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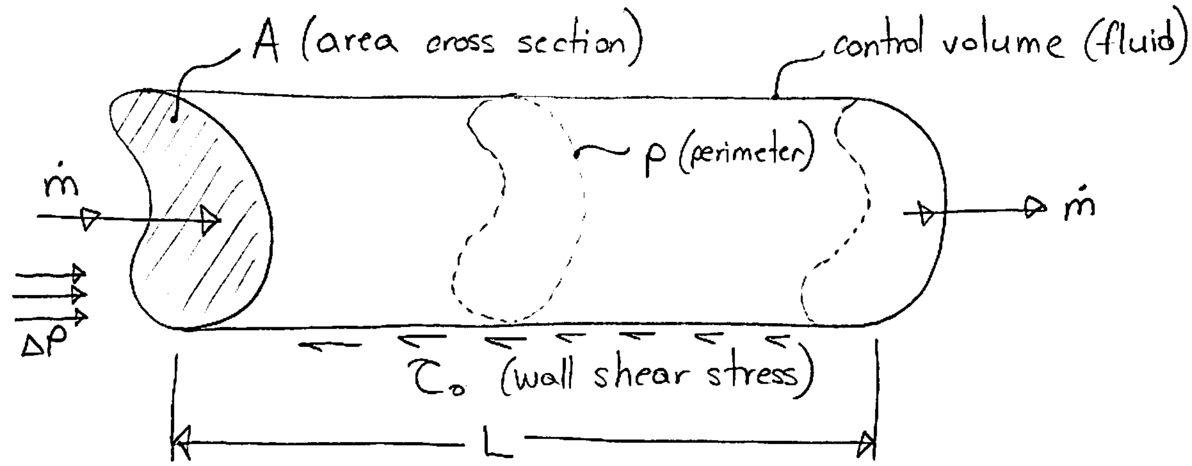
Hydraulic Diameter and Pressure Drop

The methodology of obtaining the relationship between $\dot{m}$ vs. ΔP stems from hydraulic theory of the 18'th century. This is still useful today exspecially for turbulent flows. For our case, we can obtain an analytical result:



For a fully developed flow, we can apply a force balance on our fluid CV:

$$\Sigma F_x = 0 \quad \text{(since fully developed, steady)}$$

$$A \Delta P = \rho L \tau_o \quad \Rightarrow \text{Note, } \rho \equiv \text{wetted perimeter}$$

$$\boxed{\tau_o = \frac{\Delta P}{\left(\frac{L}{A/\rho}\right)}}$$

Here, we can define a friction factor, f, as: $\boxed{f = \dfrac{\tau_o}{\frac{1}{2}\rho \bar{U}^2}}$

Note, f is similar to C_f but for fully developed flow, $f \neq f(x) = \text{constant}$

Rearanging:

$$\boxed{\Delta P = f \, \frac{\rho L}{A} \left(\frac{1}{2} \rho \bar{U}^2\right)} \Rightarrow \text{Pressure drop across the duct.}$$

Finally, note that A/ρ is the linear dimension of the cross section:

$$\boxed{r_h = \frac{A}{\rho} \equiv \text{hydraulic radius}} \; ; \; \boxed{D_h = 4 r_h = \frac{4A}{\rho} \equiv \text{hydraulic diameter}}$$

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The physical meaning of D_h can be understood as the length that accounts for how closte the wall (and its resistive forces - shear) are positioned relative to the stream.

We can now solve for f for our Hagen-Poiseulle flows: Let's use the round tube for simplicity:

$$\bar{u} = \frac{r_o^2}{8\mu}\left(-\frac{\partial P}{\partial x}\right) \Rightarrow \frac{8\mu\bar{u}}{r_o^2} = \frac{\Delta P}{L}$$

We know: $$f = \frac{\Delta P}{\left(\frac{4L}{D_h}\right)\frac{1}{2}\rho\bar{u}^2} \Rightarrow D_h = \frac{4A}{P} = \frac{\cancel{4}(\cancel{\pi} r_o^{\cancel{2}})}{\cancel{2\pi r_o}} = 2r_o$$

$$f = \frac{\Delta P}{\left(\frac{\cancel{2}\cancel{4}L}{2r_o}\right)\frac{1}{\cancel{2}}\rho\bar{u}^2} \Rightarrow \left(\frac{\Delta P}{L}\right)\cdot\frac{D}{2\rho\bar{u}^2} = \frac{8\mu\cancel{\bar{u}}}{r_o^2}\cdot\frac{D}{2\rho\bar{u}^{\cancel{2}}}$$

$$f = \frac{4\mu D}{\left(\frac{D}{2}\right)^2\cdot\rho\bar{u}} = \frac{16\mu}{\rho\bar{u}D} = \frac{16}{Re_{D_h}}$$

$$\boxed{f = \frac{16}{Re_{D_h}}}, \quad D_h = D \text{ (round tube)}, \quad \boxed{Re_D = \frac{\rho\bar{u}D}{\mu}}$$

For parallel plates, we can do a similar analysis to show:

$$\boxed{f = \frac{24}{Re_{D_h}}}, \quad D_h = 2D \quad (D = \text{gap thickness}).$$

Note, these are valid for $Re_{D_h} < 2000$. Also, note the convention. We have derived the "Fanning" friction factor. There is another convention that is used for pipes:

$$\boxed{f = \frac{\Delta P}{\left(\frac{L}{D}\right)\frac{1}{2}\rho\bar{u}^2} = \frac{64}{Re_D} \equiv \text{Darcy friction factor}}, \quad Re_D < 2300$$

Laminar flow
Round tube.

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Typically, friction factor results are tabulated as $f \cdot Re_D = \text{constant}$.

Interestingly, we can see from our previous scaling:

$$\frac{\Delta P/L}{\mu \bar{U}/D_h^2} = \frac{\frac{4f}{D_h}\left(\frac{1}{2}\rho \bar{U}^2\right)}{\mu \bar{U}/D_h^2} = \frac{f \rho \bar{U} D_h}{\mu} = f \cdot Re_{D_h}$$

So $\boxed{\frac{\Delta P/L}{\mu \bar{U}/D_h^2} = f \cdot Re_{D_h} \sim O(1)}$ $\Rightarrow$ Makes sense. As we expected.

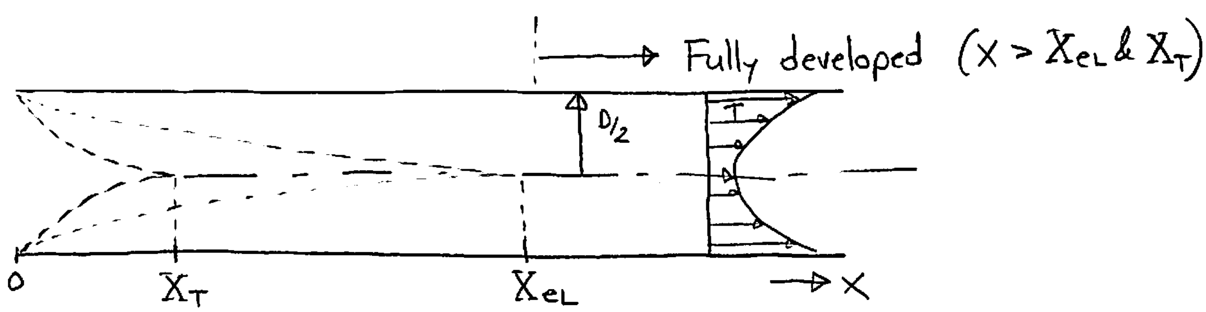
For many cross sections, $f \cdot Re_{D_h} \sim O(1)$ indeed.

$f \cdot Re_{D_h} = 13.3$ $\quad$ $f \cdot Re_{D_h} = 14.2$ $\quad$ $f \cdot Re_{D_h} = 16$ $\quad$ $f \cdot Re_{D_h} = 24$

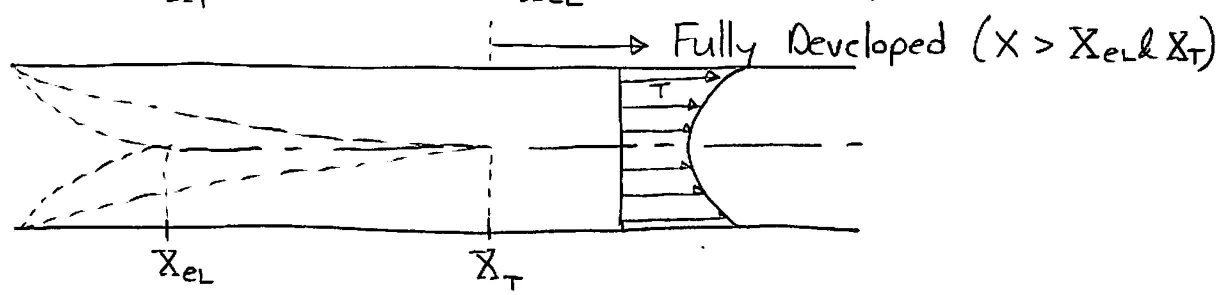
Heat Transfer in Fully Developed Duct Flow

To begin, we can first analyze when fully developed thermal conditions are met. We have 2 cases to consider:

$Pr \ll 1$



$Pr \gg 1$



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For $\underline{Pr \ll 1}$: $\delta_T \sim x\, Pr^{-1/2}\, Re_x^{-1/2}$ (pg. 42 of notes)

For our case, $x \sim X_T$ and $\delta_T \sim \frac{D_h}{2}$ (note, some books use $\delta_T \sim D_h$)

$$X_T\, Pr^{-1/2}\, Re_{X_T}^{-1/2} \sim \frac{D_h}{2} \Rightarrow \text{convert } Re_{X_T} \text{ to } Re_{D_h}$$

$$Re_{X_T} = \frac{\rho \bar{U} X_T}{\mu}\left(\frac{D_h}{D_h}\right) = \frac{\rho \bar{U} D_h}{\mu} \cdot \left(\frac{X_T}{D_h}\right)$$

$$X_T\, Pr^{-1/2}\, Re_{D_h}^{-1/2}\left(\frac{X_T}{D_h}\right)^{-1/2} \sim \frac{D_h}{2}$$

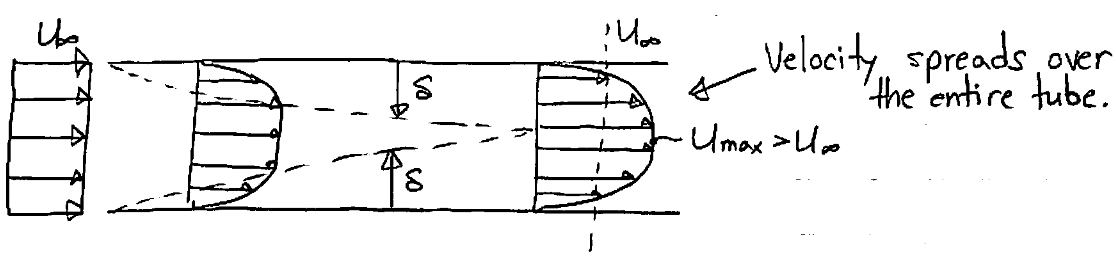
$$\boxed{\left(\frac{X_T/D_h}{Pr \cdot Re_{D_h}}\right)^{1/2} \sim \frac{1}{2}} \quad (1) \Rightarrow \text{Thermal b.l. development}$$

Note, we could have done the same thing for the hydrodynamic b.l.:

$$\boxed{\left(\frac{X_{eL}/D_h}{Re_{D_h}}\right)^{1/2} \sim \frac{1}{2}} \quad (2) \Rightarrow \text{Hydrodynamic b.l. development}$$

For $\underline{Pr \gg 1}$: It is tempting to say that $u \sim \left(\frac{\delta_T}{\delta}\right) U_\infty$ like we did before, inside a layer of thickness δ_T. However, since our channel is confined, and the velocity scale spreads over D_h, hence $u \sim U_\infty$ and:

$$\delta_T \sim x\, Pr^{-1/2}\, Re_x^{-1/2} \Rightarrow \text{Same as } Pr \ll 1.$$



Hence:

$$\boxed{\left(\frac{X_T/D_h}{Pr \cdot Re_{D_h}}\right)^{1/2} \sim \frac{1}{2}} \Rightarrow \text{For all } Pr.$$

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Dividing equations 1 and 2: $\boxed{\dfrac{X_T}{X_{eL}} \sim Pr}$ ⇒ For all Pr.

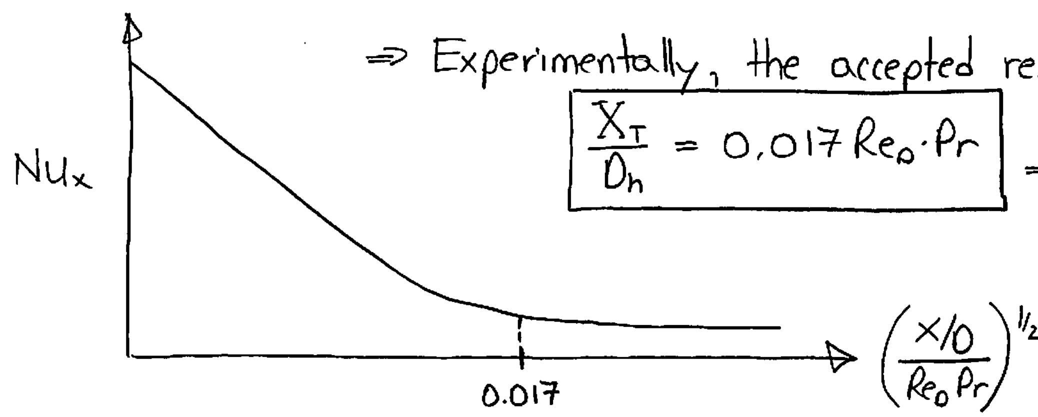
Following this up to see how heat transfer varies:

$$Nu = \frac{hD_h}{k} \sim \frac{q''}{\Delta T}\cdot\frac{D_h}{k} \sim \frac{\cancel{k}\frac{\cancel{\Delta T}}{\delta_T}\cdot D_h}{\cancel{\Delta T}\cdot\cancel{k}} \sim \frac{D_h}{\delta_T}$$

$$\delta_T \sim x\, Pr^{-1/2} Re_x^{-1/2}$$ ⇒ Convert Re_x to Re_{D_h}

$$\boxed{Nu \sim \left(\frac{x/D_h}{Re_{D_h}\cdot Pr}\right)^{-1/2}}$$ ⇒ Laminar
⇒ Developing flow, for all Pr

Typically our results are plotted as:



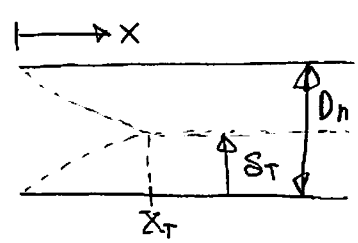
⇒ Experimentally, the accepted result is:

$$\boxed{\frac{X_T}{D_h} = 0.017\, Re_D\cdot Pr}$$ ⇒ For all Pr.

<u>Fully Developed Region</u> $(x > X_{eL}$ and $x > X_T)$

Before we do any analysis, let's try scaling:

$$Nu = \frac{hD_h}{k} \sim \frac{D_h}{\delta_T}$$

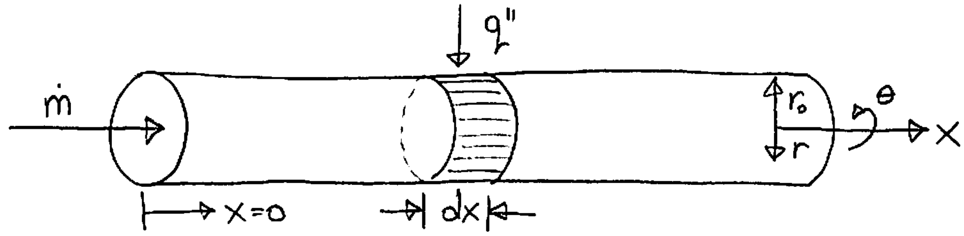


For the fully developed region, $\delta_T \sim \dfrac{D_h}{2}$

$$Nu \sim \frac{D_h}{D_h/2} \sim 2 \Rightarrow \boxed{Nu \sim 2}$$

We can now try to analyze the problem more exactly

Consider a pipe section of length dx undergoing heat transfer



We know that $\dot{m} = \rho \pi r_o^2 \bar{U}$

Applying an energy balance on our sliver (dx)

$$q'' \cdot 2\pi r_o dx = \dot{m}\left(h_{x+dx} - h_x\right) \;; \quad h_x = \text{fluid enthalpy at } x$$

For most fluids, we can model them as an ideal gas:

$$dh = c_p dT_m \Rightarrow \text{Back substituting}$$

$$\boxed{\frac{dT_m}{dx} = \frac{2}{r_o} \frac{q''}{\rho c_p \bar{U}}} \quad (1)$$

Note, T_m is defined as the mean temperature of the fluid as defined by thermodynamics, where the term "bulk" is used.

But we know that the fluid temperature varies as a function of x and r. However, we've defined our dh relation with respect to the thermodynamic bulk temperature.

$$q'' \cdot 2\pi r_o dx = \dot{m} dh = d\iint_A \rho u c_p T dA \Rightarrow \text{since both } u \text{ and } T \text{ are functions of } r.$$

Back substituting (1) into the equation above:

$$\frac{\rho c_p \bar{U} r_o}{\cancel{2}} \cdot \frac{\partial T_m}{\partial \cancel{x}} \cdot \cancel{2}\pi r_o \cancel{dx} = d\iint_A \rho u c_p T dA$$

$$\int \rho c_p \pi r_o^2 \bar{u}\, dT_m = \int d \iint \rho u c_p T\, dA$$

Aside: For more general:

$$\boxed{T_m = \frac{1}{A\bar{u}} \cdot \iint_A uT\,dA}$$

$$\rho c_p \bar{u} A T_m = \iint_A \rho u c_p T\, dA$$

For constant property fluids : $\boxed{T_m = \frac{1}{\pi r_o^2 \bar{u}} \int_0^{2\pi} \int_0^{r_o} uT\, r\,dr\,d\theta}$

We can simplify this further since θ is usually not a factor

$$\boxed{T_m = \frac{1}{\pi r_o^2 \bar{u}} \int_0^{r_o} uT\, 2\pi r\, dr} \Rightarrow \text{Pipe flow, constant prop.}$$

Since the fluid temperature varies over the cross section, it is customary to define heat transfer analysis with respect to the mean temperature, T_m :

$$\boxed{h = \frac{q''|_o}{T_o - T_m} = \frac{k \frac{\partial T}{\partial r}\big|_o}{T_o - T_m}}$$

Fully Developed Temperature Profile

Let's analyze a round tube (θ-symmetric). Our energy equation becomes:

$$\frac{1}{\alpha}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r}\right) = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial x^2}$$

For the hydrodynamic and thermally fully developed region, we know that: $v = 0$ and $u = u(r)$ only:

$$\underbrace{\frac{u(r)}{\alpha}\frac{\partial T}{\partial x}}_{\text{Convection}} = \underbrace{\underbrace{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}}_{\text{radial}} + \underbrace{\frac{\partial^2 T}{\partial x^2}}_{\text{axial}}}_{\text{Conduction}}$$

Using scaling, we can see which terms dominate and when.

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We know: $u \sim \bar{U}$

$$\frac{dT_m}{dx} \sim \frac{2q''}{r_o \rho c_p \bar{U}} \quad ; \quad \frac{\partial^2 T}{\partial x^2} \sim \frac{\partial^2 T_m}{\partial x^2} \sim \frac{1}{x}\left(\frac{2q''}{r_o \rho c_p \bar{U}}\right)$$

$$\frac{\partial^2 T}{\partial r^2} \sim \frac{\Delta T}{r_o^2} \quad ; \quad \frac{1}{r}\frac{\partial T}{\partial r} \sim \frac{1}{r_o}\frac{\Delta T}{r_o} \sim \frac{\Delta T}{r_o^2}$$

Back substituting:

$$\frac{\bar{U}}{\alpha}\left(\frac{2q''}{r_o \rho c_p \bar{U}}\right) \sim \frac{\Delta T}{r_o^2} + \frac{1}{x}\left(\frac{2q''}{r_o \rho c_p \bar{U}}\right)$$

We know the radial conduction term must be present in order for us to solve the problem (otherwise its a trivial solution). Multiplying through by $r_o^2/\Delta T$

$$\frac{\cancel{\bar{U}}}{\alpha}\frac{r_o^2}{\Delta T}\left(\frac{2q''}{\cancel{r_o} \rho c_p \cancel{\bar{U}}}\right) \sim \underbrace{\cancel{\frac{r_o^2}{\Delta T}\cdot\frac{\Delta T}{r_o^2}}}_{1} + \frac{1}{x}\frac{r_o^2}{\Delta T}\left(\frac{2q''}{\cancel{r_o} \rho c_p \bar{U}}\right)$$

$$\frac{hD}{k} \sim 1 + \underbrace{\frac{q''}{\Delta T}}_{h}\left(\frac{1}{\cancel{r_o}\underbrace{\rho c_p}_{\frac{\alpha}{k}}\bar{U}}\right)\left(\frac{2q''}{\underbrace{\rho c_p}_{\frac{\alpha}{k}}\bar{U}}\right)\underbrace{\cancel{\frac{r_o}{\Delta T}}}_{h}$$

$$\underbrace{\frac{hD}{k}}_{\text{Convection}} \sim \underbrace{1 + \left(\frac{hD}{k}\right)^2\left(\frac{\alpha}{\bar{U}D}\right)^2}_{\text{Conduction}}$$

Comparing our first and third scales, we see that, if

$$\frac{\bar{U}D}{\alpha} \gg 1 \Rightarrow \text{Longitudinal conduction is negligible}$$

We already know this number: $\boxed{Pe_D = \frac{\bar{U}D}{\alpha} \equiv \text{Peclet \#} \equiv \frac{\text{advection}}{\text{conduction}}}$

↳ convection = conduction + advection

We already know from previous scaling that: $Nu = \frac{hD}{k} \sim 2$, here, we see that $Nu \sim 1$ (makes sense). The difference is in the first scaling, we used r_o instead of D for $\delta \sim r_o$.

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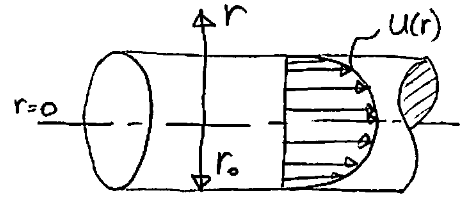
Assuming that $Pe_0 >> 1$, our governing equation becomes:

$$\frac{u(r)}{\alpha}\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial T}{\partial r}$$



sometimes written as:

$$\rho c_p u \frac{\partial T}{\partial x} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

We already solved that $u = 2\bar{u}\left(1-\frac{r^2}{r_0^2}\right)$ ⇒ for a round tube

Our B.C.'s are: $\left.\frac{\partial T}{\partial r}\right|_0 = 0$

$T(r=r_0) = T_0(x)$ ⇒ <u>Constant heat flux case</u>

Let: $\frac{T-T_m}{T_0-T_m} = \phi(r)$ ⇒ $q''|_{r_0} = -k\left.\frac{\partial T}{\partial r}\right|_{r_0} = -k\left.\frac{\partial \phi}{\partial r}\right|_{r_0}\cdot(T_0-T_m)$ = constant

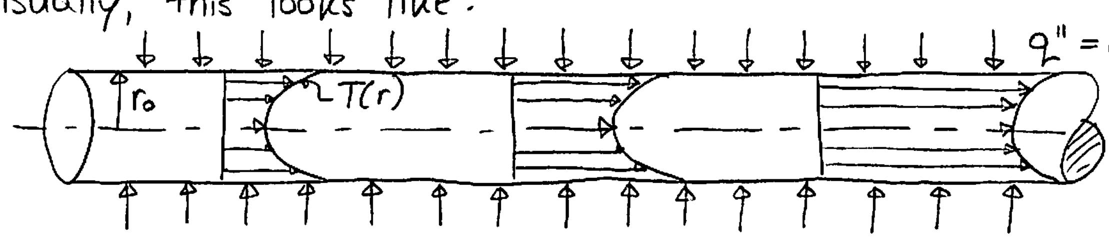
Here, we realize that the temperature profile doesn't change:

$$q''|_{r_0} = \text{constant} = \underbrace{-k\left.\frac{\partial \phi}{\partial r}\right|_{r_0}}_{\text{constant}}\underbrace{(T_0-T_m)}_{\text{has to be constant since } \left.\frac{\partial \phi}{\partial r}\right|_{r_0} = \text{constant}}$$

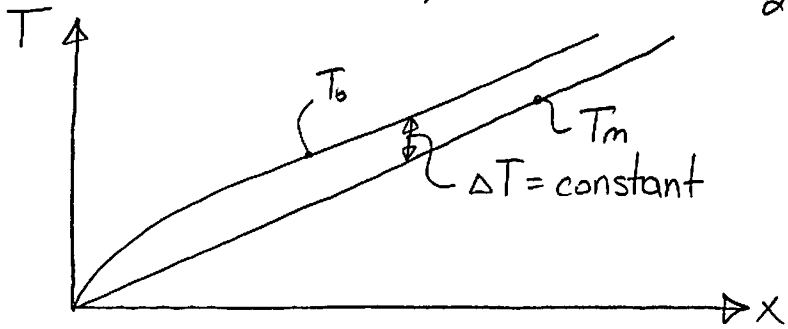
$\frac{\partial}{\partial x}(T_0-T_m) = \frac{\partial}{\partial x}(\text{const.})$

$\frac{\partial T_0}{\partial x} = \frac{\partial T_m}{\partial x}$

Visually, this looks like:



This implies that for q'' = constant, $\frac{\partial T_m}{\partial x} = \frac{\partial T_0}{\partial x}$



⇒ if $\frac{\partial T_m}{\partial x} \neq \frac{\partial T_0}{\partial x}$, then our two lines would cross and this would violate the first law of thermodynamics.