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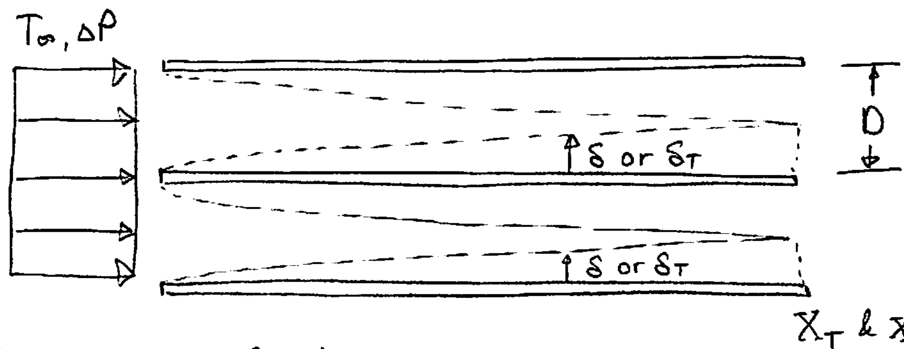
Large Spacing Limit: $D \to \infty$

For large spacing, each channel looks like the channel entrance region for the whole length:



Here, ΔP is fixed, so we need to solve for U_∞ that can achieve entrance effects for the whole plate.

A force balance on the whole control volume $(H \times L)$ reveals:

$$\underbrace{\Delta P \cdot H}_{\text{Pressure Drop}} = \underbrace{n \cdot 2\overline{\tau}_o \cdot L}_{\text{Total shear force}} \quad \Rightarrow \overline{\tau}_o = \text{averaged shear stress over } L.$$

$$\overline{\tau}_o = 1.328\, Re_L^{-1/2} \cdot \frac{1}{2} \rho U_\infty^2$$

Back substituting:

$$U_\infty = \left(\frac{1}{1.328} \cdot \frac{\Delta P\, H}{n L^{1/2} \rho \nu^{1/2}} \right)^{2/3}$$

For the overall heat transfer rate from one board

$$\frac{\overline{h} L}{k} = \frac{q''}{T_w - T_\infty} \cdot \frac{L}{k} = 0.664\, Pr^{1/3} \left(\frac{U_\infty L}{\nu} \right)^{1/2} \quad ; \quad Pr > 0.5$$

$$q_1' = q'' \cdot L = k \left(T_w - T_\infty \right) 0.664\, Pr^{1/3} \left(\frac{U_\infty L}{\nu} \right)^{1/2}$$

Assuming both sides are heating and maintained at T_w:

$$q_b' = 2n q_1' = 2nk \left(T_w - T_\infty \right) 0.664\, Pr^{1/3} \left(\frac{U_\infty L}{\nu} \right)^{1/2}$$

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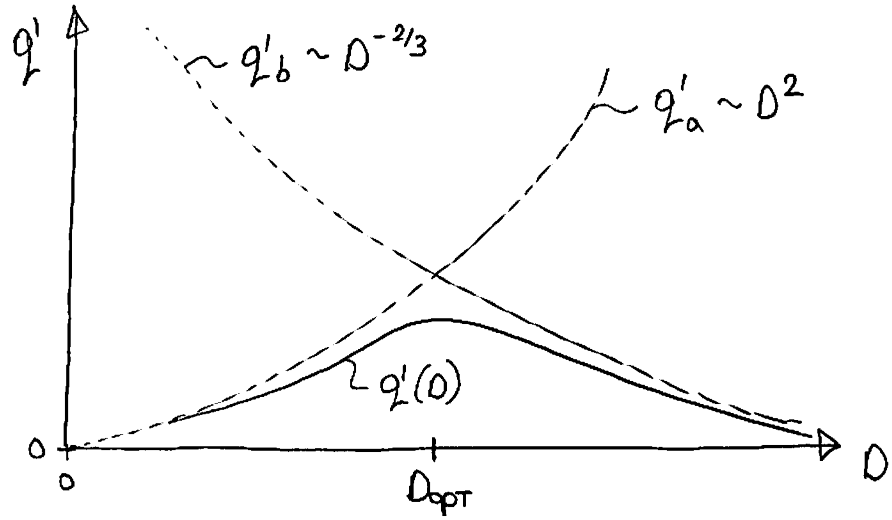
But we know that $n = \frac{H}{D}$, and we solved for U_∞

$$\boxed{q_b' = 1.208k\,(T_w - T_\infty)\,H\,\frac{Pr^{1/3}\,L^{1/3}\,\Delta P^{1/3}}{\rho^{1/3}\,\nu^{2/3}\,D^{2/3}}}$$

We see that $q'_b \sim D^{-2/3}$ in the large spacing limit.

This problem is a classical example of solution via intersection of asymptotes. We can do this since our two solutions are limiting cases only, and in between, mixed behaviour will occur:



To obtain our solution, we can equate the two solutions via scaling

$$q_a' \sim q_b' \Rightarrow \cancel{\rho H}\frac{D^2}{12\mu}\cdot\frac{\Delta P}{L}\,c_p\,\cancel{(T_w - T_\infty)} \sim 1.08k\,\cancel{(T_w - T_\infty)}\cancel{H}\frac{(Pr\,L\,\Delta P)^{1/3}}{\rho^{1/3}\nu^{2/3}D^{2/3}}$$

$$\boxed{D_{opt} \cong 2.73\,L\,Be_L^{-1/4}} \text{ for } 0.7 < Pr < 10^3$$

$$\boxed{Be_L = \frac{\Delta P L^2}{\mu\alpha}} = \text{Bejan \# (Dimensionless } \Delta P)$$

Note: $D_{opt,exp} = 3.05\,L\,Be_L^{-1/4}$.
Results show for this solution that the board length (L) is of the same order of magnitude as the thermal entrance length (X_T).

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Solving for our maximum heat transfer at D_{opt}, we obtain:

$$\boxed{q'_{max} \leq 0.62 \left(\frac{\rho \Delta P}{Pr}\right)^{1/2} \cdot H c_p (T_w - T_\infty)} \Rightarrow T_w = \text{constant}$$

Two other cases that are useful to know are $q'' = \text{constant}$

$$\boxed{D_{opt} = 3.2\, L\, Be_L^{-1/4}} \Rightarrow q'' = \text{constant}$$

$$\boxed{q'_{max} \doteq 0.4 \left(\frac{\rho \Delta P}{Pr}\right)^{1/2} \cdot H \cdot c_p (T_{w,L} - T_\infty)} \; ; \; T_{w,L} = T_w(L) \Rightarrow \text{max temp.}$$

For one side of each board $T_w = \text{constant}$, and the other adiabatic

$$\boxed{D_{opt} = 2.10\, L\, Be_L^{-1/4}}$$

$$\boxed{q'_{max} \leq 0.37 \left(\frac{\rho \Delta P}{Pr}\right)^{1/2} H c_p (T_w - T_\infty)} \Rightarrow \begin{array}{l} T_w = \text{constant on 1 side} \\ q'' = 0 \text{ on other.} \end{array}$$

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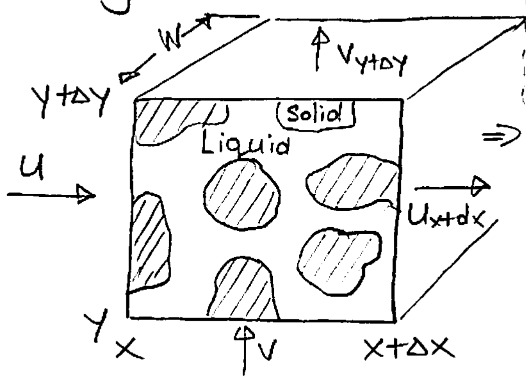
Convection in Porous Media

Relatively old problem due to need to manage the water table for irrigation systems.

Assuming we have a homogeneous porous medium:



⇒ We can estimate the flow to be 2D but locally it is always 3D in nature ⇒ just like turbulence

We assume $W \gg \Delta x$ and $W \gg \Delta y$ (2D flow). Only rates in the x & y directions are important:

$$\underbrace{\Delta x W}_{y\text{ cross-sect.}} \text{ and } \underbrace{\Delta y W}_{x\text{-cross sect.}} \gg \underbrace{\Delta x \Delta y}_{z\text{ cross section}}$$

$$\dot{m}_x = \rho \int_y^{y+\Delta y} \int_0^W u_p \, dz \, dy$$ ⇒ $u_p \equiv$ uneven x-velocity distribution over void patches in x-plane.

To make our lives simpler, we can determine the area averaged x-velocity:

$$u = \frac{1}{W \Delta y} \int_0^{y+\Delta y} \int_0^W u_p(y,z) \, dz \, dy \quad \Rightarrow \quad \dot{m}_x = \rho u (W dy) \quad (1)$$

For the y-direction, we can do the same:

$$v = \frac{1}{W \Delta x} \int_0^{x+\Delta x} \int_0^W v_p(x,z) \, dz \, dx \quad \Rightarrow \quad \dot{m}_y = \rho v (W \Delta x) \quad (2)$$

Note, we've assumed that ρ is constant in the $\Delta x \Delta y$ domain, not necessarily over the entire x,y domain.

Applying mass conservation:

$$\frac{\partial M_{cv}}{\partial t} = \sum_{inlet} \dot{m} - \sum_{outlet} \dot{m} \quad ③ \Rightarrow M_{cv} \text{ is the instantaneous mass of the CV.}$$

We can define $M_{cv} = \rho W \Delta x \Delta y \phi$ ④

$$\boxed{\phi \equiv \text{porosity or void fraction} = \frac{\text{void volume}}{\text{total volume}}}$$

Combining ③ & ④, we obtain:

$$\frac{\partial}{\partial t}(\rho \phi W \Delta x \Delta y) + \frac{\partial \dot{m}_x}{\partial x}\Delta x + \frac{\partial \dot{m}_y}{\partial y}\Delta y + HOT(\Delta x^n, \Delta y^n) = 0$$

Back substituting ① and ②

$$\frac{\partial}{\partial t}(\rho \phi W \cancel{\Delta x \Delta y}) + \frac{\partial}{\partial x}(\rho u W \cancel{\Delta y})\cancel{\Delta x} + \frac{\partial}{\partial y}(\rho v W \cancel{\Delta x})\cancel{\Delta y} + H.O.T. = 0$$

Divide through by $\Delta x \Delta y$ and let $\Delta x, \Delta y \to 0$; $H.O.T. \to 0$

$$\phi \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

In general: $\boxed{\phi \frac{\partial \rho}{\partial t} + \nabla(\rho V) = 0}$

$V \equiv$ volume averaged velocity vector (u, v, w)

Note, if medium is a pure fluid, $\phi = 1$, and we return back to our original definition of mass conservation.

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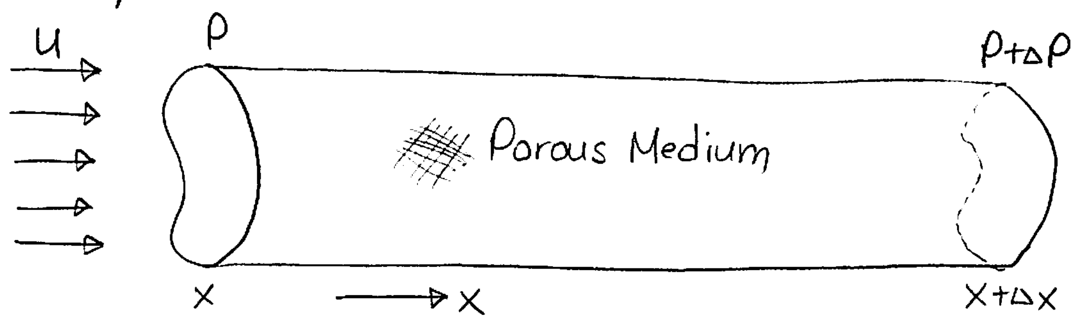
Darcy's Law

The main constitutive relation for flow through porous media is Darcy's law. Darcy, a french hydrologist emperically derived it by studying the flow in sand packed beds. It is the equivalent to Fourier's law in heat transfer:

$$\boxed{u = \frac{K}{\mu}\left(-\frac{\partial P}{\partial x}\right)} \quad (1), \quad K \equiv \text{permeability}$$

This law can be derived from Navier-Stokes equation, however it requires the permeability tensor and is skipped here.



Looking at the dimension of K:

$$[K] = \frac{[\mu][u]}{[-\partial P/\partial x]} = (\text{length})^2$$

Note the similarity of equation (1) & our Hagen-Poiseulle flow solution; For a pipe and channel

$$\underbrace{\underset{\text{Pipe}}{\bar{U} = \frac{r_o^2}{8\mu}\left(-\frac{\partial P}{\partial x}\right)} \qquad \underset{\text{Channel}}{\bar{U} = \frac{D^2}{12\mu}\left(-\frac{\partial P}{\partial x}\right)}}_{K \sim r_o^2 \sim D^2}$$

So: $K^{1/2} \equiv$ length scale of the pore diameter.

By assuming a small scale bundle of channels with H-P flow, we can derive (1) from it.

Defining our Reynolds number based on our pore scale:

$$\boxed{Re = \frac{\rho u K^{1/2}}{\mu}} \quad (2)$$

And our porous flow friction factor:

$$\boxed{f = \frac{\left(-\frac{\partial P}{\partial x}\right) K^{1/2}}{\rho u^2}} \quad (3)$$

$\Rightarrow$ Before for H-P flow, we had $f = \dfrac{\left(-\frac{\Delta P}{L}\right) D^{1/2}}{\frac{1}{2}\rho u^2}$

Back substituting our definition for u (1):

$$Re = \frac{\rho \frac{K}{\mu}\left(-\frac{\partial P}{\partial x}\right) K^{1/2}}{\mu} = \frac{\rho K^{3/2}\left(-\frac{\partial P}{\partial x}\right)}{\mu^2} \quad (4)$$

$$f = \frac{\cancel{\left(-\frac{\partial P}{\partial x}\right)} \cancel{K}^{1/2}}{\rho\left(\frac{K^{\cancel{2}3/2}}{\mu^2}\left(-\frac{\partial P}{\partial x}\right)^{\cancel{2}}\right)} = \frac{\mu^2}{\rho K^{3/2}\left(-\frac{\partial P}{\partial x}\right)} \quad (5)$$

We see that: $\boxed{f = \frac{1}{Re}}$ $\Rightarrow$ Second form of Darcy's Law.

$\hookrightarrow$ Valid for laminar flow $\Rightarrow Re \leq 10$

Note, for $Re > 10$, inertia becomes important and we can use the <u>Forchheimer</u> modification:

$$\boxed{-\frac{\partial P}{\partial x} = \frac{\mu}{K} u + b\rho |u| u}$$

$b \equiv$ emperical constant based on geometry of pores.

From this stems:

$$\boxed{f = \frac{1}{Re} + 0.55} \quad \Rightarrow Re > 10$$

If we have gravity present (body force $= \rho g_x$)

$$u = \frac{K}{\mu}\left(-\frac{\partial P}{\partial x} + \rho g_x\right)$$

$\Rightarrow$ Note, $u = 0$ when $\frac{\partial P}{\partial x} = \rho g_x$
Pressure matches hydrostatics.

or in 3D:

$$\boxed{V = \frac{K}{\mu}\left(-\nabla P + \rho g\right)} \; ; \quad V = (u, v, w), \quad g = (g_x, g_y, g_z)$$

In many typical problems involving <u>seepage flow</u> of water in the ground, ρ and μ = constant, $g = (0, -g, 0)$

$$V = -\frac{K}{\mu}\nabla \mathcal{E} \quad \Rightarrow \quad \mathcal{E} = P + \rho g y$$

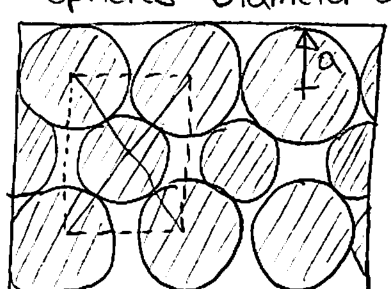
Our mass conservation for ρ = const. becomes: $\nabla \cdot V = 0$
Combining, we obtain:

$$\boxed{\nabla^2 \mathcal{E} = 0} \iff \boxed{\nabla^2 T = 0} \iff \boxed{\nabla^2 E = 0}$$

Can solve the seepage problem using steady state heat conduction.

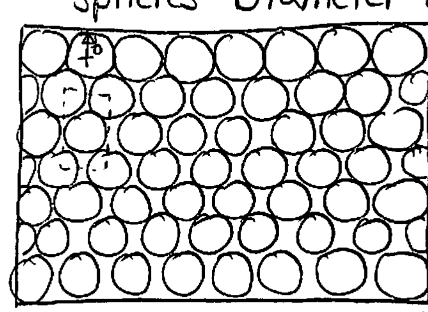
Note, one important thing to discuss is the <u>difference</u> between <u>pemeability</u> and <u>porosity.</u>

Spheres Diameter a



or

Spheres Diameter b



$$\phi_a \equiv \text{porosity} = \frac{2(\pi \cancel{a^2})}{(2\cancel{a})(3.464\cancel{a})}$$

$$\phi_a = 0.91$$
$$K_a \approx a$$

$$\phi_b \equiv \text{porosity} = \frac{2(\pi \cancel{b^2})}{(2\cancel{b})(3.464\cancel{b})}$$

$$\phi_b = 0.91 = \phi_a$$
$$K_b \approx b$$