Now the $x$-momentum equation becomes:

$$
p \frac{Du}{Dt} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ 2u \frac{\partial u}{\partial x} - \frac{2u}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ u \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + x \frac{\partial u}{\partial x}
$$

But note: $u \frac{\partial u}{\partial x} = u \frac{\partial}{\partial x} \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial y} = - \frac{2u}{3} \frac{\partial v}{\partial x}$ (cons. of mass)

$$
p \frac{Du}{Dt} = - \frac{\partial p}{\partial x} + 2u \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 v}{\partial y^2} - u \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x}
$$

Expanding the left hand side:

$$
p \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + u \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + x \frac{\partial u}{\partial x}
$$

$x$-momentum equation (20).

In general:

$$
p \frac{DV}{Dt} = - \nabla p + u \nabla^2 v + F
$$

Where $F$ is the body force per unit volume.

Similar equations can be derived for cylindrical & spherical coordinates. For example (you may use this one)

$$
p \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_r}{r} \frac{\partial v_z}{\partial \theta} + v_\theta \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z}
$$

$$
+ u \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \frac{\partial}{\partial z}
$$

$z$-momentum eqn. in cylindrical coord.
First Law of Thermodynamics
So far, we've equipped ourselves well to solve fluid mechanics problems. However, the heat transfer part of the convection problem requires a solution to the temperature distribution in the flow (especially close to the wall).

To solve, we need the energy equation (1st law)

\[
\frac{\text{Rate of energy}}{\text{accumulation in the CV}} = \frac{\text{net transfer}}{1} + \frac{\text{net heat transfer by fluid flow}}{2} + \frac{\text{heat transfer by conduction}}{3} + \frac{\text{heat}}{4} - \frac{\text{work}}{5} \text{ (of CV on its env.)}
\]

Let's draw our CV

1+2

\[ \frac{\partial (pue) \Delta y}{\partial t} \Delta x \Delta y \]

3+4

\[ q_x \Delta y \]

Note, I haven't drawn the work contribution to either CV (5)
So if we write out our terms (1) to (5)

\( (\cdots)_1 = \Delta x \Delta y \frac{2}{a_t} (pe) \; ; \; e = \text{specific internal energy} \)

\( (\cdots)_2 = - (\Delta x \Delta y) \left[ \frac{2}{a_x} (pue) + \frac{2}{a_y} (pve) \right] \)

\( (\cdots)_3 = - (\Delta x \Delta y) \left( \frac{2a_x''}{a_x} + \frac{2a_y''}{a_y} \right) \)

\( (\cdots)_4 = (\Delta x \Delta y) q'''' \)

\( (\cdots)_5 = (\Delta x \Delta y) \left( \sigma_x \frac{2h}{a_x} - 2c_y \frac{2u}{a_y} + \sigma_y \frac{2v}{a_y} - 2c_x \frac{2u}{a_x} \right) \)

\[ + \left( \Delta x \Delta y \right) \left( \frac{2a_x}{a_x} - a_x \frac{2c_y}{a_y} + \frac{2a_y}{a_y} - a_y \frac{2c_2}{a_x} \right) \]

For the work term (5) we had to use our CV from the x-momentum equation:

For example, \( \mathbf{W}_{\text{LEFT}} = \frac{\partial}{\partial y} \left( \sigma_x \Delta y u \right) \quad \) work on the environment

\( \mathbf{F} \rightarrow \text{Boundary displacement per unit time} \)

Similarly, on the right side:

\( \mathbf{W}_{\text{RIGHT}} = \left( \sigma_x + \frac{2a_x}{a_x} \Delta x \right) \Delta y \cdot (u + \frac{2u}{a_x} \Delta x) \)

The net work transfer rate due to these two contributions is
\[ W_{NET} = \left( \sigma_{x\alpha} + \frac{2\sigma_{x}}{3x} \alpha x\alpha \right) \left( u + \frac{2u}{3x} \alpha x \right) - u \sigma_{x\alpha} \alpha \alpha \\
= u \sigma_{x\alpha} \alpha \alpha + \sigma_{x\alpha} \frac{2u}{3x} \alpha x \alpha \alpha + u \frac{2\sigma_{x}}{3x} \alpha x \alpha \alpha + \frac{2\sigma_{x}}{3x} \alpha x \alpha \alpha - u \sigma_{x\alpha} \alpha \alpha \]
\[ = \sigma_{x\alpha} \frac{2u}{3x} \alpha x \alpha \alpha + u \frac{2\sigma_{x}}{3x} \alpha x \alpha \alpha + \frac{2\sigma_{x}}{3x} \frac{2u}{3x} \alpha x \alpha \alpha \]

The last term goes to zero when we divide by \( \alpha x \alpha \alpha \)
and let \( \alpha x \to 0 \) \& \( \alpha \alpha \to 0 \)

Similarly, we can do the same procedure for the other stresses.

We can also show that the 4 terms in the \( \varepsilon_{ij} \) of equat. (5) reduce to: (try deriving it \( \Rightarrow \) substitute in \( \sigma_{x}, \sigma_{y}, c_{x}, c_{y} \))

\[ \alpha x \alpha \alpha \left( u \frac{2\sigma_{x}}{3x} + u \frac{2\sigma_{x}}{3y} + v \frac{2\sigma_{x}}{2y} - v \frac{2\sigma_{x}}{2x} \right) = -\rho \frac{0}{0 \pm} \left( \frac{u^{2} + v^{2}}{2} \right) \]

This equates to the change in kinetic energy of the fluid packet.

Since \[ -\rho \frac{0}{0 \pm} \left( \frac{u^{2} + v^{2}}{2} \right) \ll \frac{\partial (p\epsilon)}{\partial \pm} \], we neglect these terms.

Assembling everything together: (back substituting constitutive rel.)

\[ \rho \frac{0}{0 \pm} \left( \frac{D_{x}}{D_{x}} + \rho \nabla \cdot v \right) = -\nabla \cdot q'' + q''_{x} - \rho \nabla \cdot v + u \Phi \]

\( \Rightarrow \) 20 energy equation, where:

\( q'' \equiv \) heat flux vector \( (q''_{x}, q''_{y}) \)

\( \Phi \equiv \) viscous dissipation function

To solve for \( \Phi \) we need to back substitute our constitutive relations into (5) and simplify.
In cartesian coordinates (30)

\[
\Phi = 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] - \frac{2}{3} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2
\]

To simplify our lives, we can assume 2D & incompressible

\[
\Phi = 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right]
\]

Now we can deal with simplifying our other terms. From Thermodynamics:

\[
h = e + \left( \frac{1}{\gamma} \right) p \implies \text{enthalpy}
\]

\[
\frac{\partial h}{\partial t} = \frac{\partial e}{\partial t} + \frac{1}{\gamma} \frac{\partial p}{\partial t} - \frac{p^2}{\gamma} \frac{\partial p}{\partial t} \quad \text{(quotient rule)}
\]

Also, via Fourier's law of heat conduction

\[
q'' = -k \nabla T
\]

Back substituting:

\[
p \frac{\partial h}{\partial t} = \nabla \cdot (k \nabla T) + q'' + \frac{\partial p}{\partial t} + u \Phi - \frac{p}{\rho} \left( \frac{\partial p}{\partial t} + \rho \nabla \cdot v \right)
\]

Finally, from mass conservation:

\[
p \frac{\partial h}{\partial t} = \nabla \cdot (k \nabla T) + q'' + \frac{\partial p}{\partial t} + u \Phi
\]
Note, it's tempting to say $h = c_p T$, however this is only correct for ideal gases. More rigorously

$$dh = Tds + \frac{1}{\rho} d\rho \quad (1) \Rightarrow T = \text{absolute temp.}$$
$$ds = \text{specific entropy change}$$

We can now write the following:

$$ds = \left(\frac{\partial s}{\partial T}\right)_\rho dT + \left(\frac{\partial s}{\partial \rho}\right)_T d\rho \quad (2)$$

From Maxwell's relations

$$\left(\frac{\partial s}{\partial \rho}\right)_T = -\left[\frac{\partial (1/\rho)}{\partial T}\right]_\rho = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial T}\right)_\rho = -\frac{\beta}{\rho} \quad (3)$$

where $\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_\rho \equiv \text{coefficient of thermal expansion} \quad (4)$

Using thermodynamic relations, we can also show that:

$$\left(\frac{\partial s}{\partial T}\right)_\rho = \frac{c_p}{T} \quad (5)$$

Putting equations $(1)$ to $(5)$ together, we obtain

$$dh = c_p dT + \frac{1}{\rho} (1-\beta T) d\rho$$

Back substituting into our energy equation, we obtain

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q'' + \beta T \frac{\partial \rho}{\partial t} + u \overrightarrow{F}$$

Temperature based 1st Law of Thermodynamics

Note, we can simplify this for certain cases
For an ideal gas: \( \beta = \frac{1}{\gamma} \)
\[
\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q'''' + \frac{\partial p}{\partial t} + \mu \Phi
\]

For an incompressible liquid (water): \( \beta = 0 \) (in real life, \( \beta_{\text{water}} \approx 0 \))
\[
\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + q'''' + \mu \Phi
\]

Note, these are general solutions but for most of the problems we'll solve in this course, we will have:

1) \( k = \text{constant} \) (constant properties)
2) \( q'''' = 0 \) (no internal energy generation)
3) \( \mu \Phi \approx 0 \) (negligible viscous dissipation)
4) \( \beta T \frac{\partial p}{\partial t} \approx 0 \) (negligible compressibility effects)

So our formulation boils down to:
\[
\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T
\]

For cartesian \((x, y, z)\)
\[
\rho c_p \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)
\]

For cylindrical \((r, \theta, z)\)
\[
\rho c_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_r}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \right.
\]

For spherical \((r, \phi, \theta)\):
\[
\rho c_p \left( \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_r}{r} \frac{\partial T}{\partial \phi} + \frac{v_\theta}{r \sin \phi} \frac{\partial T}{\partial \theta} \right) = k \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial T}{\partial \phi} \right) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 T}{\partial \theta^2} \right]
\]
Note, for extremely viscous fluids & flows (i.e. lubrication problems or piping of crude oil), the viscous dissipation must be taken into account:

\[
\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T + u \Phi
\]

For exact definitions of \( \Phi \) in other coordinate systems, see pg. 14 of Bejan (4th edition).

When the energy equation was first developed by Fourier & Poisson, they always used \( c \) instead of \( c_p \). This is ok since they were working with incompressible fluids (\( c = c_p \)). However, \( c_v \) should never be used as it ignores the PdV work done by the moving fluid packet as it expands and contracts.

**Second Law of Thermodynamics**

We don't need the second law of thermo to solve convection problems. Where it becomes important is when we design systems that have convective heat transfer with a fluid.

The second law states all processes are irreversible:

\[
\frac{dS_{cv}}{dt} = \sum \frac{q_i}{T_i} + \sum m_s - \sum m_s
\]

where; \( S_{cv} \) = instantaneous entropy in the control volume
\( q_i \) & \( T_i \) = heat transfer and absolute temperature of the boundary i. (+ means inflow)
The measure of the irreversibility of the process is quantified by the strength of the inequality, or the entropy generation rate $S_{gen}$:

$$S_{gen} = \frac{dS_{rev}}{dt} - \sum \frac{q_i}{T_i} - \sum \frac{\dot{m}_s}{inlet} - \sum \frac{\dot{m}_s}{outlet} \geq 0$$

We also know from exergy analysis that

$$W_{lost} = T_0 S_{gen} = \text{destruction of useful work due to entropy generation.}$$

$T_0 = \text{absolute temp. of ambient reservoir.}$

This shows the engineering importance of making the convection process as efficient as possible and estimating $S_{gen}$ for a fluid flow.

Based on analysis similar to previously done, we can say that the entropy generated at an arbitrary point in a flow per unit time per unit volume, $S''_{gen}$ is:

$$S''_{gen} = \frac{k}{T^2} (\nabla T)^2 + \frac{\mu}{T} \nabla \phi \geq 0 \quad \text{⇒ 1st term represents entropy gen. due to temp. gradients, while second term represents viscous dissipation, (pressure drop)}$$

$$\quad \geq 0 \quad \geq 0$$

In a 2D convective field, the local $S''_{gen}$ is:

$$S''_{gen} = \frac{k}{T^2} \left[ \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 \right] + \frac{\mu}{T} \left[ 2 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right] + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \geq 0$$

Note, this is very useful for entropy generation minimization which is a whole field in itself. We can design our flow geometry such that $S_{gen}$ is minimized along with $W_{lost}$. (18)