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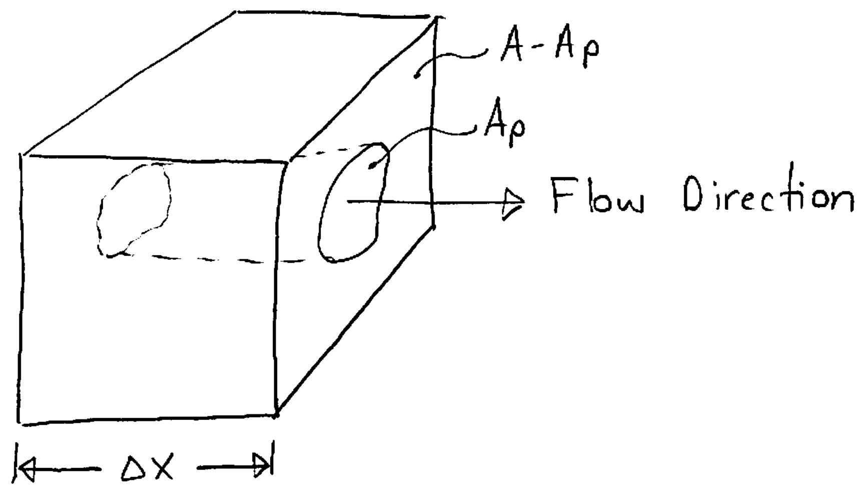
Heat Transfer in Porous Media

Let's consider a simple 1-D model in order to obtain our energy equation in porous media:



According to our previous definition: $\phi = \frac{A_p \Delta x}{A \Delta x}$ ①

For the solid domain:

heat generation/volume

$$\rho_s c_s \frac{\partial T}{\partial t} = k_s \frac{\partial^2 T}{\partial x^2} + q_s''' \Rightarrow \rho_s, c_s, k_s \Rightarrow \text{solid properties}$$

If T = constant over the solid volume, we can integrate volume-
-trically

$$\Delta x (A - A_p) \rho_s c_s \frac{\partial T}{\partial t} = \Delta x (A - A_s) k_s \frac{\partial^2 T}{\partial x^2} + \Delta x (A - A_p) q_s''' \quad ②$$

For the fluid domain:

$$\rho_f c_{p,f} \left(\frac{\partial T}{\partial t} + u_p \frac{\partial T}{\partial x} \right) = k_f \frac{\partial^2 T}{\partial x^2} + \mu \Phi \quad ③$$

Note we neglected our compressibility term $\left(\beta T \frac{DP}{Dt}\right) \Rightarrow$ incompressibl.

Note, T here is the same in the fluid as the solid. Local thermal equilibrium. Not always true though so be carifull when using it.

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For example $\Rightarrow$ nuclear reactors or electronics cooling : $T_f \neq T_s$.
If no Local Equilibrium, we must solve the fluid & solid coupled equations.
Integrating eq. 3 over the pore volume : Note : $A u = \iint_{A_p} u_p \, dA_p$

$$\Delta x A_p \rho_f c_{pf} \frac{\partial T}{\partial t} + \Delta x A \rho_f c_{pf} u \frac{\partial T}{\partial x} = \Delta x A_p k_f \frac{\partial^2 T}{\partial x^2} + \Delta x \mu \iint_{A_p} \Phi \, dA_p \quad (4)$$

Our last term represents the internal heating due to viscous dissipation.

Since this is a loss term ($E_{out} = E_{in}$), then the last term is equal to the mechanical power required to drive the flow.

We can use fluid mechanics to show that:

$$\underbrace{\Delta x \mu \iint_{A_p} \Phi \, dA_p}_{\text{Viscous heating}} = \underbrace{A u \left(-\frac{\partial P}{\partial x} + \rho_f g_x \right)}_{\text{Mechanical work input}} \Delta x \quad (5)$$

Back substituting 5 into 4 and adding 2 $\Rightarrow$ divide out by $A \Delta x$

$$\boxed{\left[\phi \rho_f c_{pf} + (1-\phi) \rho_s c_s \right] \frac{\partial T}{\partial t} + \rho_f c_{pf} u \frac{\partial T}{\partial x} = \left[\phi k_f + (1-\phi) k_s \right] \frac{\partial^2 T}{\partial x^2} + (1-\phi) q_s''' + u \left(-\frac{\partial P}{\partial x} + \rho_f g_x \right)}$$

$\hookrightarrow$ Porous media energy equation (incompressible)

$$\boxed{k = \phi k_f + (1-\phi) k_s} \Rightarrow \text{porous medium thermal conduct.}$$

Note, since we added 2 & 4, this assumes a parallel model, hence k is valid for this assumption. In general, k needs to be experimentally measured.

We can define a capacity ratio of our medium as:

$$\boxed{\sigma = \frac{\phi \rho_f c_{pf} + (1-\phi)\rho_s c_s}{\rho_f c_{pf}}}$$

⇒ capacity ratio
Doesn't mean much, more for notation as we will see.

With this new notation, we can write: Note: $q''' = (1-\phi) q_s'''$

$$\boxed{\rho_f c_{pf}\left(\sigma \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x}\right) = k \frac{\partial^2 T}{\partial x^2} + q''' + \frac{\mu}{K} u^2}$$

In general for 3D:

$$\boxed{\rho_f c_{pf}\left(\sigma \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T\right) = k \nabla^2 T + q''' + \frac{\mu}{K} (\vec{v})^2}$$

↳ $\vec{v} = (u, v, w) \equiv$ volume averaged velocity vector

If heat generation & viscous heating are negligible:

$$\sigma \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

Where:

$$\boxed{\alpha = \frac{k}{\rho_f c_{pf}}} \equiv \text{homogeneous porous medium thermal diff.}$$

<u>Note</u>: Our main assumptions here are:

1) Homogeneous medium: solid material & fluid is distributed evenly in the medium.
2) Isotropic medium: i.e. K & $k \neq f(\text{direction})$. Note, if not the case, then:

$$u = \frac{K_x}{\mu}\left(-\frac{\partial P}{\partial x} + \rho g_x\right), \quad v = \frac{K_y}{\mu}\left(-\frac{\partial P}{\partial y} + \rho g_y\right), \quad w = \frac{K_z}{\mu}\left(-\frac{\partial P}{\partial z} + \rho g_z\right)$$

$$\sigma \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \alpha_x \frac{\partial^2 T}{\partial x^2} + \alpha_y \frac{\partial^2 T}{\partial y^2} + \alpha_z \frac{\partial^2 T}{\partial z^2}$$

3) At any point in the medium, the solid & fluid are in local thermal equilibrium.
4) Local $Re = \frac{\rho u K^{1/2}}{\mu} < 10$ (Darcy's Law is Applicable) Laminar flow.

Non-LTE Heat Transfer

If the fluid and solid are not in thermal equilibrium, we must solve a coupled set of equations:

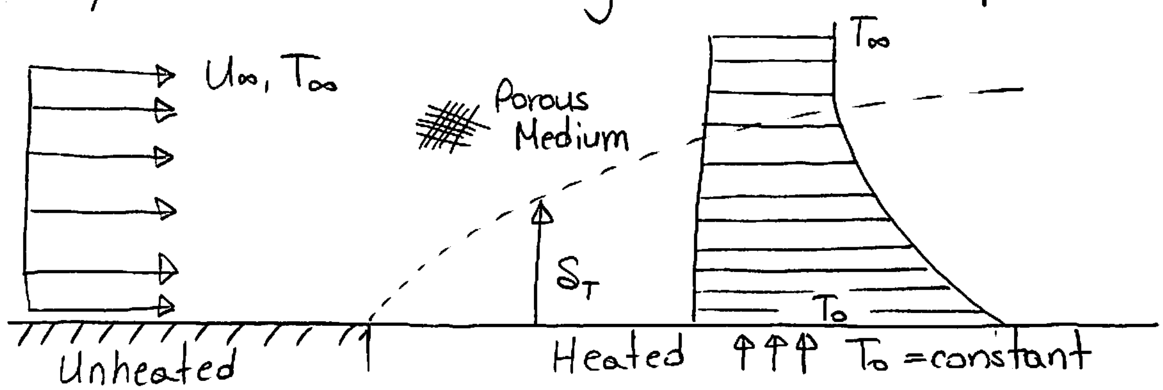
$$(1-\phi)\rho_s c_s \frac{\partial T_s}{\partial t} = (1-\phi)k_s \frac{\partial^2 T_s}{\partial x^2} + h\,(T_f - T_s) \quad ①$$

$$\phi \rho_f c_{pf} \frac{\partial T_f}{\partial t} + \rho_f c_{pf}\, u \frac{\partial T}{\partial x} = \phi k_f \frac{\partial^2 T_f}{\partial x} - h\,(T_f - T_s) \quad ②$$

Need to solve ① and ② simoutaneously. Note, this example still assumes incompressible, viscous heating is negligible, and zero volumetric heating.

Boundary Layers

Let's say we had the following situation (incompressible flow)



Our governing equations become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(mass conservation)}$$

$$u = -\frac{K}{\mu}\frac{\partial P}{\partial x} \quad ; \quad v = -\frac{K}{\mu}\frac{\partial P}{\partial y}$$

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Considering that the b.l. is slender; with uniform parallel flow:

$$u = U_\infty, \quad v = 0, \quad P(x) = -\frac{\mu}{K} U_\infty x + \text{const}$$

↳ Solved by integrating Darcy's eqn.

Applying a scaling analysis: $u \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial y^2}$ (energy eqn.)

$u \sim U_\infty$; $x \sim x$
$\partial T \sim \Delta T$; $y \sim \delta_T$
$\partial^2 T \sim \Delta T$

$$U_\infty \frac{\Delta T}{x} \sim \alpha \frac{\Delta T}{\delta_T^2} \Rightarrow \boxed{\frac{\delta_T}{x} \sim Pe^{-1/2}}$$

$$\boxed{Nu_x = h\frac{x}{k} \sim \frac{x}{\delta_T} \sim Pe^{1/2}} \qquad \boxed{Pe = \frac{U_\infty x}{\alpha}} \equiv \text{Peclet number}$$

Note, $\alpha \equiv$ thermal diffusivity of the porous medium $= \frac{k}{\rho_f c_{pf}}$

Note also the similarity to the Blasius - Pohlhausen solution for a flat plate. Very similar however no hydrodynamic boundary layer development due to the porous media.

Same as $Pr \ll 1$ case where $u \sim U_\infty$ (see pg. 42 of notes). Only difference is before, we used fluid properties, & here we use porous media properties.

<u>Similarity Solution</u> (T_0 = constant ≡ wall temperature)

Assuming the following similarity variable:

$$\eta = \frac{y}{x} Pe_x^{1/2}, \quad \theta(\eta) = \frac{T - T_0}{T_\infty - T_0}$$

Converting our energy equation

$$\left.\begin{aligned} \partial T &= (T_\infty - T_0)\partial\theta \\ \partial^2 T &= (T_\infty - T_0)\partial^2\theta \\ u &= U_\infty \end{aligned}\right\} \text{ Back substitute into energy equation}$$

$$u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2} \iff U_\infty \frac{\partial\theta}{\partial x}\cancel{(T_\infty - T_0)} = \alpha\frac{\partial^2\theta}{\partial y^2}\cancel{(T_\infty - T_0)}$$

Now we need to convert from $x \to \eta$

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} \Rightarrow \frac{\partial\eta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}Pe_x^{1/2}\right) = -\frac{y}{2}\left(\frac{U_\infty}{\alpha x^3}\right)^{1/2}$$

$$\frac{\partial\theta}{\partial x} = -\frac{y}{2}\left(\frac{U_\infty}{\alpha x^3}\right)^{1/2}\theta' \quad (1)$$

Now let's deal with the right hand side:

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}\right) = \frac{\partial}{\partial\eta}\left(\frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}\right)\cdot\frac{\partial\eta}{\partial y} = \frac{\partial^2\theta}{\partial\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2$$

$$\frac{\partial\eta}{\partial y} = \left(\frac{U_\infty}{\alpha x}\right)^{1/2} \Rightarrow \left(\frac{\partial\eta}{\partial y}\right)^2 = \frac{U_\infty}{\alpha x}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{U_\infty}{\alpha x}\theta'' \quad (2)$$

Putting (1) & (2) together:

$$-\cancel{U_\infty}\left(\frac{y}{2}\left(\frac{U_\infty}{\alpha x^3}\right)^{1/2}\right)\theta' = \cancel{\alpha}\,\frac{\cancel{U_\infty}}{\cancel{\alpha} x}\theta''$$

$$-\frac{1}{2}\frac{y}{x}\left(\frac{U_\infty x}{\alpha}\right)^{1/2}\theta' = \theta''$$

$$\boxed{\theta'' + \frac{1}{2}\eta\theta' = 0} \Rightarrow \text{ODE} \Rightarrow \text{Solvable by separation of variables}$$

B.C.'s $\Rightarrow \theta(0) = 0$

$\theta(\eta \to \infty) = 1$

error function. See tables.

Solving, we obtain: $\boxed{\theta(\eta) = \text{erf}\left(\frac{\eta}{2}\right)}$

So for heat transfer: $q'' = -k\left.\frac{\partial T}{\partial y}\right|_{y=0}$

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$$q'' = -k(T_\infty - T_0)\frac{\partial\theta}{\partial y}\bigg|_{y=0} = -k(T_\infty - T_0)\frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0}\cdot\frac{\partial\eta}{\partial y}\bigg|_{y=0}$$

$$\frac{\partial\eta}{\partial y} = \left(\frac{U_\infty}{\alpha x}\right)^{1/2} \quad ; \quad \frac{\partial\theta}{\partial\eta}\bigg|_{\eta=0} = \pi^{-1/2} \quad \text{(from tables or Wolfram)}$$

$$\boxed{q'' = -k(T_\infty - T_0)\pi^{-1/2}\cdot\left(\frac{U_\infty}{\alpha x}\right)^{1/2}}$$ $\Rightarrow$ Wall heat flux
$\Rightarrow T_0 = \text{constant}, Re_k < 10$

For a wall heating a fluid, $T_0 > T_\infty$

$$Nu_x = \frac{q''}{T_0 - T_\infty}\cdot\frac{x}{k} = \frac{\cancel{k}\pi^{-1/2}\left(\frac{U_\infty}{\alpha x}\right)^{1/2}\cdot x}{\cancel{k}} = \left(\frac{U_\infty x}{\pi\alpha}\right)^{1/2}$$

$$\boxed{Nu_x = 0.564\, Pe_x^{1/2}}$$ $\Rightarrow T_0 = \text{constant}, Re_k < 10$, Incompressible

Note, in scaling analysis we had $Nu_x \sim Pe^{1/2}$ (Good!)

$$\boxed{\overline{Nu} = \frac{\bar{h}L}{k} = 1.128\, Pe_L^{1/2}}$$

Constant Heat Flux

For the constant heat flux case, we can do another similarity analysis, but this time with:

$$\left.\begin{aligned} \frac{\partial T}{\partial y} &= -\frac{q''}{k} \quad \text{at } y = 0 \\ T &\to T_\infty \quad \text{at } y \to \infty \end{aligned}\right\} \text{B.C.'s}$$

$$\zeta = y\left(\frac{U_\infty}{\alpha x}\right)^{1/2} \quad ; \quad \tau(\zeta) = \frac{T(x,y) - T_\infty}{(q''/k)(\alpha x/U_\infty)^{1/2}}$$

Doing the math & converting our energy PDE into an ODE we obtain:

$$\tau'' + \frac{1}{2}(\zeta\tau' - \tau) = 0$$

$$\left.\begin{aligned} \tau'(0) &= -1 \\ \tau(\zeta \to \infty) &= 0 \end{aligned}\right\} \text{B.C.'s}$$

Differentiating once, we can do separation of variables

$$\frac{\partial}{\partial \zeta}\left(\tau'' + \frac{1}{2}\left(\zeta\tau' - \tau\right)\right) = 0$$

$$\tau''' + \cancel{\tfrac{1}{2}\tau'} + \frac{1}{2}\zeta\tau'' - \cancel{\tfrac{1}{2}\tau'} = 0$$

$$\tau''' + \frac{1}{2}\zeta\tau'' = 0 \Rightarrow \frac{\partial}{\partial \zeta}\left(\ln(\tau'')\right) = \frac{1}{\tau''}\cdot\frac{\partial}{\partial \zeta}(\tau'') = \frac{\tau'''}{\tau''}$$

$$\boxed{\frac{\tau'''}{\tau''} = -\frac{1}{2}\zeta} \Rightarrow \text{ODE} \Rightarrow \text{Similar to Blasius: } \frac{f'''}{f''} = -\frac{1}{2}f$$

To solve, we integrate 3 times in a row and then apply our B.C.'s: (For a similar integral, see pg. 49a of notes)

$$\boxed{\tau(\zeta) = \frac{2}{\pi^{1/2}}\exp\left(-\frac{\zeta^2}{4}\right) - \zeta\,\mathrm{erfc}\left(\frac{\zeta}{2}\right)} \quad \begin{array}{l} \Rightarrow q'' = \text{const} \\ \Rightarrow Re_x < 10 \end{array}$$

Solving for our Nusselt number (I skipped a few steps here for the sake of brevity):

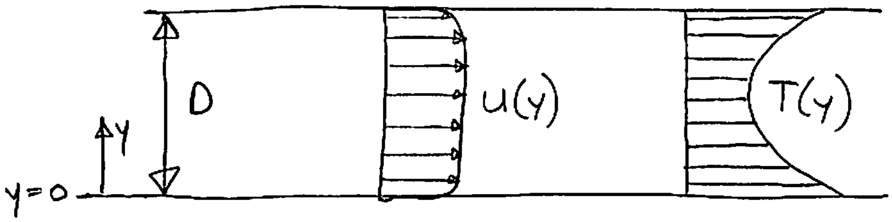
$$\boxed{Nu_x = \frac{q''x}{k\left[T_0(x) - T_\infty\right]} = 0.866\, Pe_x^{1/2}} \quad \begin{array}{l} \Rightarrow \text{Incompressible} \\ \Rightarrow q'' = \text{constant} \\ \Rightarrow Re_x < 10 \end{array}$$

$$\boxed{\overline{Nu} = \frac{q''L}{k\left[\overline{T_0} - T_\infty\right]} = 1.329\, Pe_L^{1/2}} \quad \begin{array}{l} \Rightarrow \text{Averaged Nusselt \#} \\ \Rightarrow \text{Remember, } k, \alpha \text{ are} \\ \quad \text{porous media properties} \end{array}$$

Internal Flows filled with Porous Media

For the case of fully developed flow of a fluid through a porous medium that fills a long channel, we can use our Darcy flow model ⇒ volume averaged velocity u is uniform across the whole channel.

This indicates what we call "slug flow"



⇒ Different from previous analysis where $u(y) \neq$ const.
see pg. 127 of notes

Note, our bulk or mean temperature reduces to:

$$T_m = \frac{1}{\bar{u}A} \cdot \int_A u T \, dA \quad \Rightarrow \text{since } u(y) = \bar{u} = \text{constant} \neq f(y)$$

$$= \frac{\cancel{\bar{u}}}{\cancel{\bar{u}}A} \int_A T \, dA$$

$$\boxed{T_m = \frac{1}{A} \int_A T \, dA}$$

⇒ Mean temperature for slug flow.
⇒ A = channel cross section.

To solve, we can use our previous approach of solving the energy equation for channel or pipe flows with $u = U_\infty = \text{constant} = \frac{k}{\mu}\left(-\frac{\partial P}{\partial x}\right)$

I've skipped the derivations but good to do for homework to check your understanding:

For <u>Tubes</u> (Internal Diameter = D)

$$\boxed{Nu_D = \frac{q''(x)}{T_0 - T_m(x)} \cdot \frac{D}{k} = 5.78} \quad (\text{tube, } T_0 = \text{constant})$$

$$\boxed{Nu_D = \frac{q''}{T_0(x) - T_m(x)} \cdot \frac{D}{k} = 8} \quad (\text{tube, } q'' = \text{constant})$$

For <u>channels</u> (Parallel Plates, spacing = D)

$$\boxed{Nu_D = \frac{q''(x)}{T_0 - T_m(x)} \cdot \frac{D}{k} = 4.93} \quad (\text{parallel plates, } T_0 = \text{constant})$$

$$\boxed{Nu_D = 6} \quad (\text{parallel plates, } q'' = \text{constant})$$