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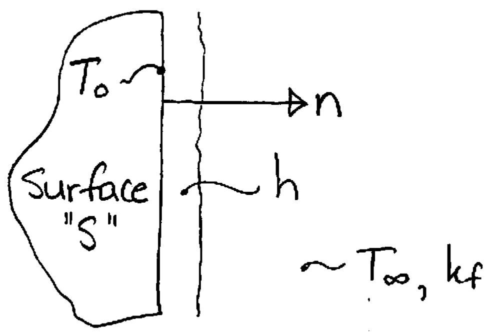
So let's see if we can get any more meaning out of Nu.



$$q'' = h(T_o - T_\infty) \quad (1)$$

or

$$q'' = -k_f \left.\frac{\partial T}{\partial n}\right|_s \quad (2)$$

Let's non-dimensionalize:

$$\theta = \frac{T - T_\infty}{T_o - T_\infty}, \quad n^* = \frac{n}{L}$$

≡ where L is some arbitrary length along the plate or surface "S".

Back substituting into (1) & (2) and equating them:

$$-k_f \frac{\cancel{(T_o - T_\infty)}}{L} \cdot \left.\frac{\partial \theta}{\partial n^*}\right|_s = h\,\cancel{(T_o - T_\infty)}$$

$$\boxed{-\left.\frac{\partial \theta}{\partial n^*}\right|_s = \frac{hL}{k_f} = Nu_L \equiv \text{Nusselt Number}}$$

So we can see that the Nusselt number also physically represents a non-dimensional temp. gradient at the surface. The larger the gradient, the larger the heat transfer.

In general, as we already showed, in order to solve for h, we first need u, v, T. Note (u,v) and (T) are coupled and we need to solve both the momentum & energy equations.

Also: $Nu \sim h = f(u, v, T)$, hence: $Nu = f($Flow conditions, geometry, fluid properties, & b.c.'s$)$

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So now we have all the tools to start solving for τ & h. Modeling our flow as incompressible & constant property:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(Mass conservation)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{(x-momentum)}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \text{(y-momentum)}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad \text{(energy)}$$

We have 4 unknowns (u, v, P, T) and the following b.c.'s

$$\left.\begin{array}{l} \text{i) No slip} \Rightarrow u = 0 \\ \text{ii) Impermeability} \Rightarrow v = 0 \\ \text{iii) Wall temp} \Rightarrow T = T_0 \end{array}\right\} \text{At wall}$$

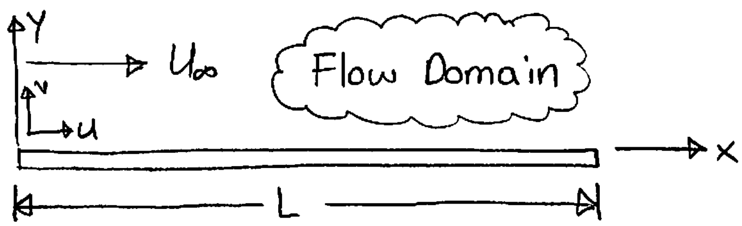
$$\left.\begin{array}{l} \text{iv) Uniform flow} \Rightarrow u = U_\infty \\ \text{v) Uniform flow} \Rightarrow v = 0 \\ \text{vi) Uniform temperature} \Rightarrow T = T_\infty \end{array}\right\} \begin{array}{l}\text{Infinitely far from the} \\ \text{wall, in both } x \text{ \& } y \text{ directions}\end{array}$$

This formulation is nice, but too cumbersome to solve without a computer, so we'll need to simplify it. To do this, we will use scaling!

The way to simplify our analysis is to realize that there is a thin region close to the plate that contains all the 'action'. Beyond this region, the free stream fluid cannot tell wether the plate is even there.

This concept was developed by Prandtl in early 1900's & it solved a very old problem called <u>D'Alembert's paradox.</u>

Basically, if you non-dimensionalize the momentum equations:



Let's work with x-momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\bar{u} = \frac{u}{U_\infty}, \quad \bar{v} = \frac{v}{U_\infty}, \quad \bar{P} = \frac{P}{\rho U_\infty^2}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}$$

Our dimensionless x-momentum equation becomes:

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{\partial \bar{P}}{\partial \bar{x}} + \frac{1}{Re_L}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

where $Re_L = \frac{\rho U_\infty L}{\mu} \equiv$ Reynolds number $= \frac{\text{inertial forces}}{\text{viscous forces}}$.

So we see as $Re_L \rightarrow \infty$, viscous forces (drag) $\rightarrow 0$. This is the source of D'Alembert's paradox.

To fix this problem, Ludwig Prandtl in the early 1900's experimentally realized that there is a very thin region adjacent to the plate where high velocity gradients exist. He called this region the boundary layer.

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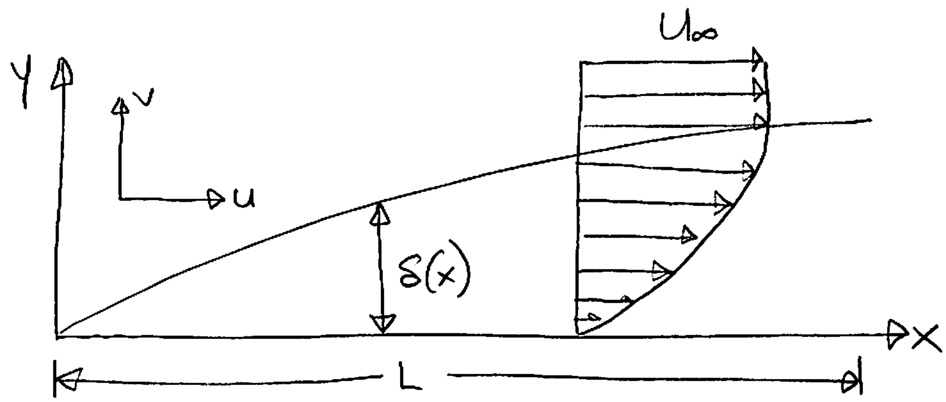

To solve D'Alembert's Paradox, we realize that we've done our scaling incorrectly. To solve the problem we must do our scaling within the boundary layer, where the flow "feels" the plate.

$$u \sim U_\infty, \quad x \sim L, \quad y \sim \delta; \text{ where } \delta \ll L$$

$$\underbrace{\underbrace{u\frac{\partial u}{\partial x}}_{U_\infty \frac{U_\infty}{L}} + \underbrace{v\frac{\partial u}{\partial y}}_{v \frac{U_\infty}{\delta}}}_{\text{Inertia}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\substack{-\frac{P}{\rho L} \\ \text{Pressure}}} + \underbrace{\nu\left[\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\nu\frac{U_\infty}{L^2}} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\nu\frac{U_\infty}{\delta^2}}\right]}_{\text{Viscosity}}$$

From continuity : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{U_\infty}{L} + \frac{v}{\delta} = 0 \Rightarrow v \sim U_\infty\left(\frac{\delta}{L}\right)$$

Back substituting into our momentum scaling:

$$\frac{U_\infty^2}{L} = -\frac{P}{\rho L} + \nu\left[\frac{U_\infty}{L^2} + \frac{U_\infty}{\delta^2}\right]$$

Dividing our viscous terms to find the dominant one:

$\cancel{\nu}\frac{\cancel{U_\infty}}{L^2} \Big/ \cancel{\nu}\frac{\cancel{U_\infty}}{\delta^2} = \left(\frac{\delta}{L}\right)^2 \ll 1 \Rightarrow$ Hence only the second viscous term is important.

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So from scaling, we reduce our x-momentum eqn. to:

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}} \quad ①$$

Now lets work with our y-momentum equation: $\left(v \sim \frac{\delta}{L}U_\infty\right)$

$$\underbrace{u\frac{\partial v}{\partial x}} + \underbrace{v\frac{\partial v}{\partial y}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial y}} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$$U_\infty\frac{\delta}{L}U_\infty\frac{1}{L} + U_\infty^2\left(\frac{\delta}{L}\right)^2\frac{1}{\delta} \sim \frac{P}{\rho\delta} + \nu\frac{\delta}{L}U_\infty\frac{1}{L^2} + \nu\frac{\delta}{L}U_\infty\frac{1}{\delta^2}$$

Rearanging our terms

$$\frac{U_\infty^2}{L}\left(\frac{\delta}{L}\right) + \frac{U_\infty^2}{L}\left(\frac{\delta}{L}\right) \sim \frac{P}{\rho\delta} + \nu\frac{U_\infty}{L}\cdot\frac{\delta}{L^2} + \nu\frac{U_\infty}{L}\frac{1}{\delta}$$

Let's check which viscous term dominates

$$\frac{\cancel{\nu}\frac{\cancel{U_\infty}}{\cancel{L}}\cdot\frac{\delta}{L^2}}{\cancel{\nu}\frac{\cancel{U_\infty}}{\cancel{L}}\cdot\frac{1}{\delta}} = \left(\frac{\delta}{L}\right)^2 << 1 \Rightarrow \text{So our second term dominates}$$

Now we need to look at which term dominates.
Re-scaling our pressure term as:

$$P \sim \rho U_\infty^2 \quad \left(\text{note we don't use } \rho v^2 \text{ since } v \sim \left(\frac{\delta}{L}\right)U_\infty\right)$$

Our scaled equation becomes:

$$\underbrace{\frac{U_\infty^2}{L}\left(\frac{\delta}{L}\right)}_{\text{Inertia (I)}} \sim \underbrace{\frac{U_\infty^2}{\delta}}_{\text{Pressure (P)}} + \underbrace{\nu\frac{U_\infty}{L}\frac{1}{\delta}}_{\text{Viscosity (V)}}$$

$$\frac{I}{P} \sim \frac{\cancel{U_\infty^2}}{L}\left(\frac{\delta}{L}\right)\cdot\frac{\delta}{\cancel{U_\infty^2}} \sim \left(\frac{\delta}{L}\right)^2 \ll 1$$ (Inertia is negligible compared to pressure)

$$\frac{V}{\rho} \sim \nu \frac{\cancel{U_\infty}}{L} \cdot \frac{1}{\cancel{\delta}} \cdot \frac{\cancel{\delta}}{\cancel{U_\infty^2}} \sim \nu \frac{1}{L}$$ (Cannot neglect viscosity)

Hence our y-momentum equation can by written as:

$$\boxed{0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \frac{\partial^2 v}{\partial y^2}} \quad (2)$$

Note, equations (1) and (2) are still too cumbersome to solve analytically. We can look at our pressure terms to help us. Comparing $\partial P/\partial x$ with $\partial P/\partial y$ using scaling:

$$\frac{\partial P}{\partial x} \sim \frac{\rho U_\infty^2}{L} \quad \text{or} \quad \frac{\partial P}{\partial x} \sim \frac{\mu U_\infty}{\delta^2}$$ (either one is valid as long as we are consistent in both)

$$\frac{\partial P}{\partial y} \sim \frac{\mu V}{\delta^2} \sim \frac{\mu U_\infty}{\delta^2}\left(\frac{\delta}{L}\right)$$

Dividing through to see which dominates:

$$\frac{\partial P}{\partial y} \Big/ \frac{\partial P}{\partial x} \sim \frac{\cancel{\mu U_\infty}}{\cancel{\delta^2}} \cdot \left(\frac{\delta}{L}\right) \cdot \frac{\cancel{\delta^2}}{\cancel{\mu U_\infty}} \sim \left(\frac{\delta}{L}\right) \ll 1$$

So from this, we can say that pressure variations in the x-direction dominate the pressure variations in the y-direction.

Hence: $\boxed{P = f(x) \text{ only}}$ (3)

This implies that inside the boundary layer, the pressure at any x is approximatelly equal to the pressure outside of it.

$$\boxed{\frac{\partial P}{\partial x} = \frac{\partial P_\infty}{\partial x}} \quad (4)$$

Back substituting (4) into (1)

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P_\infty}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}} \quad (5)$$

↳ Boundary Layer momentum equation.

We can stipulate our $\partial P_\infty/\partial x$ term by taking a streamline in the free stream (Bernoulli equation). Here we have inviscid flow, (effects of viscosity are negligible).

$$\frac{\partial}{\partial x}\left(P_\infty + \frac{1}{2}\rho_\infty U_\infty^2 = \text{constant}\right)$$

$$\frac{\partial P_\infty}{\partial x} + \frac{1}{2}\rho_\infty \frac{\partial (U_\infty^2)}{\partial x} = 0$$

$$\frac{\partial P_\infty}{\partial x} + \frac{1}{2}\rho_\infty 2 U_\infty \frac{\partial U_\infty}{\partial x} = 0$$

$$\boxed{\frac{\partial P}{\partial x} = -\rho_\infty U_\infty \frac{\partial U_\infty}{\partial x}} \quad (6)$$

From continuity:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \quad (7)$$

Equations (5) - (7) constitute the general boundary layer equations in 2-D and in cartesian co-ordinates.

In general, there are 4 methods to solve these equations for simplified cases:

1) Similarity Solution
2) Momentum Integral Methods
3) Scaling
4) Computational (we will not do this method in this class)

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Laminar External Flow over a Flat Plate

With flow over a flat plate, we know that U_∞ = constant.
Therefore:

$$\frac{\partial P}{\partial x} = -\rho_\infty U_\infty \overbrace{\frac{\partial U_\infty}{\partial x}}^{=0} = 0$$

Our boundary layer equations become:

$$\boxed{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}} \quad (8) \Rightarrow \text{Momentum}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \quad (9) \Rightarrow \text{Continuity}$$

Our boundary conditions are:

$$u = v = 0 \quad \text{at} \quad y = 0$$
$$u = U_\infty \quad \text{at} \quad y \to \infty$$

We can also at this point write out our boundary layer energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

Similar to our previous analysis, we can simplify with scaling:

$$\left.\begin{aligned}\frac{\partial^2 T}{\partial x^2} &\sim \frac{\Delta T}{L^2} \\ \frac{\partial^2 T}{\partial y^2} &\sim \frac{\Delta T}{\delta_T^2}\end{aligned}\right\} \quad \frac{\cancel{\alpha}\frac{\partial^2 T}{\partial x^2}}{\cancel{\alpha}\frac{\partial^2 T}{\partial y^2}} \sim \frac{\frac{\Delta T}{L^2}}{\frac{\Delta T}{\delta_T^2}} \sim \left(\frac{\delta_T}{L}\right)^2 \ll 1$$

Hence, our boundary layer energy equation becomes:

$$\boxed{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}} \quad (10)$$

Note here, $\delta \neq \delta_T$. Both are small compared to L.

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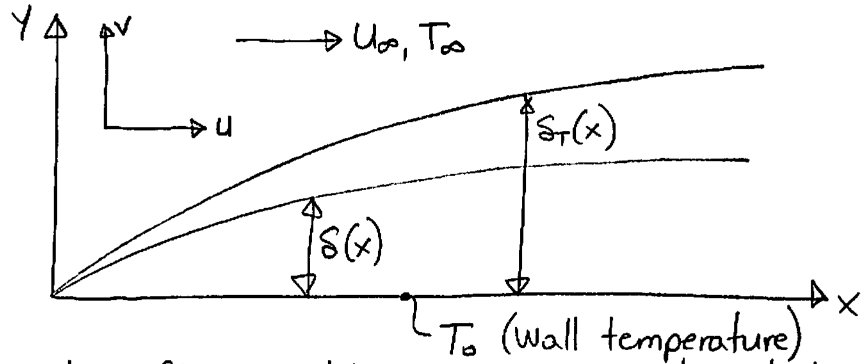
Scale Analysis



$\Delta T = T_0 - T_\infty$

For the flow problem, we are interested in shear, τ

$$\tau = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} \sim \mu \cdot \frac{U_\infty}{\delta}$$

To solve for τ, we must estimate δ using scaling.
From equation (8):

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\substack{\frac{U_\infty^2}{L} \\ \text{Inertia}}} = \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\substack{\nu\frac{U_\infty}{\delta^2} \\ \text{Friction (Viscosity)}}}$$

So from scaling, we can say that in the boundary layer, the inertial forces scale with the friction forces (are of the same order of magnitude)

$$\frac{U_\infty^2}{L} \sim \nu\frac{U_\infty}{\delta^2} \Rightarrow \delta \sim \left(\frac{\nu L}{U_\infty}\right)^{1/2} \text{ or } \boxed{\frac{\delta}{L} \sim Re_L^{-1/2}} \quad (11)$$

where $Re_L = U_\infty L/\nu$.

Note, we've defined our boundary layer analysis based on the fact that $\frac{\delta}{L} \ll 1$, hence, we can now check this condition.

So b.l. analysis is valid as long as $\boxed{Re_L^{1/2} \gg 1}$

So now we can solve for our shear stress τ

$$\tau \sim \mu \frac{U_\infty}{\delta}\left(\frac{\delta}{L}\right)\left(\frac{L}{\delta}\right) \sim \mu \frac{U_\infty}{L} Re_L^{1/2} \sim \rho U_\infty^2 Re_L^{-1/2}$$

So: $\boxed{\tau \sim \rho U_\infty^2 Re_L^{-1/2}}$

Defining a dimensionless skin friction coefficient as: C_f

$$C_f = \frac{\tau}{\frac{1}{2}\rho U_\infty^2} \Rightarrow \boxed{C_f \sim Re_L^{-1/2}}$$

We shall see later how well these scaling solutions hold up to more exact results.

Now we can look at the heat transfer aspect of the problem:

$$\underbrace{\underbrace{u\frac{\partial T}{\partial x}}_{u\frac{\Delta T}{L}} + \underbrace{v\frac{\partial T}{\partial y}}_{v\frac{\Delta T}{\delta_T}}}_{\text{Convection}} = \underbrace{\underbrace{\alpha\frac{\partial^2 T}{\partial y^2}}_{\alpha\frac{\Delta T}{\delta_T^2}}}_{\text{Conduction}}$$

Here, we are interested in the heat transfer coefficient at the wall, h:

$$h = \frac{-k\left.\frac{\partial T}{\partial y}\right|_{y=0}}{\Delta T} \sim \frac{k\left(\Delta T/\delta_T\right)}{\Delta T} \sim \frac{k}{\delta_T}$$

Now we must be a bit more carifull because we are involving flow properties (δ) with heat transfer properties (δ_T). We also can't simply say $u \sim U_\infty$ since the u considered here is within the thermal boundary layer, not the hydrodynamic.

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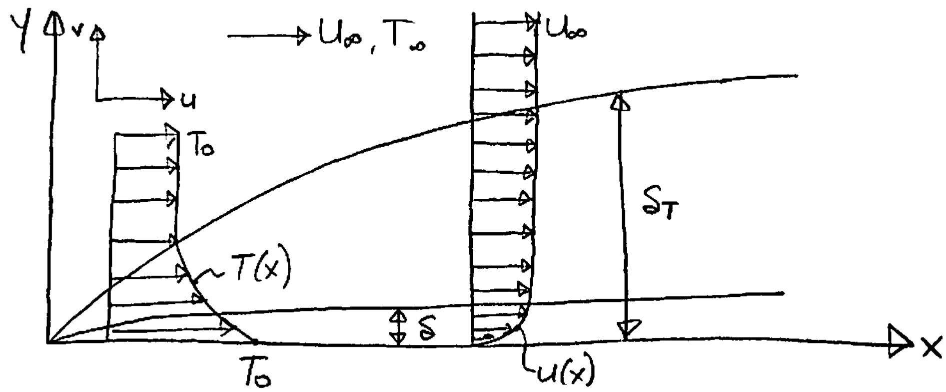
Thick Thermal Boundary Layer ($\delta_T \gg \delta$)



In this limit, the thermal boundary layer is much thicker compared to the velocity boundary layer at any x, i.e. $\frac{\delta}{\delta_T} \ll 1$

Looking back at our energy equation:

$$\underbrace{u\frac{\Delta T}{L} + v\frac{\Delta T}{\delta_T}}_{\text{convection}} \sim \underbrace{\alpha\frac{\Delta T}{\delta_T^2}}_{\text{conduction}}$$

For this case, in the thermal b.l., $u \sim U_\infty$
From continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{U_\infty}{L} + \frac{v}{\delta} \Rightarrow \underbrace{v \sim \frac{U_\infty \delta}{L}}$$

So now going back to our scaling

Note, when scaling for velocity, always use the b.l. that characterizes the flow in the thermal b.l.

$$u\frac{\Delta T}{L} \sim U_\infty\frac{\Delta T}{L}$$

$$v\frac{\Delta T}{\delta_T} \sim \frac{U_\infty \delta \Delta T}{L\ \delta_T} \sim \frac{U_\infty \Delta T}{L}\left(\frac{\delta}{\delta_T}\right) \ll 1 \quad \text{(This term is negligible)}$$

Hence, we can now write:

$$U_\infty\frac{\Delta T}{L} \sim \alpha\frac{\Delta T}{\delta_T^2} \Rightarrow \boxed{\frac{\delta_T}{L} \sim Pr^{-1/2} Re_L^{-1/2}}\ ; \ \boxed{Pr = \frac{\nu}{\alpha}}$$

↳ Prandtl #

Re-writing this as: $\boxed{Pe_L = Pr \cdot Re_L \equiv \text{Peclet Number} = U_\infty L/\alpha \equiv \dfrac{\text{Rate of thermal advection}}{\text{Rate of thermal diffusion}}}$

$$\boxed{\frac{\delta_T}{L} \sim Pe_L^{-1/2}}$$

Interestingly, we can see that:

$$\frac{\delta}{L} \sim Re_L^{-1/2} \quad (\text{equation (11)}) \quad \text{and} \quad \frac{\delta_T}{L} \sim Pe_L^{-1/2} \sim Pr^{-1/2} Re_L^{-1/2}$$

$$\frac{\delta_T}{\cancel{L}} \cdot \frac{\cancel{L}}{\delta} = \frac{\delta_T}{\delta} \sim \frac{Pr^{-1/2} \cancel{Re_L^{-1/2}}}{\cancel{Re_L^{-1/2}}} \sim Pr^{-1/2}$$

$$\therefore \boxed{\frac{\delta_T}{\delta} \sim Pr^{-1/2} \gg 1} \; ; \quad Pr = \frac{\nu}{\alpha}$$

Therefore, our first assumption of $\frac{\delta_T}{\delta} \gg 1$ is only valid if $Pr^{1/2} \ll 1$. This is true for liquid metals ($Pr \approx 0.0001$).

Now we can solve for heat transfer:

$$\boxed{h \sim \frac{k}{\delta_T} \sim \frac{k}{L} Pr^{1/2} Re_L^{1/2}} \quad \text{for } Pr \ll 1,$$

Or re-writing this in terms of Nusselt number:

$$\boxed{Nu = \frac{hL}{k} \sim Pr^{1/2} Re_L^{1/2}}$$