We also know from before that: \( \frac{\partial T}{\partial x} = \frac{2T_m}{\partial x} = \frac{2g''}{\rho c \rho_0 U} \) = constant

Back substituting \( u(r) \) into our PDE
\[
2U \left[ 1 - \frac{r^2}{r_o^2} \right] \frac{\partial T}{\partial x} = \frac{x}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \Rightarrow \text{PDE since } T(x,r)
\]

Back substituting \( \partial T/\partial x \):
\[
\left( 1 - \frac{r^2}{r_o^2} \right) \frac{4g''}{\rho c \rho_0 U} = \frac{k}{\rho c r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \Rightarrow \text{ODE} \Rightarrow \text{Solvable}
\]

\[
\left( 1 - \frac{r^2}{r_o^2} \right) \frac{4g''}{kr_o} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \tag{1}
\]

Integrating once, we obtain
\[
x \frac{\partial T}{\partial r} = \frac{4g''}{kr_o} \left[ \frac{r^2}{2} - \frac{r^4}{4r_o^2} \right] + C_1
\]
\[
\frac{\partial T}{\partial r} = \frac{4g''}{4r_o} \left[ \frac{r}{2} - \frac{r^3}{4r_o^2} \right] + C_1 \quad \Rightarrow \quad C_1 = 0 \quad \text{since} \quad \frac{C_1}{r} \to \infty \quad \text{as} \quad r \to 0
\]

Integrating once more:
\[
T(r) = \frac{4g''}{kr_o} \left[ \frac{r^2}{4} - \frac{r^4}{16r_o^2} \right] + C_2
\]

Using our second B.C.: \( T(r=r_0) = T_0 \) \( \Rightarrow \) \( C_2 = T_0 - \frac{4g''}{kr_o} \left[ \frac{3}{16} r_o^2 \right] \)

So our solution becomes:
\[
T(r) = T_0 - \frac{4g''}{kr_o} \left( \frac{3r_o^2}{16} - \frac{r^2}{4} + \frac{r^4}{16r_o^2} \right) \tag{2}
\]

\( \Rightarrow \) Quartic temperature profile for \( g''=\text{const.} \)

Solving for \( T_m \):
\[
T_m = \int_0^{r_o} u(r) T r dr = \frac{1}{\pi r_o^2 U} \int_0^{r_o} u(r) T dr
\]

\[
\Rightarrow \quad T_m = T_0 - \frac{11}{24} \cdot \frac{g''r_o}{k}
\]

Note, \( (T_m - T_0) = -\frac{11}{24} \cdot \frac{g''r_o}{k} = \text{constant} \Rightarrow \text{makes sense according to our formulation.} \)
Solving for Nusselt number:

\[ h = \frac{q''}{T_0 - T_m} = \frac{24k}{11r_o} = \frac{48k}{11D} \]

\[ \text{Nu}_0 = \frac{hD}{k} = \frac{48}{11} = 4.364 \]

\[ \text{Nu}_0 = 4.364 \Rightarrow \text{Laminar flow in tube with constant heat flux.} \]

Note, initially we solved that \( \text{Nu}_0 \approx 2 \). Pretty close right!

**Uniform Wall Temperature \((T_0 = \text{constant})\)**

For constant wall temperature, the solution proves difficult to solve analytically. Typically, numerical solutions are obtained.

\[ q(x) = h \left[ T_0 - T_m \right] \Rightarrow T_m = f(x), T_0 = \text{constant} \]

we know from before that:

\[ \frac{2T_m}{\partial x} = \frac{2q''}{r_o \rho c_p U} \Rightarrow q'' = \frac{r_o \rho c_p U}{2} \frac{\partial T_m}{\partial x} = h \left[ T_0 - T_m \right] \]

let \( \theta = T_0 - T_m \Rightarrow d\theta = -dT_m \)

\[ -\frac{r_o \rho c_p U}{2} \frac{\partial \theta}{\partial x} = h \theta \Rightarrow \text{We can solve this} \]

\[ \int_{\theta_i}^{\theta} \frac{\partial \theta}{\theta} = -\int_{x_i}^{x} \frac{2h}{r_o \rho c_p U} dx \]

\[ \ln \left( \frac{T_0 - T_m}{T_0 - T_{m,1}} \right) = -\frac{2h}{r_o \rho c_p U} (x-x_1) \]

\[ T_0 - T_m = (T_0 - T_{m,1}) \exp \left( -\frac{2h}{r_o \rho c_p U} (x-x_1) \right) \Rightarrow \frac{1}{\rho c_p} = \frac{\alpha}{k} \]

\[ T_0 - T_m = (T_0 - T_{m,1}) \exp \left( -\frac{\alpha Nu}{r_o^2 U} (x-x_1) \right) \]

The temperature difference decreases exponentially in the flow direction. As does the heat flux, \( q'' \)
Now, we can say that: \[ T = T_o - \phi (T_o - T_m) \]

\[
\frac{2T}{\delta x} = \frac{2}{\delta x} \left[ T_o - \phi (T_o - T_m) \right] = -\phi \frac{\partial T}{\partial x} (T_o - T_m) + \phi \frac{2T_m}{\delta x}
\]

But since we have a fully developed flow: \[ \frac{\partial \phi}{\partial x} = 0 \]

\[ \frac{2T}{\delta x} = \phi \frac{2T_m}{\delta x} \]

Back substituting everything into our energy equation:

\[
-2Nu (1 - r^2) \phi = \frac{2^2 \phi}{\delta r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}
\]  \( \text{Equation 2} \)

\[
\Rightarrow \text{Similar to our g''=constant case but now } \phi \text{ term appears on the LHS.}
\]

We also need to use both boundary conditions to solve this numerically

\[ \frac{\partial \phi}{\partial r} = 0 \text{ at } r = 0 \]  \( \text{Equation 3} \)

\[ \phi = 0 \text{ at } r = r_o \]  \( \text{Equation 4} \)

Our last condition needed is our definition of Nu, since \[ \phi = f(r, Nu) \text{ in this problem:} \]

\[
Nu = \frac{h_0}{k} = \frac{g''}{\Delta T} \cdot \frac{D}{k} = -\frac{h_0}{\Delta T} \frac{\partial T}{\partial r} \left|_{r_o} \right. \cdot 0 \cdot \frac{1}{(T_o - T_m)}
\]

We know \[ T = T_o - \phi (T_o - T_m) \]

\[ \frac{\partial T}{\partial r} = -\frac{\partial \phi}{\partial r} (T_o - T_m) + \phi \frac{2T_m}{\partial r} (T_o - T_m) \]

\[ Nu = \chi \frac{\partial \phi}{\partial r} (T_o - T_m) \cdot 0 \cdot \frac{1}{(T_o - T_m)} \]

\[ Nu = 2r_o \cdot \frac{\partial \phi}{\partial r} \]  \( \text{Equation 5} \)

To solve (2), we need 3, 4 \& 5. Solving numerically, we obtain:

\[ Nu_0 = \frac{h_0}{k} = 3.66 \Rightarrow \text{Laminar flow in tube with constant wall temperature.} \]
Stack of Heat Generating Plates (Electronics cooling example)

Think of a stack (or rack) of electronics circuit boards

\[ \Delta P = \frac{\rho L}{D} \]

Incoming cooling fluid has a constant \( T_\infty \), and pressure drop budget from a supply fan or pump.

Assuming the flow is laminar, and board temperature \( T_w = \text{constant} \) (< the electronics component breakdown temperature). Also: \( \Delta T \ll 0 \).

\[ n = \frac{H}{D} \quad (\text{assuming } \Delta T \ll 0) \]

Small Spacing Limit: \( D \to 0 \)

For \( D \to 0 \), we can say each channel becomes fully developed rapidly and remains fully developed for all \( L \). Also, the fluid outlet temperature approaches \( T_w \).

\[ \bar{U} = \frac{D^2}{12\mu L} \Rightarrow \rho g \] of notes \( \Rightarrow \) fully developed channel flow.

\[ m' = \rho \bar{U} H = \rho H \frac{D^2}{12\mu} \Delta \frac{p}{L} \quad (m' = \text{mass flow rate per unit depth}) \]

For total heat transfer, since \( T = T_w \) at \( x = L \)

\[ q'_a = m' \rho (T_w - T_\infty) = \rho H \frac{D^2}{12\mu} \Delta \frac{p}{L} \cdot \rho (T_w - T_\infty) \]

So \( q'_a \sim D^2 \) for the small spacing limit.