrace{\Theta_p'' + \frac{1}{2} Pr f \Theta_p' + 2 Pr (f'')^2}_{= 0 \text{ (Complementary solution)}} + \underbrace{C_1 \Theta_H'' + \frac{1}{2} Pr f C_1 \Theta_H'}_{= 0 \text{ (Homogeneous solution)}} = 0$$

Now back to our B.C.'s

$$\Theta(\eta \to \infty) = 0 \quad \text{or} \quad T(\eta \to \infty) = T_\infty$$

$$\Theta_H(\eta \to \infty) = \frac{T_\infty - T_0}{T_\infty - T_0} = 1$$

$\Theta_p(\eta \to \infty) = 0$ (From numerical solution)

$$0 = 0 + C_1(1) + C_2 \Rightarrow C_1 = -C_2$$

Our second b.c. is: $\Theta(\eta = 0)$

$$\Theta_H(\eta = 0) = \frac{T_0 - T_0}{T_\infty - T_0} = 0$$

$\Theta_p(\eta = 0) = \Theta_{AW}(\eta = 0)$ (Just a name change) or $\Theta_p(0)$

$$\frac{T_0 - T_\infty}{U_\infty^2/2C_p} = \Theta_{AW}(0) + 0 + C_2 \Rightarrow C_2 = \frac{T_0 - T_\infty}{U_\infty^2/2C_p} - \Theta_{AW}(0)$$

Back substituting everything into our complete solution

$$\Theta = \frac{T - T_\infty}{U_\infty^2/2C_p} = \Theta_p + \left(\frac{T_0 - T_\infty}{U_\infty^2/2C_p} - \frac{T_{0,AW} - T_\infty}{U_\infty^2/2C_p}\right)(1 - \Theta_H)$$

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Rearanging, we obtain:

$$T - T_\infty = \theta_p \frac{U_\infty^2}{2c_p} + (\underbrace{T_0 - T_{0,AW}}_{T_{AW}})(1-\theta_H)$$

Now we can evaluate the heat flux (note, θ_p still is solved numerically). θ_H we solved before (pg. 55 of notes).

$$q''|_0 = -k \frac{\partial T}{\partial y}\Big|_0$$

$$\frac{\partial T}{\partial y}\Big|_0 = \frac{\partial \theta_p}{\partial y}\Big|_0 \cdot \frac{U_\infty^2}{2c_p} - (T_s - T_{AW})\frac{\partial \theta}{\partial y}\Big|_0 \quad ; \text{ note } \frac{\partial \theta_p}{\partial y}\Big|_0 = 0 \text{ (Adiabatic)}$$

$$\frac{\partial T}{\partial y}\Big|_0 = 0 - (T_s - T_{AW})\frac{\partial \theta}{\partial y}\Big|_0$$

$$q''|_0 = -k\frac{\partial T}{\partial y}\Big|_s = k(T_s - T_{AW})\frac{\partial \theta_H}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$q''|_0 = k(T_s - T_{AW})\frac{\partial \theta_H}{\partial \eta}\Big|_0 \cdot \sqrt{\frac{U_\infty}{\nu x}}$$

But we've already solved for $\theta_H'|_0$ on pg. 56

$$\theta(\eta^*) = f'(\eta)$$

$$\frac{\partial \theta_H}{\partial \eta} = \frac{\partial \theta_H}{\partial \eta^*}\cdot\frac{\partial \eta^*}{\partial \eta} \Rightarrow \eta^* = \eta Pr^{1/3} \Rightarrow \frac{\partial \eta^*}{\partial \eta} = Pr^{1/3}$$

$$\frac{\partial \theta_H}{\partial \eta}\Big|_0 = \frac{\partial \theta_H}{\partial \eta^*}\Big|_0 \cdot Pr^{1/3} = f''(0)\cdot Pr^{1/3} = 0.332 Pr^{1/3}$$

$$\therefore \boxed{\frac{\partial \theta_H}{\partial \eta}\Big|_0 = 0.332 Pr^{1/3}} \Rightarrow \text{Back substitute into } q''|_0$$

$$\boxed{q''|_0 = k(T_0 - T_{AW})\frac{0.332 Pr^{1/3}}{\sqrt{\nu x/U_\infty}}} \Rightarrow \text{Same result as before but } T_\infty = T_{AW}$$

So we can say for flows with viscous dissipation

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So for viscous heating b.l.'s:

$$\boxed{q''|_0 = h(T_0 - T_{AW})}$$

$$\boxed{h = k\frac{0.332 Pr^{1/3}}{\sqrt{\nu x/U_\infty}}} \Rightarrow \text{Same as before}$$

$$\boxed{Nu_x = \frac{hx}{k} = 0.332 Pr^{1/3} Re_x^{1/2}} \Rightarrow \text{Same as before}$$

The only change is instead of using T_∞, we should use T_{AW}. Very cool!

Let's test the limits of this:

$$T_{AW} = T_\infty + r\frac{U_\infty^2}{2c_p}, \quad r = \begin{cases} Pr^{1/2} & ; 0.5 \le Pr < 47 \\ 1.9 Pr^{1/3} & ; Pr \ge 47 \end{cases}$$

For $\underline{Pr = 1}$, $r = 1$ so

$$T_{AW} = T_\infty + \frac{U_\infty^2}{2c_p} \Rightarrow \text{Same as previous solution (stagnation pt.)}$$

For: $q''|_0 = h\left(T_0 - T_\infty - \frac{U_\infty^2}{2c_p}\right) \Rightarrow \frac{U_\infty^2}{2c_p(T_0 - T_\infty)} \ll 1$

Then $T_{AW} = T_\infty$ and our old results are valid.

Here we see why we defined the Eckert number like we did:

$$\boxed{Ec = \frac{U_\infty^2}{c_p(T_0 - T_\infty)}} \Rightarrow \text{It's nothing but a measure of the negligibility of the recovery factor term.}$$

Note to be more rigorous, we should say:

$$Ec = \frac{r U_\infty^2}{2c_p(T_0 - T_\infty)} \ll 1 \text{ for } T_{AW} = T_\infty \text{ since } r = f(Pr)$$

Sometimes our condition is written as $\boxed{Pr \cdot Ec \ll 1}$

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So how about our second condition (temperature dependent properties). Well, Eckert found a miracoulous solution. As long as we take our properties at: (as long as $C_p \approx$ const)

$$\boxed{T_R = T_\infty + 0.5(T_0 - T_\infty) + 0.22(T_{AW} - T_\infty)} \quad \text{for } Ma < 20. \quad Pr < 15.$$

All properties are evaluated at T_R, including Pr, ρ, μ, k
Note, this also takes into account compressibility effects.
This also works for calculating shear stress & local skin friction coefficient. Use T_R as the reference temperature.

Example #1 For turbulent flows, it's been shown that:

$r \approx Pr^{1/3}$ for all Pr, and $0 < Ma < 8$, and gas flow

Find T_{AW} for the space shuttle re-entry. ($Ma = 5$)

$$V = 5(343\,m/s) = 1715\,m/s$$
$$C_{p,Air} \cong 1000\,J/kg\cdot K$$
$$Pr_{,Air} \cong 0.68 \quad (\text{at } T = 400°C)$$

$$r = Pr^{1/3} = 0.88 \Rightarrow T_{AW} = -40°C + (0.88)\frac{(1715\,m/s)^2}{2(1000\,J/kg\cdot K)}$$

$$\boxed{T_{AW} \approx 1254°C}$$

Note, now we should itterate to get properties at T_R. C_p won't change that much, but Pr will.

Also, at very high gas velocities ($Ma > 20$), the wall gets so hot that ionization of the gas occurs, which needs to be taken into account. This ionization is why we loose radio communication with re-entry vehicles. The charged ions act as a blocking mechanism for EM waves.

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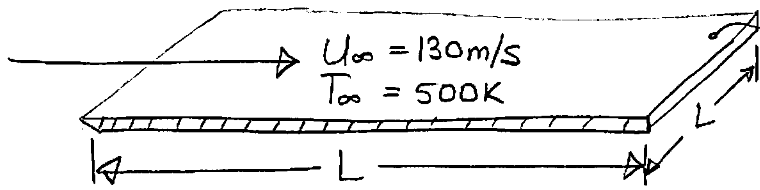
Example #2 Flow past a flat plate (air)



$T_w = 550K$

$L = 0.1m$

First let's check to see if viscous heating is important:

$$Pr \cdot Ec = Pr \frac{U_\infty^2}{C_p \cdot \Delta T} = 0.23 \Rightarrow \text{Need to consider}$$

$$Re_L = 310\,000 < 5.0 \times 10^5 \text{ (Laminar)}$$

$$Ma = 0.28 \approx \text{incompressible}$$

Now we can solve since all the conditions fit our developed correlations.

$$T_{AW} = T_\infty + \frac{r U_\infty^2}{2 C_p}, \quad r = Pr^{1/2} = (0.68)^{1/2} = 0.82$$

$$T_{AW} = 500K + \frac{0.82(130 m/s)^2}{2(1000 J/kg \cdot K)} = 506.9 K$$

$$T_R = T_\infty + 0.5(T_o - T_\infty) + 0.22(T_{AW} - T_\infty)$$
$$= 500K + 25K + 0.22(6.9K)$$

$$T_R = 526.5K \Rightarrow \rho_{Air} = 0.65 kg/m^3, \; C_{p,Air} \approx 1050 J/kg \cdot K$$
$$\mu_{Air} = 2.67 \times 10^{-5} kg/m \cdot s, \; Pr \approx 0.68, \; k_{Air} = 0.04 W/mK$$

$$q''|_o = h L^2 (T_o - T_{AW})$$

$$h = \overline{Nu_L} \cdot \frac{k_{Air}}{L} = \frac{k_{Air}}{L} \cdot \left(0.664 Re_L^{1/2} Pr^{1/3}\right) = 136.3 W/m^2 \cdot K$$

$\boxed{q''|_o = 58.84W}$ $\boxed{q''|_{o,false} = hA(T_o - T_\infty) = 68W}$ $q''_{real} < q''_{false}$ due

heat