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$\frac{1}{6}n\bar{v}$ cross plane A from below
$\frac{1}{6}n\bar{v}$ cross plane A from above

Molecules experienced their last collision (momentum exchange) a distance λ away from the plane A (λ = mean free path).

So an x-momentum balance yields:

$$m_x \uparrow = \frac{1}{6}n\bar{v}\left[m u_x(y-\lambda)\right]$$ = momentum transported per unit time per unit area across plane upwards

$$m_x \downarrow = \frac{1}{6}n\bar{v}\left[m u_x(y+\lambda)\right]$$

$$\tau = \frac{1}{6}n\bar{v}m\left[u_x(y-\lambda) - u_x(y+\lambda)\right]$$ $\Rightarrow$ Taylor series expand

$$\boxed{\tau = \frac{1}{6}n\bar{v}m\left(-2\frac{\partial u_x}{\partial y}\cdot\lambda\right) = -\mu\frac{\partial u}{\partial y}}$$ $\Rightarrow$ Shear stress on plane A.

Note here, $n \cdot m = \rho$

$$\boxed{\mu = \frac{1}{3}\rho\bar{v}\lambda}$$ $\Rightarrow$ Molecular Viscosity (for a gas)

Note, before we showed that; $\lambda = \dfrac{k_B T}{\pi d^2 p}$ (ideal gas)

$$\bar{v} = \sqrt{\frac{8k_B T}{\pi m}}$$

Interestingly, back substituting $\bar{v}$ & λ into μ

$$\mu = \frac{1}{3}\rho\sqrt{\frac{8k_B T}{\pi m}}\cdot\frac{k_B T}{\pi d^2 p} \Rightarrow n = \frac{p}{k_B T} \Rightarrow \rho = n \cdot m$$

$$\mu = \frac{1}{3}m\frac{\cancel{p}}{\cancel{k_B T}}\cdot\sqrt{\frac{8k_B T}{\pi m}}\cdot\frac{\cancel{k_B T}}{\pi d^2 \cancel{p}} = \frac{m}{3\pi d^2}\sqrt{\frac{8k_B T}{\pi m}}$$

$$\boxed{\mu \sim T^{1/2}} \neq f(p, \rho)!$$

Interesting since μ increases at higher temperatures for gas. Opposite for liquids.

So for our turbulent viscosity, we can assume a similar form as our molecular viscosity.

$$\boxed{\text{Molecules Diffusing}} \xrightleftharpoons{\text{Analogy}} \boxed{\text{Eddies Diffusing}}$$

$\mu = \frac{1}{3}\rho \bar{v} \lambda$

$\Updownarrow$ — mean free path analogous to eddy size

$\boxed{\mu_t \sim \rho u_* l} \Rightarrow u_* \equiv$ friction velocity

$l \equiv$ mixing length or Eddy size.

This approximation is known as Prandtl's Mixing Length model. Simplest approach.

Doing a wall coordinate nondimensionalization:

$$u^+ = \frac{\bar{u}}{u_*} \quad , \quad v^+ = \frac{\bar{v}}{u_*}$$

$$x^+ = \frac{x u_*}{\nu} \quad , \quad y^+ = \frac{y u_*}{\nu}$$

We can re-cast $\tau_0 = \rho \frac{\partial \bar{u}}{\partial y}(\nu + \varepsilon)$ as: $\left(u_* = \left(\frac{\tau_0}{\rho}\right)^{1/2} \right)$

$$\underbrace{\frac{\tau_0}{\rho}}_{u_*^2} = \frac{\partial \bar{u}}{\partial y}(\nu + \varepsilon) \Rightarrow \partial \bar{u} = u_* \partial u^+ \; ; \; \partial y = \frac{\partial y^+ \nu}{u_*}$$

$$\cancel{u_*^2} = \frac{\partial u^+}{\partial y^+} \cdot \frac{\cancel{u_*^2}}{\nu}(\nu + \varepsilon)$$

$$\boxed{\left(1 + \frac{\varepsilon}{\nu}\right)\frac{\partial u^+}{\partial y^+} = 1} \Rightarrow u^+(y^+) \equiv \text{velocity distribution near the wall.}$$

But we still need to know ε to solve. Lookinb back at Prandtl's mixing length model.

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$$\mu_t = \rho u_* \ell = \rho \varepsilon \Rightarrow \varepsilon = \underbrace{\ell}_{\text{need to determine}} \Big(\overbrace{\frac{\tau_0}{\rho}}^{\text{Wall shear stress}}\Big)^{1/2}$$

Prandtl assumed that the size of turbulent eddies cannot be bigger than their distance from the wall, hence:

$$\boxed{\ell = K y} \equiv \text{mixing length}$$

Back substituting

$$\left(1 + \frac{\varepsilon}{\nu}\right)\frac{\partial u^+}{\partial y^+} = 1$$

In the <u>viscous sublayer</u>, $\nu \gg \varepsilon$, $\frac{\varepsilon}{\nu} \ll 1$

$$\int \frac{\partial u^+}{\partial y^+} = \int 1$$

$$\boxed{u^+ = y^+} \equiv \text{Viscous sublayer velocity profile}$$

In the <u>turbulent layer</u>, $\varepsilon \gg \nu$, $\frac{\varepsilon}{\nu} \gg 1$

$$\frac{\varepsilon}{\nu}\frac{\partial u^+}{\partial y^+} = 1 \quad \Rightarrow \text{use mixing length model}$$

$$\frac{\ell u_*}{\nu}\frac{\partial u^+}{\partial y^+} = 1 \quad \Rightarrow \ell = K y = K \frac{y^+ \nu}{u_*}$$

$$K y^+ \frac{\cancel{\nu}}{\cancel{u_*}} \cdot \frac{\cancel{u_*}}{\cancel{\nu}} \frac{\partial u^+}{\partial y^+} = 1$$

$$K y^+ \frac{\partial u^+}{\partial y^+} = 1 \quad \Rightarrow \int \partial u^+ = \int \frac{dy^+}{y^+ K}$$

$$\boxed{u^+ = \frac{1}{K} \ln y^+ + C} \equiv \text{Log-layer velocity profile.}$$

Note, this turbulent b.l. behaviour is also applicable to internal flows.

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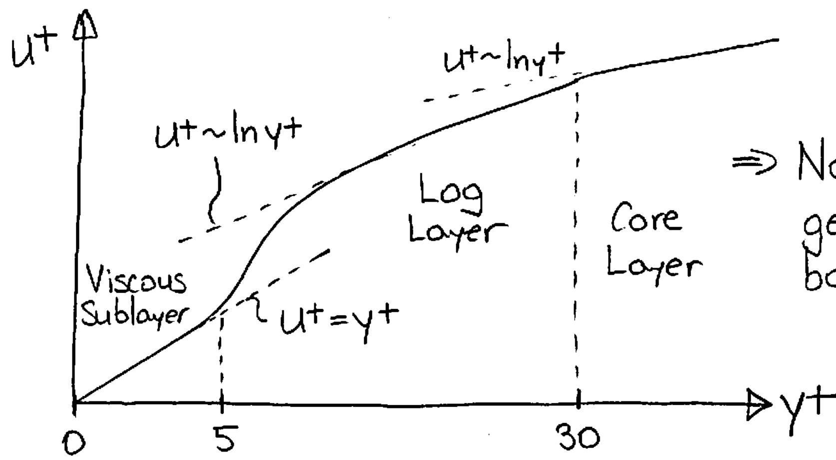

$\Rightarrow$ Note, applicable to general turbulent boundary layer flow.

Experimentally verified solutions are:

$$\left.\begin{array}{ll} u^+ = y^+ \ ; & 0 < y^+ < 11.6 \\ u^+ = 2.5 \ln y^+ + 5.5 \ ; & y^+ > 11.6 \end{array}\right\} \text{Prandtl \& Taylor}$$

$$\left.\begin{array}{ll} u^+ = y^+ & 0 < y^+ < 5 \\ u^+ = 5 \ln y^+ - 3.05 & 5 < y^+ < 30 \\ u^+ = 2.5 \ln y^+ + 5.5 & y^+ > 30 \end{array}\right\} \begin{array}{l}\text{von Karman (1939)} \\ \text{Result plotted above}\end{array}$$

So to solve for shear stress in turbulent flows, we need to solve for τ_0 using our profiles above. Note:

$$\tau_0 \sim \frac{\partial \bar{u}}{\partial y} \sim \frac{U_\infty}{\delta} \Rightarrow \text{We need } \delta \text{ for a turbulent b.l.}$$

We can approximate our piecewise profiles with one good solution: $u^+ = f(y^+)$

$$\boxed{u^+ = 8.75 (y^+)^{1/7}} \Rightarrow \text{Prandtl's } \tfrac{1}{7} \text{ power law}$$

Dimensionalizing our equation $(u^+ = f(y^+))$ for $\bar{u} = U_\infty$ at $y = \delta$

$$\frac{\bar{u}}{u_*} = f\left(\frac{y u_*}{\nu}\right) \Rightarrow u_* = \left(\frac{\tau_0}{\rho}\right)^{1/2}$$

For $\bar{u} = U_\infty$, $y = \delta$ $\Rightarrow$ $\boxed{\dfrac{U_\infty}{(\tau_0/\rho)^{1/2}} = f\left(\dfrac{\delta}{\nu}\left(\dfrac{\tau_0}{\rho}\right)^{1/2}\right)}$

Using Plandtl's 1/7 power law as f, i.e. $f = 8.75(y^+)^{1/7}$
And applying our momentum integral equation:

$$\frac{d}{dx}\int_0^\infty \bar{u}\,(U_\infty - \bar{u})\,dy = \frac{\tau_0}{\rho} \quad (1)$$

$$\frac{U_\infty}{(\tau_0/\rho)^{1/2}} = 8.75\left(\frac{\delta}{\nu}\left(\frac{\tau_0}{\rho}\right)^{1/2}\right)^{1/7} \quad (2) \Rightarrow \text{Solve for } \tau_0$$

$$\tau_0 = 0.0225\,\rho U_\infty^2\left(\frac{\delta U_\infty}{\nu}\right)^{-1/4} \quad (3)$$

Substituting (3) into (1) & integrate to $y=\delta$ ⇒ we can now solve.
Complex method since $\bar{u} = f(\tau_0)$ as well.

$$\boxed{\frac{\delta}{x} = 0.37\left(\frac{U_\infty x}{\nu}\right)^{-1/5} = \frac{0.37}{Re_x^{0.2}}} \quad (4) \Rightarrow \text{Turbulent b.l. thickness.}$$

Combining (4) and (2) yields:

$$\boxed{\frac{\tau_0}{\rho U_\infty^2} = \frac{1}{2}C_{f,x} = 0.0296\left(\frac{U_\infty x}{\nu}\right)^{-1/5}} \Rightarrow \text{Local turbulent skin friction coeff. Flat plate.}$$

$$\boxed{\frac{\overline{\tau_0}}{\rho U_\infty^2} = \frac{1}{2}\overline{C_f} = 0.037\left(\frac{U_\infty x}{\nu}\right)^{-1/5}} \Rightarrow \text{Average turbulent skin friction coeff. } 10^5 < Re_x < 10^8$$

Experiments show that:

$$\boxed{C_{f,x} = 0.37\left[\log_{10}\left(\frac{U_\infty x}{\nu}\right)\right]^{-2.584}} \quad 10^5 < Re_x < 10^{10}$$

Note, discrepancy arises between experiment & theory at $Re_x > 10^8$ due to our assumption of $u^+ = f(y^+)$ ⇒ Prandtl's 1/7 Power law.

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Turbulent Heat Transfer

If we follow the same procedure as before of time averaging our energy equation:

$$\boxed{\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y} = \alpha \nabla^2 \overline{T} - \frac{\partial}{\partial x}\left(\overline{u'T'}\right) - \frac{\partial}{\partial y}\left(\overline{v'T'}\right)}$$

↳ Time averaged energy equation for turbulent flow.

For boundary layers: $(u' \sim v')$

$$\frac{\frac{\partial}{\partial x}\left(\overline{u'T'}\right)}{\frac{\partial}{\partial y}\left(\overline{v'T'}\right)} \sim \frac{\frac{\cancel{\overline{u'T'}}}{L}}{\frac{\cancel{\overline{v'T'}}}{\delta_T}} \sim \frac{\delta_T}{L} \ll 1$$ ⇒ so we can drop the x term

Our b.l. energy equation becomes:

$$\overline{u}\frac{\partial \overline{T}}{\partial x} + \overline{v}\frac{\partial \overline{T}}{\partial y} = \underbrace{\alpha \frac{\partial^2 \overline{T}}{\partial y^2} - \frac{\partial}{\partial y}\left(\overline{v'T'}\right)}$$

$$= \frac{1}{\rho c_p}\frac{\partial \overline{T}}{\partial y}\underbrace{\left(k\frac{\partial \overline{T}}{\partial y} - \rho c_p \overline{v'T'}\right)}$$

Looks analogous to our shear solution a few pages ago.

Using the same reasoning as before, we can show:

$$\boxed{-\rho c_p \overline{v'T'} = \rho c_p \varepsilon_T \frac{\partial \overline{T}}{\partial y}}$$ ≡ Eddy Heat Flux

ε_T ≡ eddy thermal diffusivity

$$\boxed{-q''_{app} = k\frac{\partial \overline{T}}{\partial y} - \rho c_p \overline{v'T'} = \rho c_p (\alpha + \varepsilon_T)\frac{\partial \overline{T}}{\partial y}}$$ ≡ Apparent Heat Flux

We can also define: $\boxed{Pr_t = \frac{\varepsilon}{\varepsilon_t}}$ ≡ turbulent Prandtl #

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The nice thing is we dont have to solve for the heat flux. We can use the <u>Turbulent Colburn Analogy</u>.

$$\boxed{St_x \cdot Pr^{2/3} = \frac{C_{f,x}}{2}}$$ ⇒ Colburn Analogy (Did this in Laminar flow)

$$St_x = \frac{Nu_x}{Re_x Pr} = \frac{\bar{h}x/k}{\frac{\rho U_\infty x}{\mu}\cdot\frac{c_p \mu}{k}} = \frac{\bar{h}}{\rho c_p U_\infty} = \frac{q''/\Delta T}{\rho c_p U_\infty}$$

$$\boxed{Nu_x = \frac{C_{f,x}}{2} Re_x Pr^{1/3}}$$ ⇒ We already solved for $C_{f,x}$

$$C_{f,x} = 0.0592 Re_x^{-1/5}$$ ⇒ External turbulent flow over flat plate

Back substituting, we obtain:

$$Nu_x = \frac{0.0592}{2} Re_x^{-1/5} \cdot Re_x \cdot Pr^{1/3}$$

$$\boxed{Nu_x = 0.029 Re_x^{4/5} Pr^{1/3}}$$ ⇒ Colburn correlation, $Pr \geq 0.5$, $Re_x > 5\times10^5$

For the best correlation (most accurate):

$$\boxed{Nu_x = \frac{\left(\frac{C_{f,x}}{2}\right)\cdot Re_x \cdot Pr}{1 + 12.7\left(\frac{C_{f,x}}{x}\right)^{1/2}\left(Pr^{2/3} - 1\right)}}$$ ⇒ White correlation

$0.5 < Pr < 2000$

$5\times10^5 < Re_x < 10^7$

Using the Colburn result:

$$\boxed{\overline{Nu}_L = \frac{\bar{h}L}{k} = 0.037 Re_L^{4/5} Pr^{1/3}}, \quad Pr \geq 0.5, \quad Re_x > 5\times10^5$$

Note, our correlations are valid for both T_o = constant and $q''|_o$ = constant

For $q''|_o$ = constant, only a 4% difference than the results above,

Note also $Nu_x = \frac{q''|_o x}{k[T_o(x) - T_\infty]}$ ≡ constant heat flux Nusselt #

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Side Note: Proof of the Colburn Analogy:

$$St = \frac{h}{\rho C_p U_\infty} \quad , \qquad \alpha = \frac{k}{\rho C_p}$$

$$\left.\begin{aligned} \tau_o &= \mu \frac{\partial \bar{u}}{\partial y} \\ q''_o &= k \frac{\partial \bar{T}}{\partial y} \end{aligned}\right\} \frac{\tau_o}{q''_o} = \frac{\mu}{k}\frac{\partial \bar{u}}{\partial \bar{T}} \sim \frac{\mu}{k}\frac{U_\infty}{\Delta T}$$

Here, we know $k = \rho c_p \alpha$, $q''_o = h\Delta T$ and $\tau_o = \frac{1}{2}\rho U_\infty^2 C_{f,x}$

Back substituting:

$$\frac{\tau_o}{q''_o} = \frac{\frac{1}{2}\rho U_\infty^2 C_{f,x}}{h\cancel{\Delta T}} \sim \frac{\mu \cancel{U_\infty}}{k\cancel{\Delta T}}$$

$$\frac{\frac{1}{2}U_\infty C_{f,x}}{h} \sim \underbrace{\frac{\mu}{\rho}}_{\nu}\frac{1}{\rho C_p \alpha} \sim \underbrace{\frac{\nu}{\alpha}}_{Pr}\cdot\frac{1}{\rho C_p}$$

$$\frac{\frac{1}{2}C_{f,x}}{h} \sim Pr\frac{1}{\rho C_p U_\infty} \Rightarrow \frac{1}{2}C_{f,x} \sim \underbrace{\frac{h}{\rho C_p U_\infty}}_{St}\cdot Pr$$

$$\therefore \boxed{St = \frac{h}{\rho C_p U_\infty} \sim \frac{1}{2}\cdot\frac{C_{f,x}}{Pr}} \Rightarrow \text{QED.}$$