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Turbulent Internal Flows

For internal flows, it's a similar procedure as external. Turbulence acts to increase mixing an hence increase shear and heat transfer.

Our time averaged governing equations become (for a tube)

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\bar{v}) = 0 \equiv \text{Mass Conservation}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left[r(\nu+\varepsilon)\frac{\partial \bar{u}}{\partial r}\right] \equiv \text{Momentum}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\alpha+\varepsilon_t)\frac{\partial \bar{T}}{\partial r}\right] \equiv \text{Energy}$$

For turbulent flows:

$$\boxed{\frac{X_{EL}}{D} \approx \frac{X_T}{D} = 10} \Rightarrow \begin{array}{l} X_{EL} \equiv \text{hydrodynamic developing length} \\ X_T \equiv \text{thermal developing length} \end{array}$$

Note, $\left.\begin{array}{l} X_{EL,\,Laminar} >> X_{EL,\,Turbulent} \\ X_{T,\,Lam.} >> X_{T,\,Turb.} \end{array}\right\}$ Better mixing in turbulent b.l.'s.

For fully developed flow, we can use scaling to show:

$$0 = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left[r(\nu+\varepsilon)\frac{\partial \bar{u}}{\partial r}\right] \quad (1)$$

Integrating (1) from the centerline $(r=0)$ to any r:

$$\int_0^r \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} r\,dr = \int_0^r \partial\left[r(\nu+\varepsilon)\frac{\partial \bar{u}}{\partial r}\right]$$

$$\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}\frac{r^{\cancel{2}}}{2} = \cancel{r}(\nu+\varepsilon)\frac{\partial \bar{u}}{\partial r} \quad (2)$$

At the wall: $r = r_0$, $\varepsilon << \nu \Rightarrow \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x}\cdot\frac{r_0}{2} = \nu\left.\frac{\partial \bar{u}}{\partial r}\right|_0$ (3)

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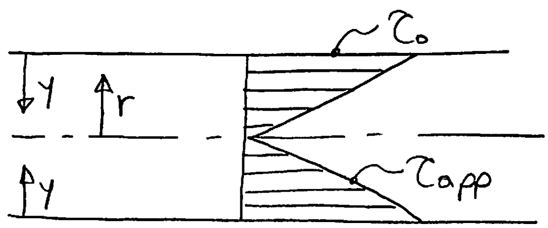
But remembering from before:

$$\tau_{app} = \rho(\nu + \varepsilon)\frac{\partial \bar{u}}{\partial y} \quad \text{and} \quad \tau_o = \rho \nu \frac{\partial \bar{u}}{\partial y}\Big|_{y=0}$$

Here, $y = r_o - r$ (distance from the wall)
Dividing ② and ③, we obtain: (and noting $r = r_o - y$)

$$\frac{\frac{1}{\cancel{\rho}}\cancel{\frac{\partial p}{\partial x}}\frac{r}{\cancel{2}}}{\frac{1}{\cancel{\rho}}\cancel{\frac{\partial p}{\partial x}}\cdot\frac{r_o}{\cancel{2}}} = \frac{\tau_{app}}{\tau_o} \Rightarrow \boxed{\frac{\tau_{app}}{\tau_o} = 1 - \frac{y}{r_o}}$$

↳ Fully developed flow momentum eqn.



Very close to the wall $\left(\frac{y}{r_o} \ll 1\right)$: $\frac{\tau_{app}}{\tau_o} = 1$ or $\tau_{app} = \tau_o = \text{const}$

We can now use the mixing length model developed previously:
Same result as before!

$$\left.\begin{array}{ll} u^+ = y^+ & 0 < y^+ < 11.6 \\ u^+ = 2.5 \ln y^+ + 5.5 & y^+ > 11.6 \end{array}\right\} \text{Prandtl \& Taylor}$$

Note our model breaks down at the pipe centerline since

$$\frac{\partial u^+}{\partial y^+}\Big|_{y=r_o} = \frac{2.5\nu}{r_o(\tau_o/\rho)^{1/2}}$$

So people usually use an emperical profile with $\frac{\partial u^+}{\partial y^+}\Big|_{y=r_o} = 0$

$$\boxed{u^+ = 2.5 \ln\left[\frac{3(1 + r/r_o)}{2[1 + 2(r/r_o)^2]}\cdot y^+\right] + 5.5} \Rightarrow \text{Pipe flow law of the wall}$$

Note, for $y \to 0$ (wall), $u^+ = 2.5\ln(y^+) + 5.5$
same as before!

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Remembering our definition of friction factor ($\bar{U}$ = average velocity)

$$f = \frac{\tau_o}{\frac{1}{2}\rho \bar{U}^2} \quad ; \quad \bar{U} = \frac{1}{\pi r_o^2}\int_0^{2\pi}\int_0^{r_o} \bar{u}\, r\, dr\, d\theta$$

Let's consider Prandtl's 1/7 power law and solve:
Note, we will assume it holds all the way to the centerline, however it doesn't in real life. It works well at the wall though.

At the centerline, $\bar{u} = u_c$, $y = r_o$

$$u^+ = 8.75(y^+)^{1/7} \quad ; \quad u^+ = \frac{\bar{u}}{u_*} = \frac{\bar{u}}{(\tau_o/\rho)^{1/2}} \quad , \quad y^+ = \frac{y u^*}{\nu}$$

$$u_c^+ = 8.75(y_c^+)^{1/7} \quad \text{(Apply at the centerline)}$$

$$\frac{u_c}{(\tau_o/\rho)^{1/2}} \cong 8.75\left[\frac{r_o(\tau_o/\rho)^{1/2}}{\nu}\right]^{1/7} \quad ①$$

Rearanging our friction factor definition:

$$f = \frac{\tau_o}{\frac{1}{2}\rho \bar{U}^2} \Rightarrow \left(\frac{\tau_o}{\rho}\right)^{1/2} = \left(\frac{1}{2} f \bar{U}^2\right)^{1/2} = \bar{U}\left(\frac{f}{2}\right)^{1/2} \quad ②$$

$$(\tau_o/\rho)^{1/2} = u_* = \bar{U}\left(\frac{f}{2}\right)^{1/2}$$

We can also say:

$$\frac{\bar{u}}{u_*} \cong 8.75\left(\frac{y u_*}{\nu}\right) = 8.75\left(\frac{y(\tau_o/\rho)^{1/2}}{\nu}\right)^{1/7} \quad ③$$

Divide ③ by ①, we obtain:

$$\frac{\bar{u}/\cancel{u_*}}{u_c/\cancel{u_*}} \cong \frac{\cancel{8.75}\left(\frac{y \cancel{u_*}}{\cancel{\nu}}\right)^{1/7}}{\cancel{8.75}\left(\frac{r_o \cancel{u_*}}{\cancel{\nu}}\right)^{1/7}} \Rightarrow \frac{\bar{u}}{u_c} = \left(\frac{y}{r_o}\right)^{1/7} \quad ④$$

Back substituting ④ into our definition for $\bar{U}$ (avg. velocity)

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$$U = \frac{1}{\pi r_o^2}\int_0^{2\pi}\int_0^{r_o} \bar{u}\, r\,dr\,d\theta$$

For a pipe, $\int_0^{2\pi} d\theta = 2\pi \Rightarrow U = \frac{1}{\pi r_o^2}\int_0^{r_o} \bar{u}\, 2\pi r\,dr$ (no-θ-dependence)

$$\bar{u} = u_c\left(\frac{y}{r_o}\right)^{1/7}$$

$$U = \frac{1}{\pi r_o^2}\int_0^{r_o} u_c\left(\frac{y}{r_o}\right)^{1/7} 2\pi r\,dr \Rightarrow y = r_o - r$$

$$= \frac{1}{\pi r_o^2}\int_0^{r_o} u_c\left(\frac{r_o - r}{r_o}\right)^{1/7} 2\pi r\,dr$$

$$= \frac{2\cancel{\pi} u_c}{\cancel{\pi} r_o^2}\int_0^{r_o} r\left(1 - r/r_o\right)^{1/7} dr = \frac{2u_c}{r_o^2}\int_0^{r_o}\left(r^7 - \frac{r^8}{r_o}\right)^{1/7} dr$$

Solving numerically or analytically, you obtain:

$$\frac{u_c}{U} = \frac{120}{98} \quad (5)$$

Now we can solve for our friction factor by back substituting into (1) and using (2) & (5)

$$\frac{u_c}{U\left(\frac{f}{2}\right)^{1/2}} \cong 8.75\left[\frac{r_o\left(U\left(\frac{f}{2}\right)^{1/2}\right)}{\nu}\right]^{1/7} \Rightarrow \text{Substitute (5) in here}$$

$$\frac{120}{98\left(\frac{f}{2}\right)^{1/2}} \cong 8.75\left[\underbrace{\frac{r_o U}{\nu}}_{Re_{r_o} = \frac{1}{2}Re_D}\left(\frac{f}{2}\right)^{1/2}\right]^{1/7}$$

$$\frac{120\cdot\sqrt{2}}{98 f^{1/2}} \cong 8.75\left(\frac{Re_D}{2\cdot\sqrt{2}}\cdot f^{1/2}\right)^{1/7} \Rightarrow \frac{0.1979}{f^{1/2}} \cong \left(0.3536\, Re_D f^{1/2}\right)^{1/7}$$

$$\frac{0.2295}{Re_D^{1/7}} = f^{1/14}\cdot f^{1/2} = f^{8/14} \Rightarrow \text{Raise both sides to power of } \frac{14}{8}$$

$$\left(0.2295\cdot Re_D^{-1/7}\right)^{14/8} = f \Rightarrow \boxed{f = 0.078\, Re_D^{-1/4}} \Rightarrow Re_D < 80{,}000 \Rightarrow \text{Turbulent}$$

↳ Smooth Pipe

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Note how close our solution was to Blasius' empirical formula:

$$\boxed{f \cong 0.079\, Re_0^{-1/4}} \Rightarrow \text{Blasius}$$

For a more accurate solution using the law at the wall ($u^+ = 2.5 \ln y^+ + 5.5$) instead of the 1/7 power law:

$$\boxed{\frac{1}{f^{1/2}} = 1.737 \ln \left[Re_0 f^{1/2} \right] - 0.396}$$

⇒ Kármán - Nikuradse
⇒ $Re_0 < 10^6$
⇒ Smooth tubes

↳ This is the smooth tube solution

Effect of Surface Roughness

Roughness important due to thin nature of laminar sublayer.

Since: $y^+_{VSL} \sim O(10)$ ⇒ For water flow

$U \sim 10 \text{m/s}$, $\nu \sim 0.01 \text{cm}^2/\text{s}$

$$\boxed{y_{VSL} \sim 0.01 \text{mm} = 10 \mu m\,!}$$

So even minute roughness not felt to our touch has big effects on the flow.

For fully rough tubes where roughness size exceeds the order of magnitude of what would have been the viscous sublayer,

$$k_s^+ = \frac{k_s (\tau_0/\rho)^{1/2}}{\nu} > O(10) \; ; \; k_s = \text{grain or rougness size}$$

Then, Nikuradse showed:

$$\boxed{f \cong \left(1.74 \ln\left(\frac{D}{k_s}\right) + 2.28\right)^{-2}}$$

⇒ rough pipes (fully rough)
⇒ Turbulent flow in pipes

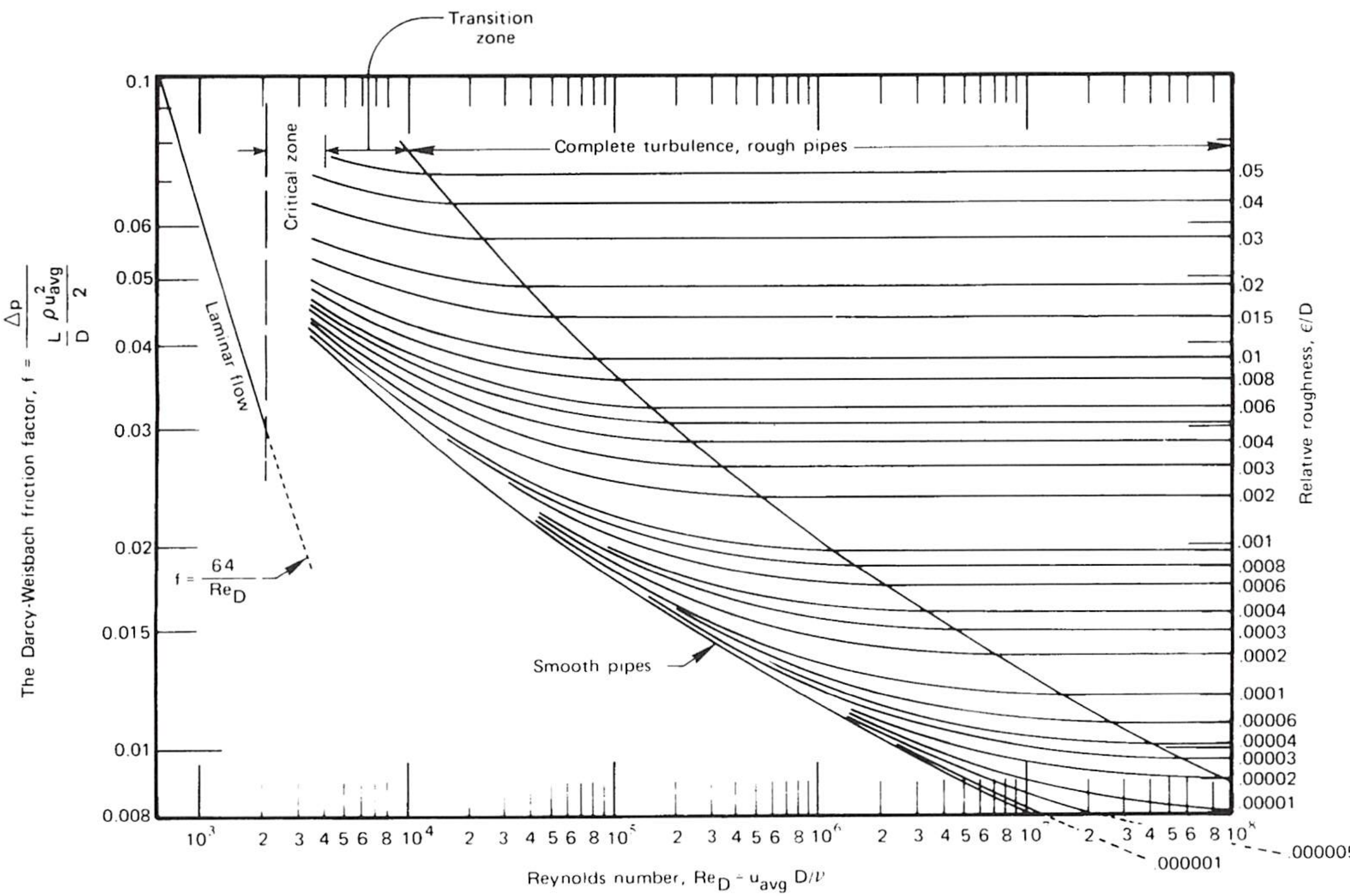

Figure 7.6 Pipe friction factors.

*Adapted from Lienhard & Lienhard, 4'Th Ed, pp. 361

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