

Now the x-momentum equation becomes:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\rho}{\partial x} \left[2u \frac{\partial u}{\partial x} - \frac{2u}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{\rho}{\partial y} \left[u \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]$$

$$= -\frac{\partial p}{\partial x} + 2u \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial y^2} + u \frac{\partial}{\partial y} \cdot \frac{\partial v}{\partial x} + X$$

But note: $u \frac{\partial}{\partial y} \cdot \frac{\partial v}{\partial x} = u \frac{\partial}{\partial x} \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ (cons. of mass)

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + 2u \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^2 u}{\partial y^2} - u \frac{\partial^2 u}{\partial x^2} + X$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + X$$

Expanding the left hand side:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + X$$

↳ X-momentum equation (20).

In general:

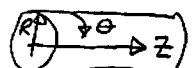
$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + F$$

Where F is the body force per unit volume. Similar equations can be derived for cylindrical & spherical coordinates. For example (you may use this one):

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z}$$

$$+ \mu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + Z$$

↳ z-momentum eqn. in cylindrical coord.



First Law of Thermodynamics

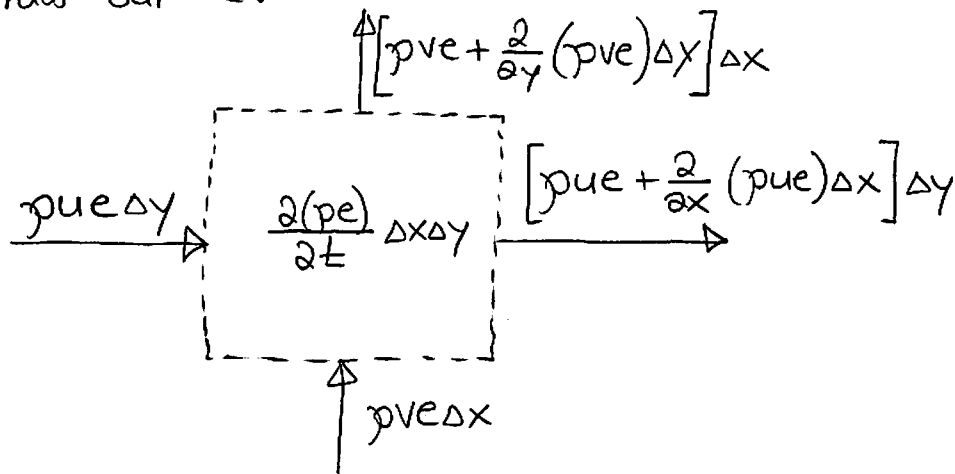
So far, we've equipped ourselves well to solve fluid mechanics problems. However, the heat transfer part of the convection problem requires a solution to the temperature distribution in the flow (especially close to the wall).

To solve, we need the energy equation (1st law)

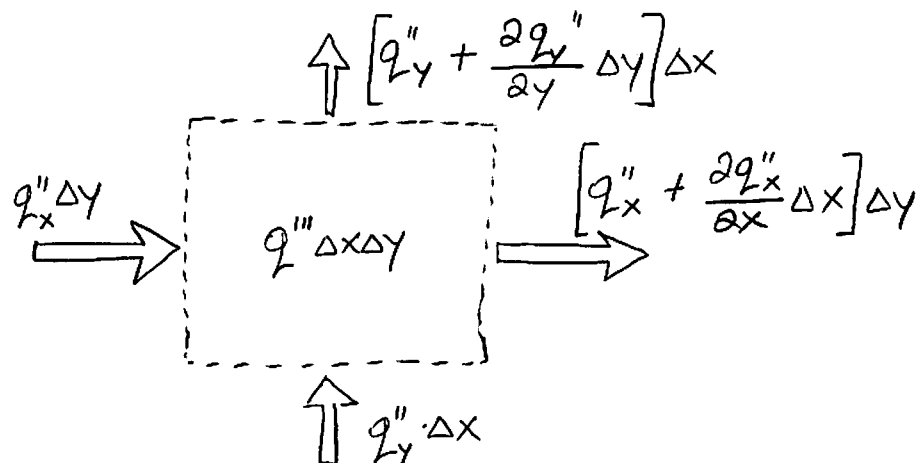
$$\left(\text{Rate of energy accumulation in the CV} \right)_1 = \left(\text{net transfer of energy by fluid flow} \right)_2 + \left(\text{net heat transfer by conduction} \right)_3 + \left(\text{heat gener.} \right)_4 - \left(\text{work of CV on its env.} \right)_5$$

Let's draw our CV

①+②



③+④



Note, I haven't drawn the work contribution to either CV. (5) (11)

So if we write out our terms (1) to (5)

$$(\dots)_1 = \Delta x \Delta y \frac{\partial}{\partial t} (\rho e) \quad ; \quad e \equiv \text{specific internal energy}$$

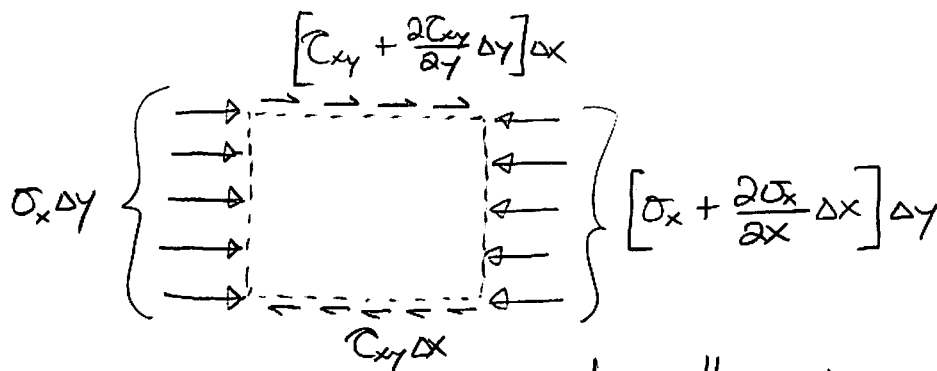
$$(\dots)_2 = -(\Delta x \Delta y) \left[\frac{\partial}{\partial x} (\rho u e) + \frac{\partial}{\partial y} (\rho v e) \right]$$

$$(\dots)_3 = -(\Delta x \Delta y) \left(\frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y} \right)$$

$$(\dots)_4 = (\Delta x \Delta y) q'''$$

$$(\dots)_5 = (\Delta x \Delta y) \left(\sigma_x \frac{\partial u}{\partial x} - \tau_{xy} \frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} - \tau_{yx} \frac{\partial v}{\partial x} \right) + \left\{ (\Delta x \Delta y) \left(u \frac{\partial \sigma_x}{\partial x} - u \frac{\partial \tau_{xy}}{\partial y} + v \frac{\partial \sigma_y}{\partial y} - v \frac{\partial \tau_{yx}}{\partial x} \right) \right\}$$

For the work term (5) we had to use our CV from the x-momentum equation.



For example, $W_{\text{LEFT}} = \underbrace{(\sigma_x \Delta y)}_{\text{force on the bound.}} \cdot \underbrace{u}_{\text{boundary displacement per unit time}}$ work on the environment

Similarly, on the right side:

$$W_{\text{RIGHT}} = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x \right) \Delta y \cdot \left(u + \frac{\partial u}{\partial x} \Delta x \right)$$

The net work transfer rate due to these two contributions is

$$\begin{aligned}
 W_{\text{NET}} &= \left(\sigma_x \Delta y + \frac{\partial \sigma_x}{\partial x} \Delta x \Delta y \right) \left(u + \frac{\partial u}{\partial x} \Delta x \right) - u \sigma_x \Delta y \\
 &= \cancel{u \sigma_x \Delta y} + \sigma_x \frac{\partial u}{\partial x} \Delta x \Delta y + u \frac{\partial \sigma_x}{\partial x} \Delta x \Delta y + \frac{\partial \sigma_x}{\partial x} \frac{\partial u}{\partial x} \Delta x^2 \Delta y - \cancel{u \sigma_x \Delta y} \\
 &= \sigma_x \frac{\partial u}{\partial x} \Delta x \Delta y + u \frac{\partial \sigma_x}{\partial x} \Delta x \Delta y + \frac{\partial \sigma_x}{\partial x} \cdot \frac{\partial u}{\partial x} \Delta x^2 \Delta y
 \end{aligned}$$

The last term goes to zero when we divide by $\Delta x \cdot \Delta y$ and let $\Delta x \rightarrow 0$ & $\Delta y \rightarrow 0$

Similarly, we can do the same procedure for the other stresses.

We can also show that the 4 terms in the $\{ \}$ of equat. (5) reduce to: (try deriving it \Rightarrow substitute in $\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx}$)

$$\Delta x \Delta y \left(u \frac{\partial \sigma_x}{\partial x} + u \frac{\partial \tau_{xy}}{\partial y} + v \frac{\partial \sigma_y}{\partial y} - v \frac{\partial \tau_{yx}}{\partial x} \right) = -\rho \frac{D}{Dt} \left(\frac{u^2 + v^2}{2} \right)$$

This equates to the change in kinetic energy of the fluid packet.

Since $-\rho \frac{D}{Dt} \left(\frac{u^2 + v^2}{2} \right) \ll \frac{\partial(\rho e)}{\partial t}$, we neglect these terms.

Assembling everything together: (back substituting constitutive rel.)

$$\rho \frac{De}{Dt} + e \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} \right) = -\nabla \cdot \mathbf{q}'' + \mathbf{q}'' \cdot \nabla - \rho \nabla \cdot \mathbf{v} + u \Phi$$

\hookrightarrow 2D energy equation, where:

$\mathbf{q}'' \equiv$ heat flux vector (q''_x, q''_y)

$\Phi =$ viscous dissipation function

To solve for Φ we need to back substitute our constitutive relations into (5) and simplify.

In cartesian coordinates (3D)

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 \right] - \frac{2}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2$$

To simplify our lives, we can assume 2D & incompressible

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2$$

Now we can deal with simplifying our other terms.
From Thermodynamics:

$$h = e + \left(\frac{1}{\rho} \right) P \Rightarrow \text{enthalpy}$$

$$\frac{Dh}{Dt} = \frac{De}{Dt} + \frac{1}{\rho} \frac{DP}{Dt} - \frac{P}{\rho^2} \frac{D\rho}{Dt} \quad (\text{quotient rule})$$

Also, via Fourier's law of heat conduction

$$q'' = -k \nabla T$$

Back substituting:

$$\rho \frac{Dh}{Dt} = \nabla \cdot (k \nabla T) + q''' + \frac{DP}{Dt} + \mu \Phi - \underbrace{\frac{P}{\rho} \left(\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} \right)}_0$$

Finally, from mass conservation:

$$\rho \frac{Dh}{Dt} = \nabla \cdot (k \nabla T) + q''' + \frac{DP}{Dt} + \mu \Phi$$

Note, it's tempting to say $h = c_p T$, however this is only correct for ideal gases. More rigorously

$$dh = T ds + \frac{1}{\rho} dP \quad (1) \Rightarrow T = \text{absolute temp.}$$

$ds = \text{specific entropy change}$

We can now write the following:

$$ds = \left(\frac{\partial s}{\partial T} \right)_P dT + \left(\frac{\partial s}{\partial P} \right)_T dP \quad (2)$$

From Maxwell's relations

$$\left(\frac{\partial s}{\partial P} \right)_T = - \left[\frac{\partial (1/\rho)}{\partial T} \right]_P = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial T} \right)_P = - \frac{\beta}{\rho} \quad (3)$$

where $\beta = - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_P \equiv \text{coefficient of thermal expansion}$ (4)

Using thermodynamic relations, we can also show that:

$$\left(\frac{\partial s}{\partial T} \right)_P = \frac{c_p}{T} \quad (5)$$

Putting equations (1) to (5) together, we obtain

$$dh = c_p dT + \frac{1}{\rho} (1 - \beta T) dP$$

Back substituting into our energy equation, we obtain

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + \dot{q}''' + \beta T \frac{DP}{Dt} + \mu \Phi$$

\hookrightarrow Temperature based 1st Law of Thermodynamics
Note, we can simplify this for certain cases

For an ideal gas: $\beta = \frac{1}{T}$

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + q''' + \frac{DP}{Dt} + \mu \Phi$$

For an incompressible liquid (water): $\beta = 0$ (in real life, $\beta_{\text{water}} \approx 0$)

$$\rho c \frac{DT}{Dt} = \nabla \cdot (k \nabla T) + q''' + \mu \Phi$$

Note, these are general solutions but for most of the problems we'll solve in this course, we will have:

- 1) $k = \text{constant}$ (constant properties)
- 2) $q''' = 0$ (no internal energy generation)
- 3) $\mu \Phi \approx 0$ (negligible viscous dissipation)
- 4) $\beta T \frac{\partial P}{\partial T} \approx 0$ (negligible compressibility effects)

So our formulation boils down to:

$$\rho c_p \frac{DT}{Dt} = k \nabla^2 T$$

For cartesian (x, y, z)

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

For cylindrical (r, θ, z)

$$\rho c_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right.$$

For spherical (r, ϕ, θ):

$$\left. + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\phi}{r} \frac{\partial T}{\partial \phi} + \frac{v_\theta}{r \sin \phi} \frac{\partial T}{\partial \theta} \right) = k \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \dots \right] \quad (16)$$

Note, for extremely viscous fluids & flows (i.e. lubrication problems or piping of crude oil), the viscous dissipation must be taken into account:

$$\rho c_p \frac{DT}{Dt} = k \nabla^2 T + \mu \Phi$$

For exact definitions of Φ in other coordinate systems, see pg. 14 of Bejan (4th edition).

When the energy equation was first developed by Fourier & Poisson, they always used c instead of c_p . This is ok since they were working with incompressible fluids ($c = c_p$). However, c_v should never be used as it ignores the PdV work done by the moving fluid packet as it expands and contracts.

Second Law of Thermodynamics

We don't need the second law of thermo to solve convection problems. Where it becomes important is when we design systems that have convective heat transfer with a fluid.

The second law states all processes are irreversible:

$$\frac{dS_{cv}}{dt} \geq \sum \frac{q_i}{T_i} + \sum_{\text{inlet}} \dot{m}s - \sum_{\text{outlet}} \dot{m}s$$

where; $S_{cv} \equiv$ instantaneous entropy in the control volume
 q_i & $T_i \equiv$ heat transfer and absolute temperature of the boundary i . (+ means inflow)

The measure of the irreversibility of the process is quantified by the strength of the inequality, or the entropy generation rate S_{gen} :

$$S_{gen} = \frac{dS_{scr}}{dt} - \sum \frac{q_i}{T_i} - \sum_{inlet} \dot{m}s + \sum_{outlet} \dot{m}s \geq 0$$

We also know from exergy analysis that

$$W_{lost} = T_0 S_{gen} \equiv \text{destruction of usefull work due to entropy generation.}$$

$T_0 = \text{absolute temp. of ambient reservoir.}$

This shows the engineering importance of making the convection process as efficient as possible and estimating S_{gen} for a fluid flow.

Based on analysis similar to previously done, we can say that the entropy generated at an arbitrary point in a flow per unit time per unit volume, S_{gen}''' is:

$$S_{gen}''' = \underbrace{\frac{k}{T^2} (\nabla T)^2}_{\geq 0} + \underbrace{\frac{\mu}{T} \Phi}_{\geq 0} \geq 0 \Rightarrow \text{1st term represents entropy gen. due to temp. gradients, while second term represents viscous dissipation, (pressure drop)}$$

In a 2D convective field, the local S_{gen}''' is:

$$S_{gen}''' = \frac{k}{T^2} \left[\left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 \right] + \frac{\mu}{T} \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \geq 0$$

Note, this is very usefull for entropy generation minimization which is a whole field in itself. We can design our flow geometry such that S_{gen} is minimized along with W_{lost} . (18)