Nenad Miljkovic

We also know from before that: $\frac{\partial T}{\partial x} = \frac{\partial T_m}{\partial x} = \frac{2q''}{\rho c_p r_0 \bar{u}} = \text{constant}$

Back substituting $u(r)$ into our PDE

$$2\bar{u}\left[1 - \frac{r^2}{r_0^2}\right]\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \Rightarrow \text{PDE since } T(x,r)$$

Back substituting $\partial T/\partial x$:

$$\left(1 - \frac{r^2}{r_0^2}\right)\frac{4q''\cancel{\bar{u}}}{\cancel{\rho c_p} r_0 \cancel{\bar{u}}} = \frac{k}{\cancel{\rho c_p} r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \Rightarrow \text{ODE} \Rightarrow \text{Solvable}$$

$$\boxed{\left(1 - \frac{r^2}{r_0^2}\right)\frac{4q''}{k r_0} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)} \quad (1)$$

Integrating once, we obtain

$$\cancel{r}\frac{\partial T}{\partial r} = \frac{4q''}{kr_0}\left[\frac{r^{\cancel{2}}}{2} - \frac{r^{\cancel{4}3}}{4r_0^2}\right] + C_1$$

$$\frac{\partial T}{\partial r} = \frac{4q''}{4r_0}\left[\frac{r}{2} - \frac{r^3}{4r_0^2}\right] + \frac{C_1}{r} \leftarrow C_1 = 0 \text{ since } \frac{C_1}{r} \to \infty \text{ as } r \to 0$$

Integrating once more:

$$T(r) = \frac{4q''}{kr_0}\left[\frac{r^2}{4} - \frac{r^4}{16r_0^2}\right] + C_2$$

Using our second B.C.: $T(r=r_0) = T_0 \Rightarrow C_2 = T_0 - \frac{4q''}{kr_0}\left[\frac{3}{16}r_0^2\right]$

So our solution becomes:

$$\boxed{T(r) = T_0 - \frac{4q''}{kr_0}\left(\frac{3r_0^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_0^2}\right)} \quad (2)$$

$\Rightarrow$ Quartic temperature profile for $q'' = \text{const.}$

Solving for T_m:

$$T_m = \frac{\int_0^{r_0} u T 2\pi r\, dr}{\pi r_0^2 \bar{u}} \Rightarrow \boxed{T_m = T_0 - \frac{11}{24}\cdot\frac{q'' r_0}{k}}$$

Note, $(T_m - T_0) = -\frac{11}{24}\cdot\frac{q'' r_0}{k} = \text{constant}$ $\Rightarrow$ makes sense according to our formulation.

Nenad Miljkovic

Solving for Nusselt number:

$$h = \frac{q''}{T_o - T_m} = \frac{24k}{11 r_o} = \frac{48k}{11D}$$

$$Nu_o = \frac{hD}{k} = \frac{48}{11} = 4.364$$

$\boxed{Nu_o = 4.364}$ ⇒ Laminar flow in tube with constant heat flux.

Note, initially we solved that $Nu_o \sim 2$. Pretty close right!

<u>Uniform Wall Temperature</u> (T_o = constant)

For constant wall temperature, the solution proves difficult to solve analytically. Typically numerical solutions are obtained.

$$q(x) = h\left[T_o - T_m\right] \Rightarrow T_m = f(x),\ T_o = \text{constant}$$

we know from before that:

$$\frac{\partial T_m}{\partial x} = \frac{2q''}{r_o \rho c_p \bar{u}} \Rightarrow q'' = \frac{r_o \rho c_p \bar{u}}{2}\frac{\partial T_m}{\partial x} = h\left[T_o - T_m\right]$$

let $\theta = T_o - T_m \Rightarrow d\theta = -dT_m$

$$-\frac{r_o \rho c_p \bar{u}}{2}\frac{\partial \theta}{\partial x} = h\theta \Rightarrow \text{We can solve this}$$

$$\int_{\theta_1}^{\theta} \frac{\partial \theta}{\theta} = -\int_{x_1}^{x} \frac{2h}{r_o \rho c_p \bar{u}}\, dx$$

$$\ln\left(\frac{T_o - T_m}{T_o - T_{m,1}}\right) = -\frac{2h}{r_o \rho c_p \bar{u}}(x - x_1)$$

$$T_o - T_m = (T_o - T_{m,1}) \exp\left(-\frac{2h}{r_o \rho c_p \bar{u}}(x - x_1)\right) \Rightarrow \frac{1}{\rho c_p} = \frac{\alpha}{k}$$

$$\boxed{T_o - T_m = (T_o - T_{m,1}) \exp\left(-\frac{\alpha Nu}{r_o^2 \bar{u}}(x - x_1)\right)} \quad (1)$$

↳ The temperature difference decreases exponentially in the flow direction. As does the heat flux, q''

Nenad Miljkovic

Now, we can say that: $T = T_o - \phi(T_o - T_m)$ ← $\phi(r)$ only $\neq f(x)$ scaling term.

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[T_o - \phi(T_o - T_m)\right] = -\frac{\partial \phi}{\partial x}(T_o - T_m) + \phi\frac{\partial T_m}{\partial x}$$

But since we have a fully developed flow; $\frac{\partial \phi}{\partial x} = 0$

$$\frac{\partial T}{\partial x} = \phi\frac{\partial T_m}{\partial x}$$

Back substituting everything into our energy equation:

$$\boxed{-2Nu(1-r^2)\phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial \phi}{\partial r}} \quad ②$$

↳ Similar to our q'' = constant case but now ϕ term appears on the LHS.

We also need to use both boundary conditions to solve this numerically

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{at } r = 0 \quad ③$$

$$\phi = 0 \quad \text{at } r = r_o \quad ④$$

Our last condition needed is our definition of Nu, since $\phi = f(r, Nu)$ in this problem:

$$Nu = \frac{hD}{k} = \frac{q''}{\Delta T}\cdot\frac{D}{k} = \frac{-k\left.\frac{\partial T}{\partial r}\right|_{r_o}}{\Delta T}\cdot\frac{D}{k} = -\left.\frac{\partial T}{\partial r}\right|_{r_o}\cdot D\cdot\frac{1}{T_o - T_m}$$

We know $T = T_o - \phi(T_o - T_m)$

$$\frac{\partial T}{\partial r} = -\frac{\partial \phi}{\partial r}(T_o - T_m) + \phi\underbrace{\cancel{\frac{\partial}{\partial r}(T_o - T_m)}}_{0 \Rightarrow T_o,\, T_m \neq f(r)}$$

$$Nu = \cancel{-}\left(\cancel{-}\frac{\partial \phi}{\partial r}\right)\cancel{(T_o - T_m)}\cdot D\cdot\frac{1}{\cancel{(T_o - T_m)}}$$

$$\boxed{Nu = 2r_o\cdot\frac{\partial \phi}{\partial r}} \quad ⑤$$

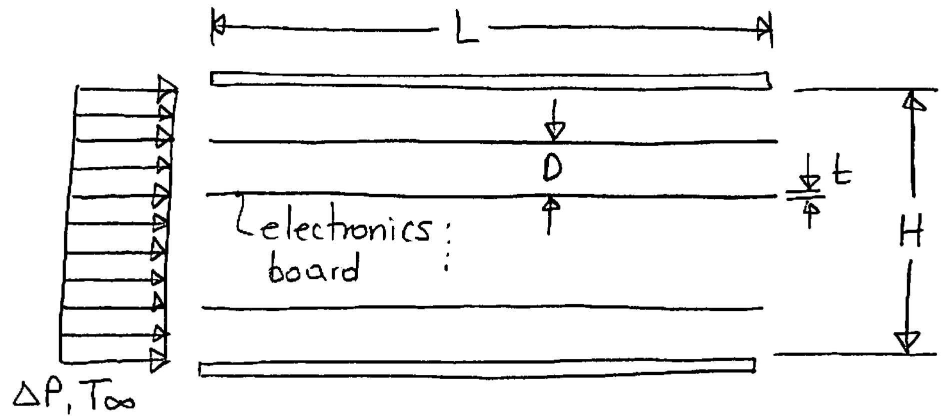
To solve ②, we need ③, ④ & ⑤. Solving numerically, we obtain:

$$\boxed{Nu_o = \frac{hD}{k} = 3.66}$$ ⇒ Laminar flow in tube with constant wall temperature.

Nenad Miljkovic

Stack of Heat Generating Plates (Electronics cooling example)

Think of a stack (or rack) of electronics circuit boards



Incoming cooling fluid has a constant T_∞, and pressure drop budget from a supply fan or pump.

Assuming the flow is laminar, and board temperature T_w = constant (< the electronics component breakdown temperature). Also: $t \ll D$.

$$n = \frac{H}{D} \quad (\text{assuming } t \ll D)$$

Small Spacing limit: $D \to 0$

For $D \to 0$, we can say each channel becomes fully developed rapidly and remains fully developed for all L. Also, the fluid outlet temperature approaches T_w.

$$\bar{u} = \frac{D^2}{12\mu} \cdot \frac{\Delta P}{L} \Rightarrow \text{pg. 123 of notes} \Rightarrow \text{fully developed channel flow.}$$

$$\dot{m}' = \rho \bar{u} H = \rho H \frac{D^2}{12\mu} \cdot \frac{\Delta P}{L} \quad (\dot{m}' = \text{mass flow rate per unit depth})$$

For total heat transfer, since $T = T_w$ at $x = L$

$$\boxed{q'_a = \dot{m}' c_p (T_w - T_\infty) = \rho H \frac{D^2}{12\mu} \cdot \frac{\Delta P}{L} \cdot c_p (T_w - T_\infty)}$$

So $q'_a \sim D^2$ for the small spacing limit.