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## Turbulence

Looking at our governing eqn's for flow over a flat plate:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \mu\Phi \quad (2)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \mu\Phi \quad (3)$$

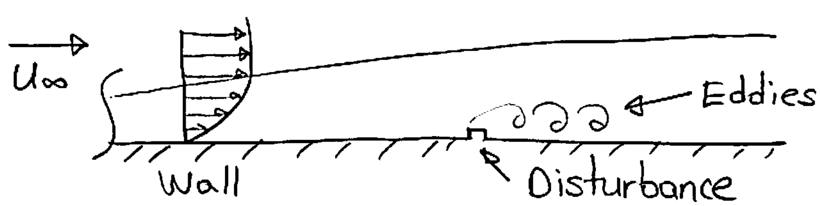
Nothing in Eqn's (1)-(3) suggests that a breakdown in the solution will occur at some $Re_{CR}$. In fact, (1)-(3) are solvable for $0 < Re < \infty$.

We also know that the transition is reversible:

$$\boxed{\text{Laminar}} \rightleftharpoons \boxed{\text{Turbulent}}$$

Turbulence can be understood as a spectrum of vortices (eddies) which form to dissipate kinetic energy from large to small, untill the smallest eddies dissipate via viscous shear stress.

A key to transition is <u>small disturbances</u> in the flow.




Note, these <u>"disturbances"</u> need not be physical. They may be fluctuations in the free stream velocity ($U_\infty$), pressure gradient ($\partial P/\partial x$), or very small scale surface rougness.

Viscous Forces act to dampen the disturbances and keep laminar. Inertial Forces associated with velocity changes do the opposite.

Hence, Reynolds # is typically used as a measure of when transition will happen.

$$Re = \frac{\text{Inertial Forces}}{\text{Viscous Forces}} \sim O(10^2) \text{ at transition.}$$

on the order of.

<u>Viscous stability</u> says that if we perterb the flow, the perturbation can be either damped out by the existing flow characteristics through viscous dissipation. If the existing flow does not have the ability to dampen the perturbation, transition to turbulence will occur, which has a 3D nature & more effective means to dissipate disturbances.

<u>Laminar external flows</u>: $Re_{CR,L} = \frac{\rho U_\infty L_{CR}}{\mu} = 3\times10^5 - 5\times10^5$

If we have large disturbances: $Re_{CR,L} = 6\times10^4$

<u>Internal Flows</u>: $Re_{CR,D_h} = \frac{\rho \bar{U} D_h}{\mu} = 2300$

If we have no disturbances, smooth pipe; $Re_{CR,D_h} \approx 1\times10^5$

In general for external or internal flows, the turbulent boundary layer consists of 2 regions:

1) Viscous sublayer adjacent to the wall, where viscous shear & molecular diffusion govern transport

2) Fully turbulent zone (most of the b.l.) where velocity is <u>not</u> independent of time, eddies exist, and momentum & heat transfer normal to the flow is much greater than possible with viscous shear & molecular diff.

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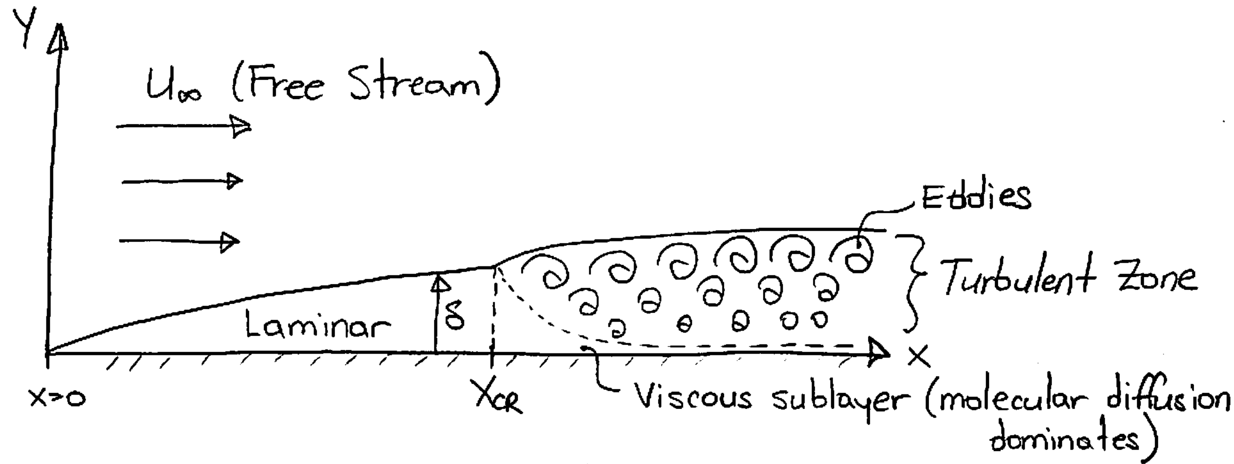
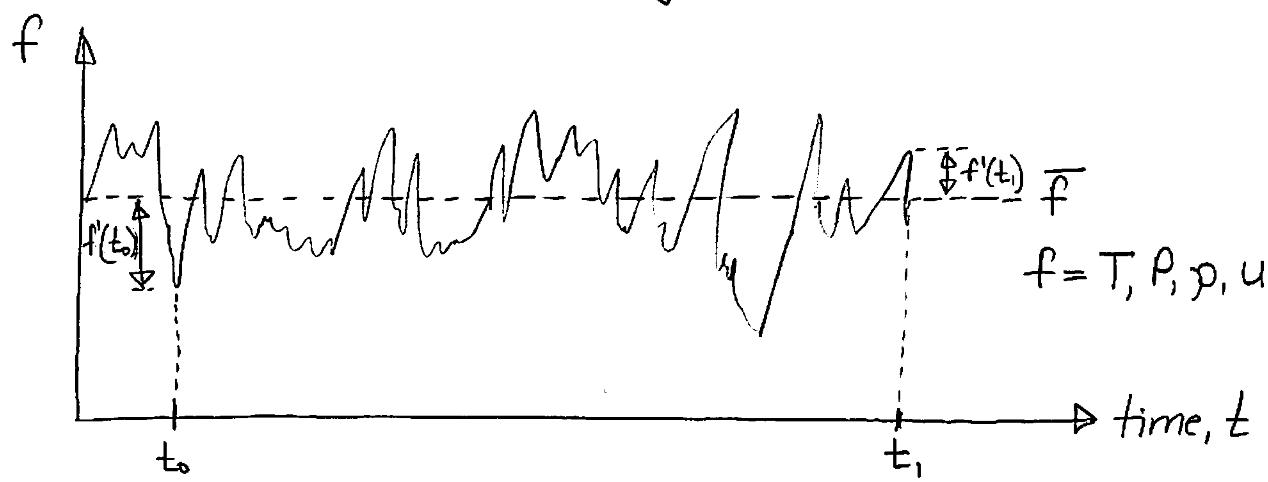



Let's deal with the fluctuating turbulent zone first:




$f = \bar{f} + f'$ $\Rightarrow$ $\bar{f} \equiv$ mean component (time averaged)
$f' \equiv$ fluctuating component (time dependent)

For the flows we will study:

$$u = \bar{u} + u' \qquad \rho = \bar{\rho} + \rho'$$
$$v = \bar{v} + v' \qquad T = \bar{T} + T'$$
$$w = \bar{w} + w' \qquad p = \text{constant (incompressible).}$$

$$\bar{u} = \frac{1}{\text{period}} \int_0^{\text{period}} u\,dt = \frac{1}{\tau} \int_{t_0 - \tau/2}^{t_0 + \tau/2} u\,dt$$

Note also, $\int_0^{\text{period}} u'\,dt = 0$ $\Rightarrow$ fluctuating components average to zero over time.

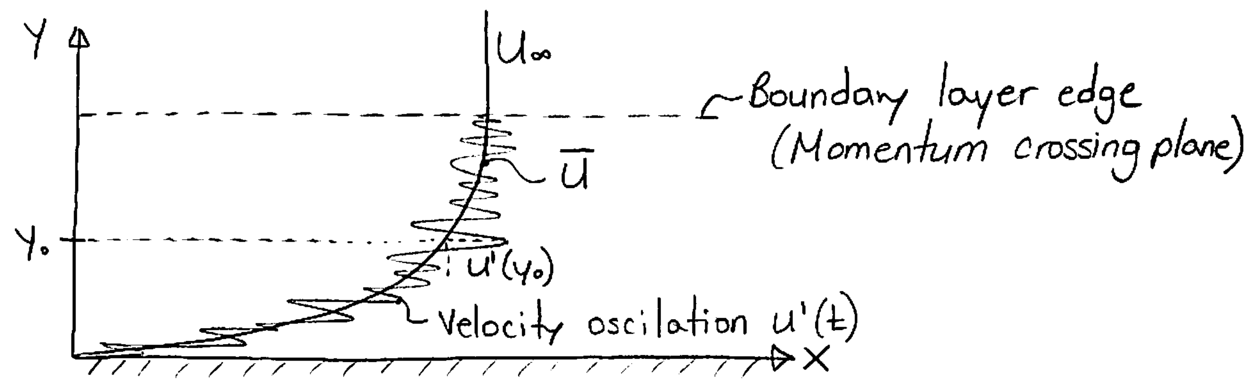
We can show this with the additive law of expectation:

$$\overline{f} = \overline{\overline{f} + f'} = \overline{\overline{f}} + \overline{f'} = \overline{f} + \overline{f'} \Rightarrow \overline{f'} = 0$$

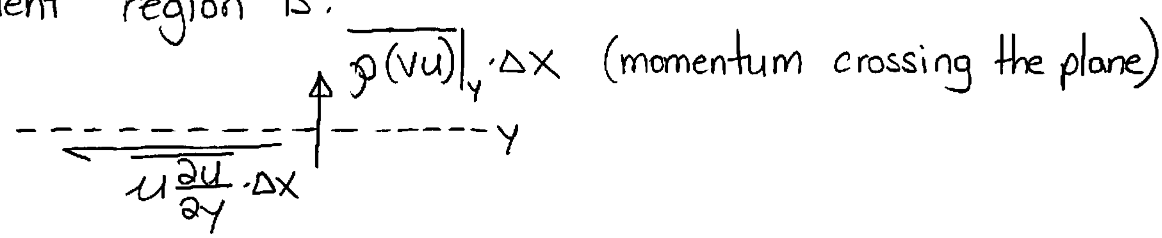
$\hookrightarrow$ since $\overline{f}$ = constant $(\overline{\overline{f}} = \overline{f})$

Also: $\overline{\overline{f} \cdot f'} = \overline{f} \cdot \overline{f'} = 0$

Let's look at our velocity boundary layer: $u = \overline{u} + u'$




The average shear stress at any y-location in the turbulent region is:




$$\overline{\tau} = \underbrace{\overline{\mu \frac{\partial u}{\partial y}}}_{\text{Laminar Component}} - \underbrace{\overline{\rho u v}}_{\text{Turbulent Component}}$$

$\Rightarrow$ Note, we always wrote before that $\tau = \mu \frac{\partial u}{\partial y}\Big|_0$ $\Rightarrow$ correct since $v = 0$ at $y = 0$, so $\rho u v$ component drops. Also, $v \ll u$ in b.l. flows.

If the equation above is still confusing, think of conservation of momentum and newtons second law. Passangers hopping on two trains.

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Since $\rho, \mu = \text{constant}$ (incompressible, constant property)

$$\overline{\tau} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{uv}$$

$$u = \overline{u} + u'$$
$$v = \overline{v} + v'$$

For external boundary layer flow, we can say:

$$\overline{v} \ll \overline{u} \;, \text{ and } \underbrace{\overline{v} \ll v'}_{\text{emperically shown for turbulent flows}} \;, \text{ and } \overline{v'} = 0$$

$$\overline{uv} = \overline{(\overline{u} + u')(\overline{v} + v')} = \overline{\overline{u}\,\overline{v}} + \overline{\overline{u}v'} + \overline{u'\overline{v}} + \overline{u'v'}$$
$$= \overline{u}\,\overline{v} + \underbrace{\overline{u}\,\overline{v'}}_{0\,(\overline{v'}=0)} + \underbrace{\overline{u'}\,\overline{v}}_{0\,(\overline{u'}=0)} + \overline{u'v'}$$

$$\overline{uv} = \overline{u}\,\overline{v} + \overline{u'v'} \Rightarrow$$ Note: $\overline{u'} = 0$ and $\overline{v'} = 0$ but $\overline{u'v'} \neq 0$ since $u'$ & $v'$ are correlated. Will show this below.

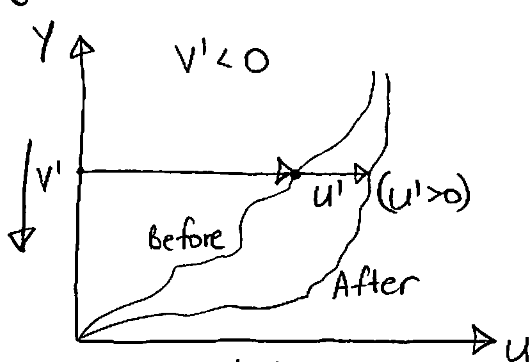
Let's see which term dominates:

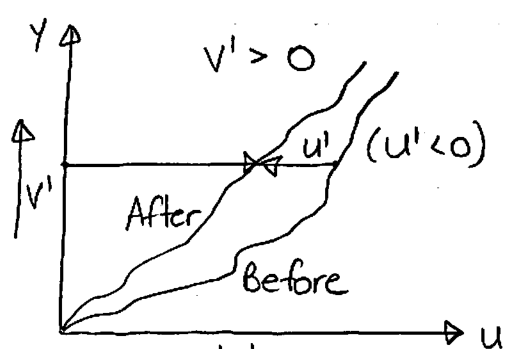
$$\frac{\overline{u}\,\overline{v}}{\overline{u'v'}} \Rightarrow \left.\begin{matrix} \overline{v} \ll \overline{v'} \text{ and } \overline{v} \ll \overline{u} \\ \overline{u} \sim v' \sim u' \end{matrix}\right\} \text{Hence } \frac{\overline{u}\,\overline{v}}{\overline{u'v'}} \ll 1$$

So we can rewrite our equation as:

$$\boxed{\overline{\tau} = \mu \frac{\partial \overline{u}}{\partial y} + \rho \overline{u'v'}}$$

But can we say something about the product $\overline{u'v'}$?
Looking at 2 possibilities:



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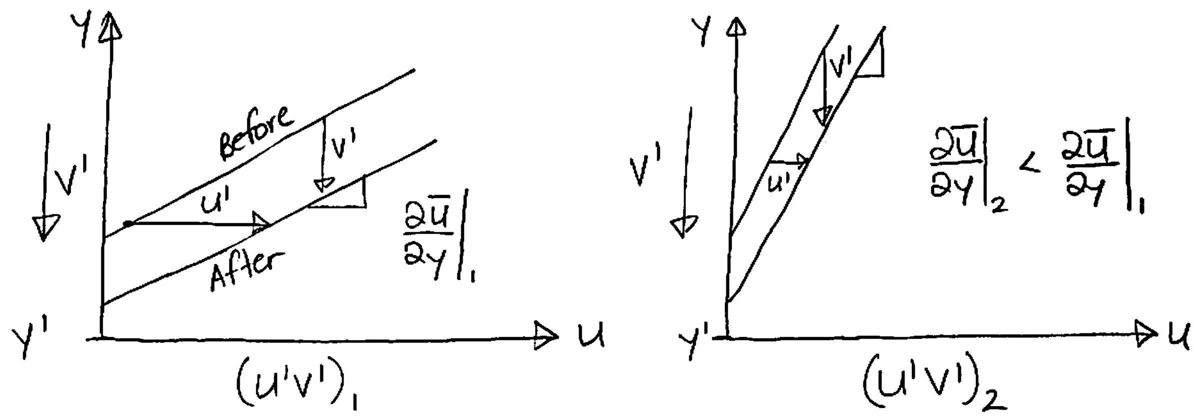
We see from our two cases that $u'v' < 0$ so $\overline{u'v'} < 0$ always.

Also, since $|\overline{u'v'}| \propto$ steepness of the profile, i.e.




We can see from above that $|(u'v')_1| > |(u'v')_2|$

Hence, we can now say: $\overline{u'v'} \sim \frac{\partial \bar{u}}{\partial y}$

$$\bar{\tau} = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}$$

$$= \mu \frac{\partial \bar{u}}{\partial y} - \underbrace{\begin{pmatrix}\text{factor reflecting} \\ \text{turbulent mixing}\end{pmatrix}}_{\equiv -\rho\varepsilon} \cdot \frac{\partial \bar{u}}{\partial y}$$

negative since we know $\overline{u'v'} < 0$. so $\varepsilon > 0$.

$$\boxed{\bar{\tau} = \rho(\nu + \varepsilon)\frac{\partial \bar{u}}{\partial y}}$$ $\Rightarrow$ $\varepsilon \equiv$ eddy diffusivity $[m^2/s]$

We can also rewrite this as:

$$\bar{\tau} = (\underbrace{\rho\nu}_{\mu} + \underbrace{\rho\varepsilon}_{\mu_t})\frac{\partial \bar{u}}{\partial y}$$

$\mu_t \equiv$ turbulent viscosity

$\bar{\tau} = \tau_{app} \equiv$ apparent shear stress.

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We can also say:

$$\tau_R = \rho\overline{u'v'} = \rho\varepsilon\frac{\partial\bar{u}}{\partial y} \equiv \text{turbulent shear stress or Reynolds stress}$$

$$u' \text{ and } v' \sim \sqrt{\frac{\tau_R}{\rho}}$$

We can also say that the total shear stress $\tau_{app}$ at the wall $(\tau_R = 0)$ is $\tau_o$

$$\boxed{u_* = \left(\frac{\tau_o}{\rho}\right)^{1/2} \equiv \text{friction velocity}}$$ $\Rightarrow$ Will become important later.

So now the problem is to find $\varepsilon$ or $\mu_t$. We have no idea about where to start.

Before we do that, we can look back at our initial equation we started with & see more clearly how we got it:

$$\bar{\tau} = \mu\frac{\partial\bar{u}}{\partial y} + \rho\overline{uv}$$ $\Rightarrow$ Can show this more rigorously

Start with mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial x}(\bar{u} + u') + \frac{\partial}{\partial y}(\bar{v} + v') = 0$$

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0 \quad (1)$$

Integrating eq. (1) wrt time and noting: $\overline{\frac{\partial u}{\partial x}} = \frac{\partial\bar{u}}{\partial x}$

$$\frac{1}{\text{period}}\int_0^{\text{period}}\frac{\partial\bar{u}}{\partial x}dt = \overline{\frac{\partial\bar{u}}{\partial x}} = \frac{\partial\bar{\bar{u}}}{\partial x} = \frac{\partial\bar{u}}{\partial x}$$ $\Rightarrow$ same for $\frac{\partial\bar{v}}{\partial y}$

$$\frac{1}{\text{period}}\int_0^{\text{period}}\frac{\partial u'}{\partial x}dt = \overline{\frac{\partial u'}{\partial x}} = \frac{\partial\overline{u'}}{\partial x} = \frac{\partial}{\partial x}(\overline{u'}) = 0$$

$$\boxed{\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} = 0}$$ $\Rightarrow$ Turbulent conservation of mass (20)

Now let's consider x-momentum:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (2)$$

Note, this is just another way of writing it since:

$$\left.\begin{aligned} \frac{\partial}{\partial x}(u^2) &= 2u\frac{\partial u}{\partial x} \\ \frac{\partial}{\partial y}(uv) &= \frac{\partial u}{\partial y}v + u\frac{\partial v}{\partial y} \end{aligned}\right\} \quad \begin{aligned} & 2u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + u\overbrace{\frac{\partial v}{\partial y}}^{=-\frac{\partial u}{\partial x}} \\ & \cancel{2}u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \cancel{u\frac{\partial u}{\partial x}} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \end{aligned}$$

Averaging each term in (2) over time and knowing that:

$$\overline{\frac{\partial u}{\partial x}} = \frac{\partial \bar{u}}{\partial x} \; ; \; \frac{\partial \bar{u}}{\partial t} = 0 \; ; \; \overline{\frac{\partial u}{\partial t}} = 0$$

$$\frac{\partial}{\partial x}(\overline{u^2}) + \frac{\partial}{\partial y}(\overline{uv}) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} \quad (3)$$

Now we know from before that:

$$\overline{uv} = \bar{u}\bar{v} + \overline{u'v'} \Rightarrow \text{showed this a few pages ago}$$
$$\overline{u^2} = \bar{u}^2 + \overline{u'^2}$$

Apply to eqn. (3):

$$\frac{\partial}{\partial x}(\bar{u}^2) + \frac{\partial}{\partial y}(\bar{u}\bar{v}) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \frac{\partial}{\partial x}(\overline{u'^2}) - \frac{\partial}{\partial y}(\overline{u'v'})$$

We know that $\overline{\bar{u}\cdot\bar{u}} = \bar{\bar{u}}\cdot\bar{\bar{u}} = \bar{u}\cdot\bar{u}$

$$\cancel{2}\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \cancel{\bar{u}\underbrace{\frac{\partial \bar{v}}{\partial y}}_{-\frac{\partial \bar{u}}{\partial x}\text{ (from continuity (1))}}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \frac{\partial}{\partial x}(\overline{u'^2}) - \frac{\partial}{\partial y}(\overline{u'v'})$$

So the time averaged x-momentum equation becomes: (20)

$$\boxed{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\nabla^2\bar{u} - \frac{\partial}{\partial x}(\overline{u'^2}) - \frac{\partial}{\partial y}(\overline{u'v'})} \quad (4)$$

So for a boundary layer, we can simplify eq. (4)

We know that $u' \sim v'$, and $x \sim L$, $y \sim \delta$

Scaling our two added terms:

$$\frac{\frac{\partial}{\partial x}(\overline{u'^2})}{\frac{\partial}{\partial y}(\overline{u'v'})} \sim \frac{\frac{\cancel{u'^2}}{L}}{\frac{\cancel{u'v'}}{\delta}} \sim \frac{\delta}{L} \ll 1 \Rightarrow \text{hence } \frac{\partial}{\partial x}(\overline{u'^2}) \ll \frac{\partial}{\partial y}(\overline{u'v'})$$

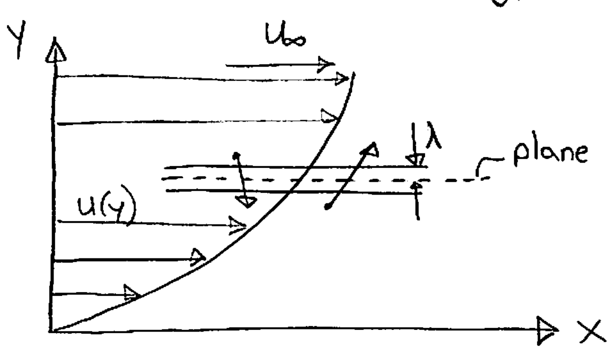
So our b.l. equation becomes:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial}{\partial y}(\overline{u'v'})$$

Rewriting this:

$$\boxed{\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\underbrace{\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)}_{\text{Looks familiar!}}}$$

OK, now back to our problem. We need to solve for $\mu_t$.
We can make an analogy to molecular diffusion:




Assuming:
1) n molecules per unit volume
2) 1/3 of molecules have velocities along the y-dir. ($\bar{v}$)
3) $\frac{1}{6}n$ travel in +y-dir., and $\frac{1}{6}n$ travel in -y-dir. ($\bar{v}$)
4) mean velocity $\bar{v}$ in the y-direction for above averages.

Note, molecules will be randomly distributed with more than $\frac{1}{3}n$ having velocity components in y-direction, however their y-velocities will also be randomized and not all $\bar{v}$. Hence if we average out direction & speed, we will get $\frac{1}{3}n\bar{v}$ in y-dir.