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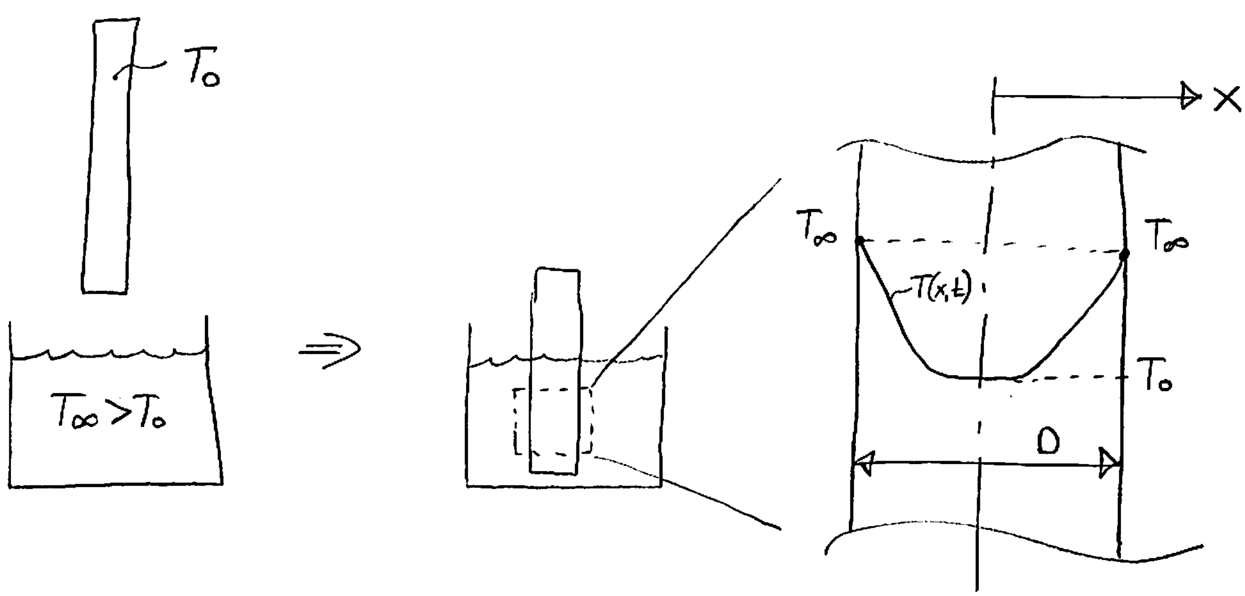
Scaling (or Scale Analysis)

This method will provide most information per unit of intellectual effort.

Different from dimensional analysis (Buckingham π theorem) as you usually know the dimensional differential equations governing the problem at hand.

Where the two meet is when determining key parameters.

Let's do an example to see how scaling works. Consider a metal bar plunged into a hot fluid.



Assuming that the bar surface temperature assumes the fluid temperature instantaneously. What is the time required for the thermal front to penetrate the plate, i.e. the time for the center of the plate to "feel" the heating imposed on the outer surfaces.

Note, for $T(x=\frac{D}{2}, t=0) = T_\infty$ to be true, is Bi low or high?

Here $Bi = \frac{\text{conduction resistance}}{\text{convection resistance}} = \frac{L/k\cancel{A}}{1/h\cancel{A}} = \frac{hL}{k_{BAR}}$

We know if $Bi < 0.1$, $T_{Bar} \neq f(x)$ & only $f(t)$, hence $Bi >> 0.1$ or h is very high.

Remember, $q = \frac{\Delta T}{R}$, so for $q = \text{constant}$ $\frac{\Delta T_{conv}}{\frac{1}{h\cancel{A}}} = \frac{\Delta T_{cond}}{\frac{L}{k\cancel{A}}}$

$$h \Delta T_{conv} = k \frac{\Delta T_{cond}}{L}$$

So for h very high, $\Delta T_{conv} << \Delta T_{cond}$ & $\boxed{T\left(x = \frac{D}{2}, t\right) = T_\infty}$

QED

Now back to our problem. We can begin by writing our energy equation in cartesian:

$$\rho C_p \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$
$$+ q''' + \beta T \frac{DP}{Dt} + \mu \Phi$$

We can drop many of our terms since its a stationary CV $u = v = w = 0$.

Also, no temperature gradients in the y, z direction. Hence:

$$\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad (1)$$

Scaling requires us to estimate the order of magnitude of each term appearing in our differential equation (1)

LHS: $\rho C_p \frac{\partial T}{\partial t} \Rightarrow$ $\rho \sim \rho$ (Notation: "$\sim$" $\equiv$ is of the same order of magnitude as)

$$C_p \sim C_p$$

$$\frac{\partial T}{\partial t} \sim \frac{\Delta T}{t} \sim \frac{T_\infty - T_0}{t}$$

$$\rho C_p \frac{\partial T}{\partial t} \sim \rho C_p \frac{\Delta T}{t}$$

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RHS: $k \sim k$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) \sim \frac{1}{D/2} \cdot \frac{\Delta T}{D/2}$$

$$k\frac{\partial^2 T}{\partial x^2} \sim k\frac{\Delta T}{(D/2)^2}$$

Now equating our two terms:

$$\rho c_p \frac{\Delta T}{t} \sim k\frac{\Delta T}{(D/2)^2} \Rightarrow \boxed{t \sim \frac{(D/2)^2 \rho c_p}{k} \sim \frac{(D/2)^2}{\alpha}}$$

$$\boxed{\alpha = \frac{k}{\rho c_p} \equiv \text{thermal diffusivity}}$$

So very simple scaling has shown us that:

$\boxed{D \sim \sqrt{\alpha t}}$ $\Rightarrow$ where $D \equiv$ thermal penetration depth.

Rules of Scaling

1. Define the spatial extent of the region in which scaling is performed (this is the common thread to dimensional analysis). For example, we just used $D/2$ for the bar problem. For a boundary layer problem, lengths like L or δ are important.

2. Any equation constitutes an equivalence between the scales of the two dominant terms appearing in the equation. From our last example, the eqn. has only 2 terms, hence they must be same order of magnitude. In general, eqn's can have more terms, not all of them important.

③ If we sum two terms (a & b)

$$c = a + b$$

and the order of one term is greater than the other

$$O(a) > O(b) \quad \Rightarrow O(\cdot\cdot) \text{ denotes "order of magnitude"}$$

then the order of the sum is dominated by the dominant term:

$$O(c) \sim O(a)$$

④ If we sum two terms (a & b)

$$c = a + b$$

and the two terms are the same order of magnitude

$$O(a) \sim O(b)$$

then the sum is also of the same order of magnitude

$$O(c) \sim O(a) \sim O(b)$$

⑤ In any product : $p = ab$

The order of magnitude of the product is equal to the product of the orders of magnitude:

$$O(p) \sim O(a) \cdot O(b)$$

If ratio : $r = \frac{a}{b}$, then $O(r) \sim \frac{O(a)}{O(b)}$

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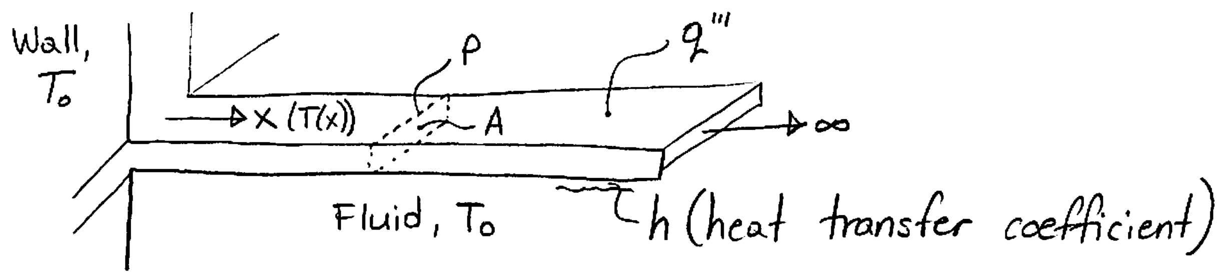
Example Scaling with a fin with heat generation.



Writing out our energy equation, we will obtain:

$$kA\frac{\partial^2 T}{\partial x^2} - hP(T - T_0) + q'''A = 0$$

a) Let T_∞ be the fin temperature sufficiently far from the wall. Show if x is large enough that longitudinal conduction becomes negligible.

$$kA\frac{\partial^2 T}{\partial x^2} \sim kA\frac{\Delta T}{\delta^2} \quad \text{where } \Delta T = T - T_0$$

$$hP(T - T_0) \sim hP\Delta T$$

$$q'''A \sim q'''A$$

Our equation becomes:

$$kA\frac{\Delta T}{\delta^2} - hP\Delta T + q'''A = 0$$

Normalizing by the conduction term $\left(kA\frac{\Delta T}{\delta^2}\right)$

$$\frac{kA\frac{\Delta T}{\delta^2}}{kA\frac{\Delta T}{\delta^2}} - \frac{hP\cancel{\Delta T}}{kA\frac{\cancel{\Delta T}}{\delta^2}} + \frac{q'''\cancel{A}}{k\cancel{A}\frac{\Delta T}{\delta^2}} = 0$$

$$1 - \frac{hP\delta^2}{kA} + \frac{q'''\delta^2}{k\Delta T} = 0$$ $\Rightarrow$ Hence as $\delta \to \infty$, the first term becomes negligible.

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b) Using scaling determine the fin temperature far from the wall.

We just showed conduction is negligible there, hence:

$$hP\Delta T \sim q'''A$$

$$\boxed{T_\infty - T_0 \sim \frac{q'''A}{hP}}$$

c) How far away from the wall is the heat transfer dominated by a balance of heat conduction & heat generation only?

We have from our equation in (a) that:

$$1 - \underbrace{\frac{hP\delta^2}{kA}}_{\text{negligible compared to 1}} + \frac{q'''\delta^2}{k\Delta T} = 0$$

or $\frac{hP\delta^2}{kA} \ll 1$

$$\boxed{\delta \ll \sqrt{\frac{kA}{hP}}}$$

Another way to do this is to say near the wall:

$$1 \sim \frac{q'''\delta^2}{k\Delta T}$$

$$\boxed{\delta \sim \sqrt{\frac{k\Delta T}{q'''}}} \Rightarrow \text{Not the best way since } \Delta T = T - T_0$$

Note if we balance the two:

$$\sqrt{\frac{k\Delta T}{q'''}} \ll \sqrt{\frac{kA}{hP}} \Rightarrow \frac{\Delta T}{q'''} \ll \frac{A}{hP} \quad \text{or} \quad \boxed{h \ll \frac{q'''A}{P\Delta T}}$$

Makes sense, if P is large, inequality is hard to obtain.

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Dimensional Analysis (Buckingham π Theorem)
Allows us to find the interdependence of a system's physical variables by consideration of appropriate balances, (i.e. forces, energies, etc...)

$\Rightarrow$ Requires physical insight, a knowledge of the physics and experience in the field.

The theorem: For a system with M physical variables (eg. density, speed, length, viscosity) describable in terms of N fundamental units (i.e. mass, length, time, temperature), there are M-N dimensionless groups.

1. If $M-N=P$, then the system is characterized by expressing the interdependency of the P dimensionless groups.

2. If $M-N=1$, then the single dimensionless parameter must be constant π

Obtaining Dimensionless Groups

1. List all physical variables (M) and fundamental units (N) and identify their dimensional form.

   Denote "dimensions of x" by [x]

   eg. $\underset{\text{velocity}}{[v]} = \frac{L}{T}$, $\underset{\text{gravity}}{[g]} = \frac{L}{T^2}$, $\underset{\text{density}}{[\rho]} = \frac{M}{L^3}$, $\underset{\text{force}}{[F]} = \frac{ML}{T^2}$

2. Determine the number of dimensionless groups (M-N)

3. Form a set of (M-N) dimensionless groups by assuming arbitrary exponents for each of the physical variables

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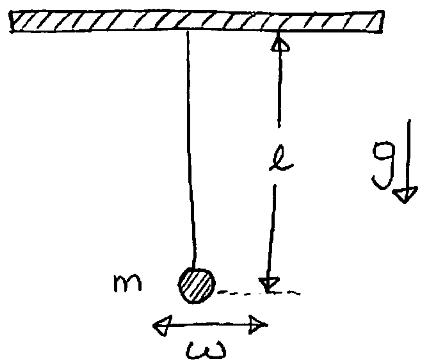
Be sure that each of the physical variables appears in at least one group.

This last part is where intuition is very usefull.

<u>Example #1</u> Pendulum (with small amplitude θ)



Physical Variables: m, ℓ, g, ω
Fundamental Units: M, L, T

Buckingham Theorem: $M - N = 4 - 3 = 1$ independent dimensionless group

Now we can solve:

$$\pi = \omega \cdot \ell^a \cdot g^b \cdot m^c$$

$$[\omega] = \frac{1}{T}, \quad [\ell] = L, \quad [g] = \frac{L}{T^2}, \quad [m] = M$$

$$[\pi] = \frac{1}{T} \cdot L^a \cdot \frac{L^b}{T^{2b}} \cdot M^c = 1$$ (since only 1 dimensionless group)

$$\left.\begin{aligned} T&: \; 2b + 1 = 0 \Rightarrow b = -\frac{1}{2} \\ L&: \; a + b = 0 \Rightarrow a = \frac{1}{2} \\ M&: \; c = 0 \end{aligned}\right\} \quad \boxed{\pi = 1 = \omega \left(\frac{\ell}{g}\right)^{1/2}}$$

Now we can say: $\boxed{\omega \sim \left(\frac{g}{\ell}\right)^{1/2}}$

→ Scaling (this is where the two meet)

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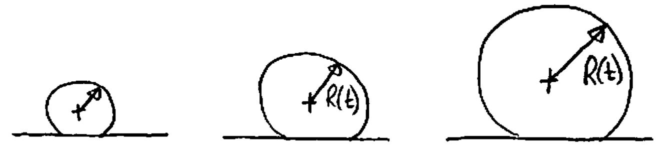
Example #2 / Atom-bomb blast cloud

What is the time-dependence of the radius of the blast cloud following the detonation of an atom bomb?

i.e. What is $R(t)$?



Physical variables: energy released, E
radius, R
time, t
air density, ρ

$\Big\}\ M = 4$

<u>Aside:</u>
Why not viscosity, μ? Consider the length & time scales. Inertia dominates any viscous effects, so no need to consider it. We could have included it, but it would end up coming out as $Re_R \gg 1$

Fundamental units: $M, L, T \Rightarrow N = 3 \Rightarrow 1$ dimensionless group

$$[E] = \frac{ML^2}{T^2}, \quad [R] = L, \quad [\rho] = \frac{M}{L^3}, \quad [t] = T$$

$$\Pi = E \cdot t^a \cdot \rho^b \cdot R^c = \frac{ML^2}{T^2} \cdot T^a \cdot \frac{M^b}{L^{3b}} \cdot L^c$$

$$\left.\begin{aligned} M&: \ 1 + b = 0 \Rightarrow b = -1 \\ L&: \ 2 + c = 3b \Rightarrow c = -5 \\ T&: \ a - 2 = 0 \Rightarrow a = 2 \end{aligned}\right\} \quad \boxed{\Pi = \frac{E t^2}{\rho R^5} = \text{constant}}$$

$$\therefore \boxed{R \sim \left(\frac{E}{\rho}\right)^{1/5} t^{2/5}} \quad \text{or} \quad \boxed{R(t) = c\left(\frac{E}{\rho}\right)^{1/5} t^{2/5}}$$

c → constant of proportionality

G.I. Tailor did this scaling for an atom bomb blast in 1945 & published it, finding the energy of the blast. He was arrested!

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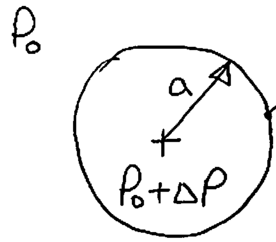
Example #3 | What is the pressure inside a bubble?



$\sigma \equiv$ surface tension $\quad [\sigma] = \dfrac{\text{Force}}{\text{Length}} = \dfrac{M}{T^2}$

Physical variables : ΔP, σ, a ⇒ note ΔP counts as 2 variables
Fundamental units : M, L, T

$$\left.\begin{aligned} [\Delta P] &= \left[\frac{F}{L^2}\right] = \frac{ML}{T^2}\cdot\frac{1}{L^2} \\ [\sigma] &= \left[\frac{F}{L}\right] = \frac{M}{T^2} \\ [a] &= L \end{aligned}\right\} P = M - N = 1$$

⇒ Means 1 dimensionless group will exist.

$$\Pi = \frac{\Delta P \cdot a}{\sigma} = \text{constant}$$

$$\boxed{\Delta P \sim \frac{\sigma}{a}}$$

So small bubbles will have higher pressure. This is why champagne is louder than beer. This is also how you tell the quality of a good champagne. Bubbles will rise slowly & pop loudly, since good champagnes are high pressure.

So far, these examples are meant to show you how important it is to chose your physical variables correctly. This also applies to scaling analysis.

For a more formal proof of why Buckingham Π theorem works, look at rank-nullity theorem.

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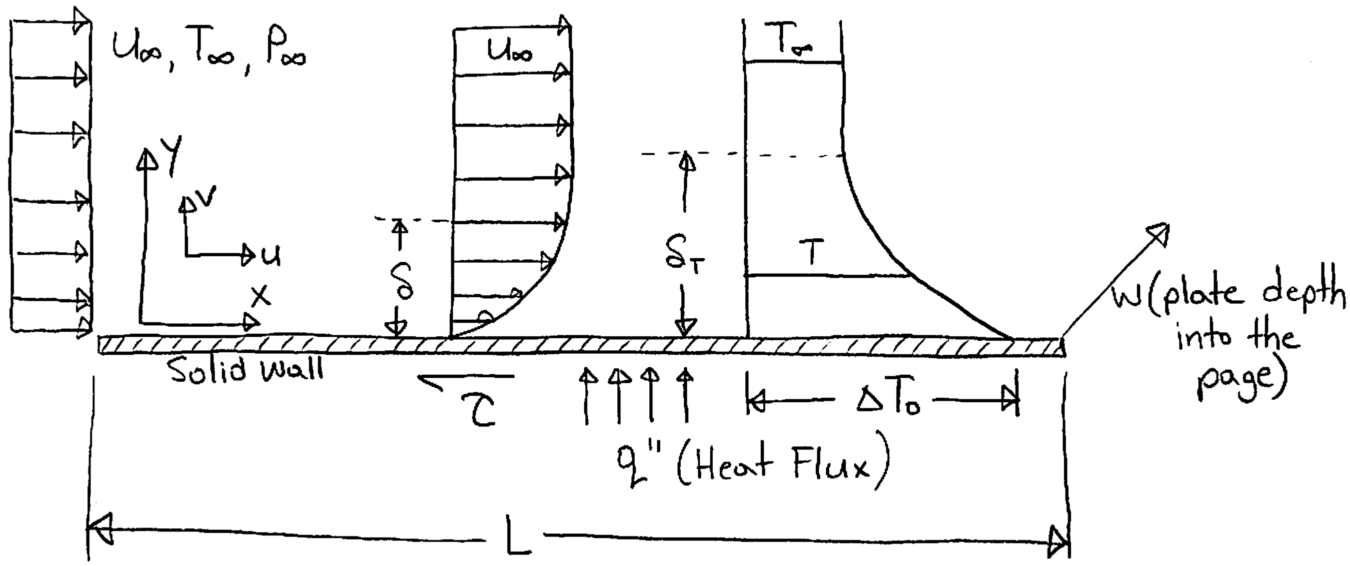
Laminar Boundary Layer Flow

So far; we have been developing tools to analyze convection problems.

Convection = advection + diffusion

Lets consider the simplest possible problem to analyze:



As engineers, we want to know:

1) The net force exerted by the stream on the plate
2) The resistance to heat transfer from the plate to stream.

We know we need the following: ($\tau \equiv$ skin friction)

$$F = \int_0^L \tau w \, dx \quad \text{and} \quad q = \int_0^L q'' w \, dx$$

For the Newtonian fluids we have been dealing with so far:

$$\boxed{\tau = \mu\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0}} \quad \text{and} \quad q'' = h(T - T_0)$$

$\hookrightarrow$ Heat Transfer coeff.

Aside:

Note, many people attribute $q = h\Delta T$ to Newton as his "law of cooling". However, this is not true. Newton found the following:

$$\frac{dT}{dt} \sim \Delta T$$ $\Rightarrow$ At that time, there was no concept of h or C_p.

The real person to write it as $q = h\Delta T$ was Fourier!

At this point, we can accept emperically that the no slip condition at the wall is true (i.e. $u(y=0)=0$). Therefore, in the fluid layer immediatelly adjacent to the wall, we have pure conduction:

$$q'' = -k_f\left(\frac{\partial T}{\partial y}\right)\bigg|_{y=0}$$ (note, heat flux is positive when the wall heats the stream)

Hence we can now write:

$$\boxed{h = \frac{-k_f(\partial T/\partial y)|_{y=0}}{T_0 - T_\infty}}$$

So we see that in order to solve for F & q, we must first solve for the velocity and temperature fields adjacent to our wall: (u, v, T).

But before we do that, it's important to consider:

$$\frac{h}{k_f} = \frac{-(\partial T/\partial y)|_{y=0}}{T_0 - T_\infty}$$ $\Rightarrow$ Making this dimensionless $\cdot \frac{L}{L}$

$$\frac{hL}{k_f} = \frac{-(\partial T/\partial y)|_{y=0}}{\left(\frac{T_0 - T_\infty}{L}\right)} = Nu_L \equiv \text{Nusselt Number}$$

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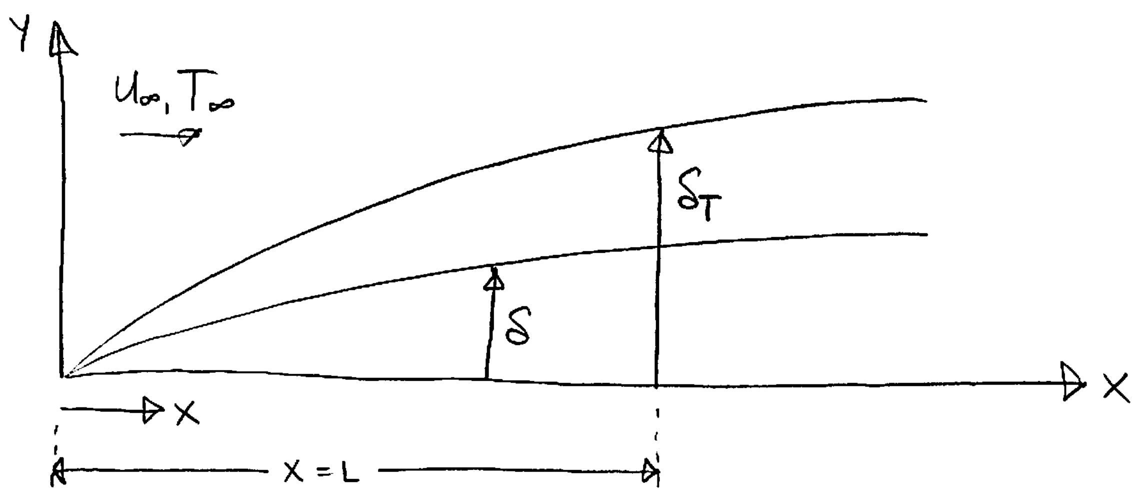
In this case, we can see that:

$$\boxed{Nu_L \equiv \frac{\text{Heat transfer due to convection (advection+diffusion)}}{\text{Heat transfer due to conduction only (diffusion)}}}$$

Note, the dimension L is chosen to be the distance in the direction of flow and is somewhat arbitrary since the denominator is not a "real" heat transfer due to conduction only.



So when you say: $Nu_L = \frac{hL}{k_f} = \frac{\text{convection h.t. at location L}}{\text{conduction h.t. of a water slab of thickness L.}}$

Many books say Nu is ratio of convection to conduction if the flow would stop, but this is incorrect since it would then become a transient problem. This is why sometimes Biot number is much easier to understand because it usually deals with finite shapes:

$$\boxed{Bi = \frac{ha}{k_{BODY}} \equiv \frac{\text{conduction resistance inside the body}}{\text{convection resistance outside the body}}}$$

a = body dimension.