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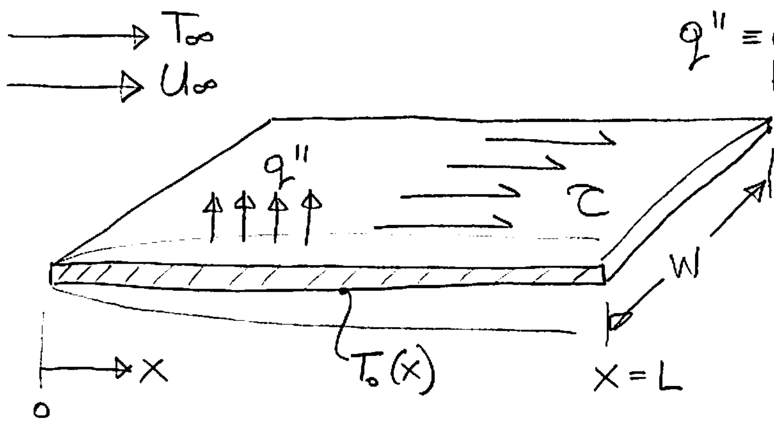
Entropy Generation Minimization in External Laminar B.L. Flow

We now have an excellent understanding of heat transfer and fluid flow (friction) in laminar b.l.'s. So how can we now design them such that the entropy generated during the process is minimal (i.e. what geometry will result in $S_{gen,min}$?).

Consider the following case:



q'' ≡ uniform heat flux on both sides of plate.

We know from the first part of the class that for any point in the flow field:

$$S'''_{gen} = \underbrace{\frac{k}{T^2}(\nabla T)^2}_{\geq 0} + \underbrace{\frac{\mu}{T}\Phi}_{\geq 0} \geq 0$$

If we volumetrically integrate this expression over the entire fluid domain, we obtain a fundamental result for entropy generation due to heat transfer between a body and a flow (U_∞, T_∞) surrounding the body:

$$\boxed{S_{gen} = \underbrace{\frac{1}{T_\infty^2}\int_A q''(T_0 - T_\infty)\, dA}_{\text{Heat Transfer}} + \underbrace{\frac{F_D U_\infty}{T_\infty}}_{\text{Friction (Shear)}}}$$

Note: $T_\infty \gg (T_0 - T_\infty)$;

F_D ≡ Drag force on the body ; $q''(T_0 - T_\infty) \geq 0$ always

A = Body surface area

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So let's solve our problem: We can already intuitively tell that it will involve optimization since

$$q''(T_o - T_\infty)\downarrow \text{ as } L\uparrow \Rightarrow \underbrace{(\overline{T}_o - T_\infty)}_{\text{For constant } q''}\downarrow \text{ or } \underbrace{q''}_{\text{For constant } (T_o - T_\infty)}\downarrow$$

$$F_D \uparrow \quad \text{as} \quad L\uparrow$$

We can simplify our integral to the following:

$$\int q''(T_o - T_\infty)\,dA = q''(\overline{T}_o - T_\infty)(2LW)$$

We can solve for $(\overline{T}_o - T_\infty)$ since we arleady solved the constant heat flux case

$$Nu_x = \frac{q''|_o}{T_o(x) - T_\infty}\cdot\frac{x}{k} = 0.455\,Pr^{1/3}\,Re_x^{1/2} \quad (0.5 \le Pr \le 10)$$

From pg. 102 of notes.

Rearanging:

$$(\overline{T} - T_\infty) = \frac{1}{L}\int_0^L \frac{q''\cdot x}{0.455\,Pr^{1/3}Re_x^{1/2}k}\,dx$$

$$= \frac{q''\mu^{1/2}}{0.455\rho^{1/2}U_\infty^{1/2}kPr^{1/3}L}\underbrace{\int_0^L x^{1/2}\,dx}_{\frac{2}{3}L^{3/2}}$$

$$= \frac{q''L}{0.455\,\frac{\rho^{1/2}U_\infty^{1/2}L^{1/2}}{\mu^{1/2}}\cdot k\,Pr^{1/3}}\cdot\frac{2}{3} = \frac{q''L}{0.455Pr^{1/3}Re_L^{1/2}k}\left(\frac{2}{3}\right)$$

So now we can compute our entropy due to heat transfer:

$$\int q''(T_o - T_\infty)\,dA = q''(\overline{T} - T_\infty)(2LW) = \frac{0.732(q')^2\cdot W}{k\,Re_L^{1/2}\,Pr^{1/3}\,T_\infty^{\,2}} \quad (1)$$

Where we've defined $\boxed{q' = 2Lq''}$ ⇒ Heat transfer rate per unit depth (w-direction).

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Now we can deal with the shear term:

$$\frac{F_D U_\infty}{T_\infty} = \frac{(2LW)\overline{\tau} U_\infty}{T_\infty}$$ $\Rightarrow$ For a flat plate in laminar flow: $\overline{\tau} = 0.664 \rho U_\infty^2 Re_L^{-1/2}$

$$= \frac{1.328 \rho U_\infty^3 w L}{Re_L^{1/2}} = \frac{1.328 \rho^{1/2} U_\infty^{5/2} w L^{1/2} \mu}{\mu^{1/2} T_\infty}$$

So:

$$\frac{F_D U_\infty}{T_\infty} = \frac{1.328 Re_L^{1/2} U_\infty^2 W \cdot \mu}{T_\infty} \quad (2)$$

Putting (1) & (2) together:

$$\frac{S_{gen}}{W} = \frac{S_{gen,HT}}{W} + \frac{S_{gen,SHEAR}}{W} = \frac{0.736 (q')^2}{T_\infty^2 k Pr^{1/3} Re_L^{1/2}} + 1.328 \frac{\mu U_\infty^2 Re_L^{1/2}}{T_\infty}$$

So now we can optimize our plate design.
The $Re_L^{1/2}$ term appears in both terms. We can differentiate and solve for $L_{optimum}$.

$$\frac{\partial S_{gen}}{\partial Re_L} = 0 \Rightarrow \text{Solve for } Re_{L,opt}$$

$$\boxed{Re_{L,opt} = 0.554 B^2} \Rightarrow \boxed{B = \frac{q'}{U_\infty (k \mu T_0 Pr^{1/3})^{1/2}} \equiv \text{Bejan \#}}$$

$$\boxed{B = \frac{\text{Heat transfer rate}}{\text{Flow speed}}} \text{ (Dimensionless)}$$

↳ Governs the entropy generation characteristics in laminar boundary layer flows.

If: $Re_L \ll B^2 \Rightarrow$ Heat transfer dominates entropy generation

$Re_L \gg B^2 \Rightarrow$ Fluid friction dominates entropy generation

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In conclusion, if a plate (fin) is to transfer a constant heat transfer per unit depth ($q' = \text{const.}$) to a stream with constant U_∞ and T_∞, then:

$$\boxed{L_{opt} = 0.554 \frac{(q')^2}{k T_\infty \rho U_\infty^3 Pr^{1/3}}} \Rightarrow \text{Optimum length for min. entropy generation.}$$

$$\boxed{S_{gen,min} = 1.98 \frac{q U_\infty}{(k/\mu)^{1/2} T_\infty^{3/2} Pr^{1/6}}} \Rightarrow S_{gen} \text{ for } L = L_{opt}$$

Note, the above analysis is fine and dandy but see if you can spot the one key assumption.

Answer: We've assumed in our solution that:

$$q' = \text{constant} = 2q''L$$

OK, but that means in order for each fin length we analyze, q'' is changing. If we wanted an optimization with $q'' = \text{constant}$, things would get complicated.

This is the nice part about entropy generation minimization. You can design heat transfer components in ways not possible before.

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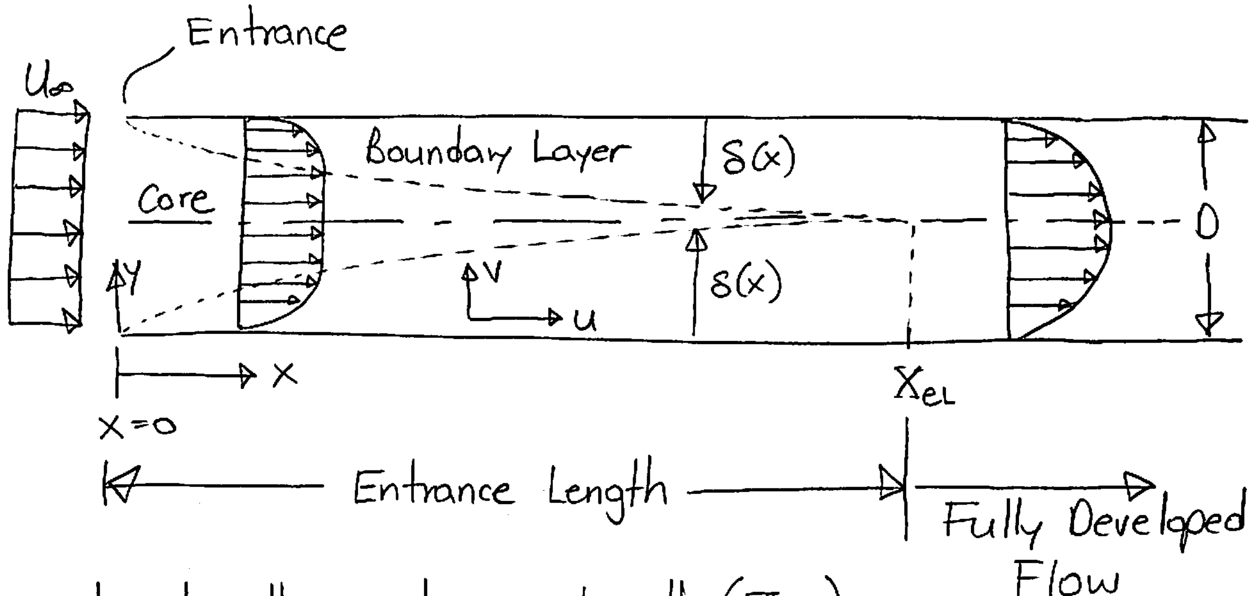
Internal Flow

Consider 2 parallel plates forming a 2D duct.



Let's estimate the entrance length (X_{eL}).
We expect here that:

For: $x < X_{eL} \Rightarrow \overline{h}_x \uparrow, \ \overline{\tau}_x \uparrow$

$x > X_{eL} \Rightarrow \overline{h}_x \downarrow, \ \overline{\tau}_x \downarrow$

To estimate X_{eL}, we can use scaling. We know from previous derivation that for a flat plate laminar flow:

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \quad \text{(Blasius solution, pg. 51 of notes)}$$

When the two boundary layes merge: $\delta = \frac{D}{2}, \ x = X_{eL}$

$$\frac{D}{2X_{eL}} = \frac{5\mu^{1/2}}{\rho^{1/2}U_\infty^{1/2}X_{eL}^{1/2}} \Rightarrow \frac{D}{X_{eL}} = \frac{10\mu^{1/2}}{\rho^{1/2}U_\infty^{1/2}X_{eL}^{1/2}} \cdot \frac{D^{1/2}}{D^{1/2}}$$

$$= \frac{10}{\frac{\rho^{1/2}U_\infty^{1/2}D^{1/2}}{\mu^{1/2}}} \cdot \left(\frac{D}{X_{eL}}\right)^{1/2}$$

So: $\left(\frac{D}{X_{eL}}\right)^{1/2} = \frac{10}{Re_D}$ or: $\boxed{\frac{X_{eL}}{D} = 0.01\, Re_D}$

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But note, this is only an estimate as our core flow is not $U_\infty = \text{constant}$. Here, the core is being squeezed and accelerates with the flow. (i.e. $U_\infty = f(x)$).

For a more accurate estimate, we can use integral methods to solve for X_{en}. From pg. 82 of our notes: (momentum integral eqn).

$$\frac{\partial}{\partial x}\int_0^Y u(U_\infty - u)\,dy = \frac{1}{\rho}\frac{\partial P_\infty}{\partial x}\cdot Y + \nu\frac{\partial u}{\partial y}\Big|_0 - \frac{\partial U_\infty}{\partial x}\int_0^Y u\,dy \quad (1)$$

Since our core flow is inviscid, we can apply Bernoulli's equation along the core streamline: ($U_c = U_\infty$ in the core).

$$\rho U_c^2/2 + P_\infty = \text{constant} \quad \Rightarrow \text{Differentiate since we need } \partial P_\infty/\partial x$$

$$\frac{\rho}{2}\frac{\partial}{\partial x}U_c^2 + \frac{\partial P_\infty}{\partial x} = 0$$

$$U_c\frac{\partial U_c}{\partial x} + \frac{1}{\rho}\frac{\partial P_\infty}{\partial x} = 0 \quad (2)$$

$$\frac{\partial U_c}{\partial x}\int_0^\delta U_c\,dy = U_c\frac{\partial U_c}{\partial x}\delta$$

Back substituting (2) into (1):

$$\frac{\partial}{\partial x}\Big[\int_0^\delta (U_c - u)u\,dy + \frac{\partial U_c}{\partial x}\int_0^\delta (U_c - u)\,dy = \nu\frac{\partial u}{\partial y}\Big|_0 \quad (3)$$

Note, I let $Y = \delta$, and only integrated up to the channel centerline due to symmetry.

To solve the integrals, we can first apply conservation of mass:

$$\boxed{\int_0^\delta \rho u\,dy + \int_\delta^{D/2}\rho U_c\,dy = \rho U_\infty\frac{D}{2}} \quad (4)$$

Now we can solve equations (3) & (4) by assuming a velocity profile. As before:

$$\frac{u}{U_c} = \frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

Back substituting and solving, we obtain:

$$\frac{x}{D \cdot Re_D} = \frac{3}{40}\left(9\frac{U_c}{U_\infty} - 2 - 7\frac{U_\infty}{U_c} - 16\ln\left(\frac{U_c}{U_\infty}\right)\right) \quad ⑤$$

and:

$$\frac{2\delta}{D} = 3\left[1 - \frac{U_\infty}{U_c}\right] \quad ⑥$$ $\Rightarrow$ Note $\delta = f(x)$ and $U_c = f(x)$

When our two b.l.'s merge, $\delta(X_{eL}) = \frac{D}{2}$

$$\frac{2D}{2D} = 3\left[1 - \frac{U_\infty}{U_c}\right] \Rightarrow U_c(X_{eL}) = \frac{3}{2}U_\infty \quad ⑦$$

Back substitute ⑦ into ⑤ and solve for X_{eL}/D:

$$\boxed{\frac{X_{eL}}{D} = 0.026\, Re_D}$$ $\Rightarrow$ More accurate solution.

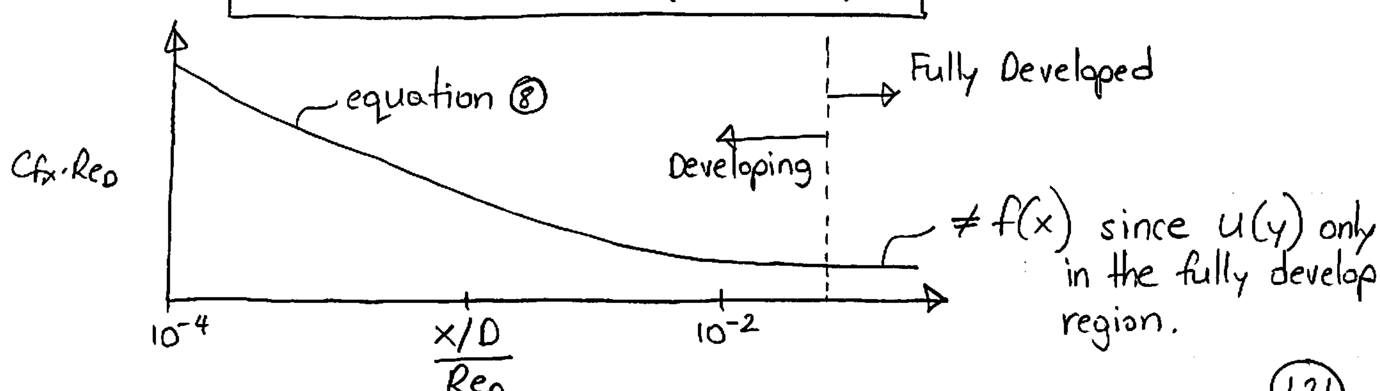
Experiments show that: $\boxed{\frac{X_{eL}}{D} \simeq 0.04\, Re_D \text{ to } 0.05\, Re_D}$

Minor difference is due to edge effects where the b.l. analysis breaks down.

Fundamental differences between the entrance and fully developed region is the wall shear stress: $C_{fx} = \tau_x / \frac{1}{2}\rho U_\infty^2$

Defining $\tau_x = \mu \frac{\partial u}{\partial y}\big|_0$ and using our solution (eqn's ⑤ & ⑥)

$$\boxed{C_{f,x} \cdot Re_D = \frac{8}{3}\frac{U_c}{U_\infty}\left(1 - \frac{U_\infty}{U_c}\right)^{-1}} \quad ⑧$$ $\Rightarrow$ Plot below:



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Fully - Developed Flow

At any point in our channel, our mass & momentum conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad ① \Rightarrow \text{mass conservation}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad ② \Rightarrow x\text{-momentum}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad ③ \Rightarrow y\text{-momentum}$$

Using scaling for a fully developed flow:

$$\left.\begin{array}{l} x \sim L \\ y \sim D \\ u \sim U_\infty \end{array}\right\} \text{From eq. ①} \quad \frac{U_\infty}{L} + \frac{V}{D} = 0 \Rightarrow V \sim \frac{D U_\infty}{L} \quad ④$$

For fully developed flow, $x > L$ such that $V \sim \frac{DU_\infty}{L} \ll U_\infty$
So far enough from the entrance such that the scale of V is negligible. or $\frac{D}{L} \ll 1$

So for fully developed flow:

$$\boxed{V = 0} \quad \text{and} \quad \frac{\partial V}{\partial y} = 0 \quad \text{so} \quad \boxed{\frac{\partial u}{\partial x} = 0} \Rightarrow \text{From continuity}$$

Note in the entrance region $(x < X_{EL})$, $y \sim \delta$ so $V \neq 0$
Applying this to the y-momentum equation: (eq. ③)

$$u\cancel{\frac{\partial v}{\partial x}} + v\cancel{\frac{\partial v}{\partial y}} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right) \Rightarrow V = 0 = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2}$$

$$\therefore \boxed{\frac{\partial P}{\partial y} = 0} \Rightarrow P = f(x) \text{ only}$$

$$\boxed{\frac{\partial P}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2}} \quad ⑤$$

Looking at x-momentum (eq. ②):

$$\cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right) \Rightarrow \frac{\partial u}{\partial x} = 0,\ V = 0$$

Note, since $P = f(x)$ only and $u = f(y)$ only (i.e. $\frac{\partial u}{\partial x} = 0$)

$$\boxed{\frac{\partial P}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} = \text{constant}} \quad (6)$$

Now we can solve eq. (6) subject to our b.c.'s: (note y=0=centerline)

$$\left.\begin{aligned} u\left(y = -\tfrac{D}{2}\right) &= 0 \Rightarrow \text{no-slip} \\ u\left(y = +\tfrac{D}{2}\right) &= 0 \Rightarrow \text{no-slip} \end{aligned}\right\} \text{Integrate (6) twice \& apply these.}$$

$$\boxed{u = \frac{3}{2}\,\overline{u}\left[1 - \left(\frac{y}{D/2}\right)^2\right]} \Rightarrow \text{velocity profile, } u(y)$$

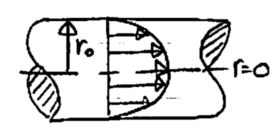
$$\boxed{\overline{u} = \frac{D^2}{12\mu}\left(-\frac{\partial P}{\partial x}\right)} \Rightarrow \text{Average velocity: } \overline{u} = \frac{1}{D}\int_{-\frac{D}{2}}^{+\frac{D}{2}} u\,dy$$

Here, y is the distance away from the centerline of the channel.

In general, we can say: $\boxed{\frac{\partial P}{\partial x} = \mu \nabla^2 u = \text{constant}}$

Where $\nabla^2 \equiv$ Laplacian operator.

For example, for a round tube with radius r_o:



$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \Rightarrow \text{Note, I've neglected the } \theta \text{ \& } z \text{ terms.}$$

$$\frac{\partial P}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \Rightarrow \text{Solving this with: } u(r = r_o) = 0, \quad \left.\frac{\partial u}{\partial r}\right|_0 = 0$$

$$\boxed{u = 2\overline{u}\left[1 - \left(\frac{r}{r_o}\right)^2\right]} \Rightarrow \text{velocity profile, } u(r)$$

$$\boxed{\overline{u} = \frac{r_o^2}{8\mu}\left(-\frac{\partial P}{\partial x}\right)} \Rightarrow \text{Average velocity: } \overline{u} = \frac{1}{\pi r_o^2}\int_0^{r_o} (2\pi r)\,u\,dr$$

$$\boxed{\dot{m} = \frac{\pi r_o^4}{8\nu}\left(-\frac{\partial P}{\partial x}\right)} \Rightarrow \text{Mass flow rate [kg/s]: } \dot{m} = \rho\,\overline{u}\,\pi r_o^2$$

These solutions were first reported by Hagen in 1839 and Poiseulle in 1840.

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These pressure-driven flows are called Hagen-Poiseulle flows. We commonly assign a Reynolds number to these flows

$$Re = \frac{\bar{U} D}{\nu} \equiv \frac{\text{inertial force}}{\text{viscous force}}$$

But when we started our analysis with equation ②

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{Inertial Forces}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\text{Pressure Force}} + \underbrace{\nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{\text{Viscous Forces}}$$

Since we showed for fully developed flow that: $v = 0, \frac{\partial u}{\partial x} = 0$

$\therefore$ Inertial Forces = 0!

Indeed, looking at our governing equation for these flows:

$$\left.\underbrace{\frac{\partial P}{\partial x}}_{\text{Pressure Forces}} = \underbrace{\nabla^2 u}_{\text{Viscous Forces}}\right\}$$

So the concept of Reynolds number in Hagen-Poiseulle flows is <u>nonsence!</u>

These flows are governed by Pressure ~ Viscosity, or

$$\boxed{-\frac{\partial P/\partial x}{\mu\, \partial^2 u/\partial r^2} \sim \frac{\Delta P/L}{\mu \bar{U}/D^2} \sim \overset{\text{On the order of:}}{O(1)} \equiv \frac{\text{longitudinal pressure force}}{\text{friction force}}}$$

Note, if you want to prove this to yourself, for a flow with $Re_D \approx 2000 \Rightarrow$ The inertia is 2000X larger than the friction force! So theoretically, if I turn off my pump which generates $\partial P/\partial x$, the flow should keep going for a long time. In reality, as soon as you turn off a pump, the channel flow will stop almost immediotelly.